Unifying Preference and Judgment Aggregation

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ABSTRACT

The paper proposes a unification of the two main frameworks commonly used for the analysis of collective decision-making: the framework of preference aggregation, developed from the seminal work of K. Arrow on social choice theory; and the more recent framework of judgment aggregation. Such unification provides several original insights on collective decision-making problems. The methods used are based on logic and, in particular, on formal semantics.

Categories and Subject Descriptors

I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multiagent Systems*

General Terms

Economics, Theory

Keywords

Collective decision-making, social-choice theory, preference aggregation, judgment aggregation, many-valued logic

1. INTRODUCTION

In recent years, theories and techniques originally developed in economics have been systematically deployed in the analysis of several problems arising from the organization, coordination and management of Multiagent Systems. One of these theories, which has been obtaining increasing attention, is social choice theory [7]. Social choice was born with the pioneering work of K. Arrow [1] and is broadly concerned with designing and analyzing methods for collective decision-making, and in particular with charting the contours of the possibility (and impossibility) of such methods.

Preference aggregation (PA), and the younger judgment aggregation (JA), are two related sub-disciplines of social choice theory which are of particular relevance for Multiagent Systems [3]. While PA studies the aggregation of a profile of agents' preferences into one collective preference, JA (see [12] for an overview of the field) studies the aggregation of profiles of agents' judgments concerning the acceptance/rejection of a set of issues displaying logical form. The problem, in both these research areas, is for the aggregation process to preserve, in a non-trivial way, some characteristic 'rational' aspects of the individual to-be-aggregated stances, e.g., transitivity in the case of preferences, and logical consistency in the case of judgments.

The paper aims at displaying the common formal roots of PA and JA, showing how the two frameworks can be fruitfully unified. This unification is motivated by the conviction that the development of techniques and formal systems, has to be backed by an appropriate analysis of the correspondences between such systems, in order to make it possible to: (i) easily compare results and transfer them across different systems; (ii) provide generalizations of available systems in a principled way. The present paper offers examples of both these points. After briefly introducing the two frameworks of PA and JA (Section 2), the paper provides in Section 3 a correspondence result between the PA framework and the JA framework (Theorem 1). It will also be shown how, via such correspondence, both Arrow's theorem and Sen's impossibility of Paretian Liberal [13] can be imported in JA. On the ground of theses findings, Section 4 shows how PA can be extended in order to deal with preferences exhibiting logical structures exactly like the judgments in JA. Finally, Section 5 investigates this extended version of the PA framework, providing a characterization of dictatorship (Theorem 2). Section 6 briefly concludes. Sketch of proofs of Theorems 1 and 2 are given in the Appendix.

2. PRELIMINARIES

This section is devoted to an introduction of PA and JA.

2.1 Preference Aggregation

PA concerns the aggregation of the preferences of several agents into one collective preference. A preference relation $\preceq$ on a set of issues $\mathtt{Iss}^P$ is a total preorder, i.e., a binary relation which is reflexive, transitive, and total. $\mathfrak{P}(\mathtt{Iss}^P)$ denotes the set of all total preorders of a set $\mathtt{Iss}^P$. As usual, on the ground of $\preceq$ we can define its asymmetric and symmetric parts: $x \prec y$ iff $(x, y) \in \preceq$ & $(y, x) \notin \preceq$; $x \approx y$ iff $(x, y) \in \preceq$ & $(y, x) \in \preceq$. The notion of PA structure can now be defined.

Definition 1. *(Preference aggregation structure) A PA structure is a quadruple $\mathfrak{S}^P = \langle Agn^P, Iss^P, Prf^P, Agg^P \rangle$ such that: Agn^P is a finite set of agents such that $1 \leq |Agn^P|$; Iss^P is a finite set of issues such that $3 \leq |Iss^P|$; Prf^P is the set of all preference profiles, i.e., $|Agn^P|$-tuples $\mathfrak{p} = (\preceq_i)_{i \in Agn^P}$ where each $\preceq_i$ is a total preorder over Iss; Agg^P*

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is a function taking each $\mathfrak{p} \in \mathit{Prf}^P$ *to a total preorder over* Iss, *i.e.,* $\mathit{Agg}^P : \mathit{Prf}^P \longrightarrow \mathfrak{P}(\mathit{Iss}^P)$. *The value of* Agg^P *is denoted* $\preceq$.

Typically, PA studies the aggregation function $\mathtt{Agg}^P$ under the assumption that it satisfies specific conditions. The most common of such conditions are the following ones:

Unanimity (**U**). If all agents strictly prefer x over y, so does the aggregated preference: $(\forall x, y \in \mathtt{Iss}^P)(\forall \mathfrak{p} \in \mathtt{Prf}^P)[[(\forall i \in \mathtt{Agn}^P)[y \prec_i x]] \Rightarrow [y \prec x]]$.

Independence (**I**).[1] The aggregated preference over x and y depends only on the agents' preferences over x and y: $(\forall x, y \in \mathtt{Iss}^P)(\forall \mathfrak{p}, \mathfrak{p}' \in \mathtt{Prf}^P)[[(\forall i \in \mathtt{Agn}^P)[y \preceq_i x \Leftrightarrow y \preceq'_i x]] \Rightarrow [y \preceq x \Leftrightarrow y \preceq' x]]$.

Non-dictatorship (**NoDict**). There is no agent i such that, for any profile, the aggregated preference relation agrees with the asymmetric part of i's preferences: $(\nexists i \in \mathtt{Agn}^P)(\forall x, y \in \mathtt{Iss}^P)(\forall \mathfrak{p} \in \mathtt{Prf}^P)[y \prec_i x \Rightarrow y \prec x]$.

Notice that the definition of $\mathtt{Agg}^P$ in Definition 1 directly incorporates the aggregation conditions usually referred to as Universal domain and Collective rationality. In the rest of the paper the superscript P will be often omitted when this does not give rise to confusion.

2.2 Judgment Aggregation

Judgment aggregation (JA) concerns the aggregation of judgments about the acceptance or rejection of a set of interrelated issues (i.e., logical formulae) into one collective judgment. This section introduces a framework for JA built on the language of propositional logic. The way the issues to be judged upon are interrelated with one another depends on the logic underlying the aggregation problem.

2.2.1 Logic

The logic underlying the aggregation problem defines the notion of "rationality" by which the individual and collective judgments should abide. For instance, presupposing propositional logic, if p and q are accepted as true, so should $p \wedge q$.

In this paper we assume the semantics of the propositional language to be based on many-valued propositional valuation functions f on the $[0, 1]$ interval. As to the meaning of $\{\neg, \wedge, \vee\}$ the standard algebraic operations of complementation, max, and min are assumed:

$$\begin{aligned} f^*(\top) &= 1 \\ f^*(\neg\phi) &= 1 - f^*(\phi) \\ f^*(\phi \wedge \psi) &= \mathtt{min}(f^*(\phi), f^*(\psi)) \\ f^*(\phi \vee \psi) &= \mathtt{max}(f^*(\phi), f^*(\psi)) \end{aligned}$$

where f is a valuation function and f^* is its inductive extension. These operations will be of relevance only in Sections 4 and 5 where we will also discuss the algebraic meaning of implication.

The designated value for the notion of satisfaction is 1: $f \models \phi$ iff $f^*(\phi) = 1$. To put it in the JA jargon, issue ϕ is accepted iff it is attributed value 1. Note that it does not hold in general that $f \not\models \phi$ iff $f \models \neg\phi$, or equivalently that $f^*(\phi) \neq 1$ iff $f^*(\phi) = 0$, where f is a valuation function for the propositional language and f^* its inductive extension. In the remaining of the paper, in order to denote that a formula ϕ is assigned a value different from 1, we will use the notation $\overline{\phi}$.

This setting slightly generalizes the JA framework built on classic propositional logic [11].[2] We can introduce now the two central notions of JA: agendas and judgment sets.

2.2.2 Agendas and Judgment Sets

As anticipated above, the issues $\mathtt{Iss}^J$ of a JA problem are propositional formulae. The set $\mathtt{Iss}_0^J$ denotes the set of propositional atoms in $\mathtt{Iss}$. Note that it is not necessarily the case that $\mathtt{Iss}_0^J \subseteq \mathtt{Iss}^J$. In other words, the issues might all be complex formulae. In what follows we will often refrain from using the superscript J when possible.

Agendas. Intuitively, an agenda is a syntactic entity denoting all the possible positions that agents can assume about some issue. The fact that an agent can accept an issue ϕ is denoted by ϕ belonging to the agenda, and the fact that it can reject an issue ϕ is denoted by the fact that $\overline{\phi}$ belongs to the agenda. Formally, the set $\mathfrak{ag}(\mathtt{Iss}) = \{\phi \mid \phi \in \mathtt{Iss}\} \cup \{\overline{\phi} \mid \phi \in \mathtt{Iss}\}$ denotes the *agenda* of $\mathtt{Iss}$.

Semantically, a JA agenda consists of a set of pairs $(\phi, \overline{\phi})$, each member in such pairs stating that ϕ is assigned value 1 and, respectively, a different value from 1, which is 0 in the standard propositional case. Put it yet otherwise, an agenda is a set of properties of many-valued propositional valuation functions, which is closed under negation: ϕ is the property enjoyed by those functions f such that $f \models \phi$; and $\overline{\phi}$ is the property enjoyed by those functions f such that $f \not\models \phi$.

At this point it is worth stressing that we have assumed a slightly different perspective on agendas than the standard literature on JA. Normally, an agenda is viewed as a set of position/negation pairs $(\phi, \neg\phi)$. In this view, the judgment themselves can be seen as formulae of the propositional language from which the issues are drawn. Instead, we see judgments as meta-formulae stating whether a given issue is accepted (true) or rejected (not true), i.e., as properties or constraints of valuation functions.

Judgment sets. A judgment set picks one element out of all position/negation pairs in the agenda. More formally, a *judgment set* for the agenda $\mathfrak{ag}(\mathtt{Iss})$ is a set $J \subseteq \mathfrak{ag}(\mathtt{Iss})$ which is non-contradictory and complete ($\forall \phi \in \mathtt{Iss}$ either $\phi \in J$ or $\overline{\phi} \in J$ but not both), and which keeps consistency with respect to the underlying semantics, i.e., which is satisfiable by at least one valuation function f of the atoms in $\mathtt{Iss}_0$. The set of all judgment sets for the agenda built on $\mathtt{Iss}$ is denoted by $\mathfrak{J}(\mathtt{Iss})$.

The following simple observations are relevant for our purposes. Given an agenda $\mathfrak{ag}(\mathtt{Iss})$ and a valuation f of $\mathtt{Iss}_0$, the set $\mathbb{T}_{\mathtt{Iss}}(f) := \{\phi \mid \phi \in \mathtt{Iss}, f \models \phi\} \cup \{\overline{\phi} \mid \phi \in \mathtt{Iss}, f \not\models \phi\}$ is clearly a judgment set. Now, two valuations can be said to be equivalent with respect to an agenda $\mathfrak{ag}(\mathtt{Iss})$ when they give rise to the same judgment sets:

$$f \sim_{\mathtt{Iss}} f' \quad \text{iff} \quad \mathbb{T}_{\mathtt{Iss}}(f) = \mathbb{T}_{\mathtt{Iss}}(f') \tag{1}$$

Now, let $\mathfrak{F}^{\mathtt{Iss}_0}$ be the set of all valuation functions under consideration. An agenda $\mathfrak{ag}(\mathtt{Iss})$ partitions $\mathfrak{F}^{\mathtt{Iss}_0}$ into equivalence classes: $|f|_{\mathtt{Iss}} = \{f' \mid f \sim_{\mathtt{Iss}} f'\}$. On the other hand, a judgment set J always determines a set of valuations $\widehat{J}$ for

[1] This condition is more commonly named **IIA** (Independence of Irrelevant Alternatives).

[2] Recall that, in propositional logic, a formula ϕ is assigned value 0 (i.e., it is false) iff ϕ is assigned a value different from 1 (i.e., it is not true).

the set of issues $\texttt{Iss}$, which is so defined:

$$\widehat{J} := \{f \mid f^*(\phi) = 1 \text{ iff } \phi \in J\} \quad (2)$$

The following semantic characterization follows from the definition of judgment set, the definition of $\widehat{J}$ in Formula 2, and of $\sim_{\texttt{Iss}}$ in Formula 1.

FACT 1 (CONTENT OF JUDGMENT SETS). *Let $\mathfrak{ag}(Iss)$ be an agenda, $J \in \mathfrak{J}(Iss)$ and $f \in \mathfrak{F}^{Iss_0}$ such that $\mathbb{T}_{Iss}(f) = J$. It holds that:*

$$\widehat{J} \quad = \quad |f|_{Iss} \quad (3)$$

In other words, each judgment set for a given agenda corresponds to an equivalence class in the partition of the valuation space yielded by the agenda.[3] On the ground of these considerations, in what follows we will often use the semantically connoted "$\widehat{J} \models \phi$", instead of "$\phi \in J$", and "$\widehat{J} \not\models \phi$" instead of "$\overline{\phi} \in J$".

2.2.3 JA structures

We can now define the structure of the JA problem.

DEFINITION 2. *(Judgment aggregation structure) A JA structure is a quadruple $\mathfrak{S}^J = \langle Agn^J, Iss^J, Prf^J, Agg^J \rangle$ where: Agn^J is a finite set of agents such that $1 \leq |Agn^J|$; Iss^J is a finite set of issues consisting of propositional formulae and containing at least two atoms, i.e., $2 \leq |Iss_0^J|$; Prf^J is the set of all judgment profiles, i.e., $|Agn^J|$-tuples $\mathtt{j} = (J_i)_{i \in Agn^J}$ where each J_i is a judgment set for the agenda $\mathfrak{ag}(Iss^J)$; Agg^J is a function taking each $\mathtt{j} \in Prf^J$ to a judgment set for $\mathfrak{ag}(Iss^J)$, i.e., $Agg^J : Prf^J \longrightarrow \mathfrak{J}(Iss^J)$. J denotes the value of Agg^J.*

Just like PA, JA studies aggregation functions under specific conditions. The following conditions are JA variants of the ones presented for PA in the previous section:[4]

Unanimity ($\mathbf{U}^{\models}$). If all agents accept (or reject) x, then so does the aggregated judgment: $(\forall x \in \texttt{Iss}^J)(\forall \mathtt{j} \in \texttt{Prf}^J)$ $[[[(\forall i \in \texttt{Agn}^J)\widehat{J}_i \models x] \Rightarrow \widehat{J} \models x]]$ & $[[(\forall i \in \texttt{Agn}^J)\widehat{J}_i \not\models x] \Rightarrow \widehat{J} \not\models x]]$.

Independence ($\mathbf{I}^{\models}$). The aggregated judgment on x depends only on the individual judgments on x: $(\forall x \in \texttt{Iss}^J)(\forall \mathtt{j}, \mathtt{j}' \in \texttt{Prf}^J)$ $[(\forall i \in \texttt{Agn}^J)[\widehat{J}_i \models x \Leftrightarrow \widehat{J'}_i \models x] \Rightarrow [\widehat{J} \models x \Leftrightarrow \widehat{J'} \models x]]$.

Systematicity ($\mathbf{Sys}^{\models}$). If the agents' judgments on x are interdependent on the agents' judgments on y, then so are the aggregated judgments: $(\forall x, y \in \texttt{Iss}^J)(\forall \mathtt{j}, \mathtt{j}' \in \texttt{Prf}^J)$ $[[[(\forall i \in \texttt{Agn}^J)[\widehat{J}_i \models x \Leftrightarrow \widehat{J'}_i \models y]] \Rightarrow [\widehat{J} \models x \Leftrightarrow \widehat{J'} \models y]]$ & $[[(\forall i \in \texttt{Agn}^J)[\widehat{J}_i \models x \Leftrightarrow \widehat{J'}_i \not\models y]] \Rightarrow [\widehat{J} \models x \Leftrightarrow \widehat{J'} \not\models y]]]$.

Non-dictatorship ($\mathbf{NoDict}^{\models}$). There is no agent i such that the value of the aggregation function is always the i^{th}-projection of its argument: $(\nexists i \in \texttt{Agn}^J)(\forall x \in \texttt{Iss}^J)(\forall \mathtt{j} \in \texttt{Prf}^J)[\widehat{J}_i \models x \Leftrightarrow \widehat{J} \models x]$.

Notice that Definition 2 incorporates the conditions usually referred to as Universal domain and Collective rationality.

[3]Notice that if $\texttt{Iss}$ is closed under atoms, i.e., $\texttt{Iss}_0 \subseteq \texttt{Iss}$, and we assume the standard semantics of propositional logic, then J corresponds exactly to one propositional valuation.

[4]To avoid confusion, the names of the JA conditions will contain $\models$ as a superscript.

2.3 Setting the stage

The paper builds on three simple observations: (i) preferences are actually defined by sets of statements —judgments— of the type: $(x, y) \in \preceq$ and $(x, y) \notin \preceq$; (ii) preferences can be studied in terms of numerical ranking functions u, e.g., on the $[0, 1]$ interval [4]; (iii) numerical functions can ground logical semantics, like it happens in many-valued logic [8]. In such logics, as well as in propositional logic, the semantic clause $u(x) \leq u(y)$ typically defines the satisfaction by u of the implication $x \rightarrow y$:

$$u \models x \rightarrow y \text{ iff } u(x) \leq u(y). \quad (4)$$

Intuitively, implication $x \rightarrow y$ is true (or accepted, or satisfied) iff the rank of x is at most as high as the rank of y.[5] These observations suggest, first of all, that preferences can be viewed as special kind of judgments in many-valued logic (Section 3). In addition, they suggest that the semantics of many-valued logic provides a viable ground for extending the framework of preference aggregation in order to include preferences ranging over logically complex issues, thus incorporating a characteristic feature of judgment aggregation (Sections 4 and 5). The paper systematically explores this idea and the light it sheds on the theory of aggregation.

3. PREFERENCES AS JUDGMENTS

This section establishes a correspondence between the PA structures as introduced in Definition 1 and a subclass of the JA structures introduced in Definition 2. We will proceed as follows. First of all, in Section 3.1, the simple fact is noted that every total preorder specifies a set of ranking functions with the same ordinal content. In Sections 3.2 and 3.3, it is shown that the judgment sets obtained by appropriately translating a total preorder, specify the very same set of ranking functions which is specified by the translated total preorder. This leads us to the desired correspondence. Finally, in Section 3.4, PA impossibility results are imported to JA thus obtaining interesting new interpretations.

3.1 Preferences and ranking functions

Let us first briefly recall the following well-known fact (see, for instance [4]), which follows from a simple argument based on the quotient yielded by a total preorder.

FACT 2 (REPRESENTATION OF $\preceq$ BY u). *Let $\preceq \in \mathfrak{P}(X)$. There exists a ranking function $u : X \longrightarrow [0, 1]$ such that $\forall x, y \in X$:*

$$x \preceq y \quad \textit{iff} \quad u(x) \leq u(y). \quad (5)$$

Such a function is unique up to ordinal transformations.[6]

This fact plays a central role in the present section. Notice that each ranking function u on a finite set X determines a linear order $\langle u(X), \leq \rangle$, where $u(X)$ is the set of values of u for X. In other words, Fact 2 states that each total preorder $\preceq$ specifies a non-empty set of ranking functions all determining isomorphic linear orders. Given a ranking function u, let us denote $|u|_{\langle u(X), \leq \rangle}$ the set of functions determining isomorphic linear orders with respect to u. Now, each total preorder also specifies a set of ranking functions.

[5]Notice that we do not commit at this stage to any precise semantics for $\rightarrow$, the only requirement on it being that $u \not\models x \rightarrow y$ iff $u^*(x \rightarrow y) < 1$.

[6]We recall that an ordinal transformation t is a function such that for all rankings m and n, $t(m) \leq t(n)$ iff $m \leq n$.

Definition 3 (Content of preferences). *Let $\preceq$ be a total preorder on a finite set X. The semantic content $\mathfrak{u}(\preceq)$ of $\preceq$ is defined as follows:*

$$\mathfrak{u}(\preceq) \quad = \quad \{u \mid \forall x \in X, u(x) \leq u(y) \text{ iff } x \preceq y\} \qquad (6)$$

where $u : X \longrightarrow [0, 1]$.

In other words $\mathfrak{u}(\preceq)$ is nothing but the set of all ranking functions agreeing with $\preceq$ on the order of the elements in X. The following simple fact follows from from Definition 3 and Fact 2.

Fact 3 ($\preceq$-equivalent ranking functions). *Let $\preceq \in \mathfrak{P}(X)$, $\mathfrak{u}(\preceq)$ be the content of $\preceq$, and u a ranking function on X preserving $\preceq$. It holds that:*

$$\mathfrak{u}(\preceq) \quad = \quad |u|_{\langle u(X), \leq \rangle} \qquad (7)$$

To sum up, any total preorder can be associated to a non-empty set of ranking functions which expresses exactly the same ordinal information. As a consequence, the set $\mathfrak{P}(X)$ of all total preorders over X yields a partition of the set of all ranking functions u of X.

3.2 Condorcet's paradox as a JA paradox

In Condorcet's paradox, pairwise majority voting on issues generates a collective preference which is not transitive. From Fact 2 we know that any preference relation which is a total preorder can be represented by an appropriate ranking function u with codomain $[0, 1]$. The left part of Table 1 depicts the standard version of the paradox in relational notation, and the middle part depicts the version which makes use of a ranking function u. The grey line displays the outcome obtained by pairwise majority.

The basic intuition underlying this section consists in reading the middle part of Table 1 as if u was an interpretation function of propositions x, y, z on the real interval $[0, 1]$, like Formula 4 suggests. It is then just a matter of closing the circle drawn by Formulae 4 and 5. Given a total preorder $\preceq$, there always exists a ranking function u, unique up to order-preserving transformations, such that:

$$x \preceq y \quad \text{iff} \quad u(x) \leq u(y) \quad \text{iff} \quad u \models x \rightarrow y. \qquad (8)$$

We thus obtain a direct bridge between preferences and judgments via ranking functions. So, by exploiting Formula 8 the middle part of Table 1 can be rewritten as the right part.

The type of JA paradox we obtain from Condorcet's is not just a mathematical diversion, since it relates to the aggregation of judgments in the context of fuzzy classifications. As a matter of fact, ranking functions can be viewed as fuzzy interpretation functions, and fuzzy implications[7] lie at the ground of the semantics of concept subsumption statements in fuzzy description logics [9]. In fuzzy logic, an implication denotes the relative strength of the truth-degrees of antecedent and consequent. The following example illustrates a fuzzy reading of a variant of Condorcet's paradox.

Example 1 (Fuzzy classifications). *A committee of three prosecutors has to decide whether to initiate legal action against a physician in the case of a terminally-ill patient who voluntarily refused medical treatments. The prosecutors cannot make up their minds about the juridical category to be applied. The case at hand looks part-murder, part-suicide, part-death by natural causes. An exact answer seems hard to reach, but they want to set at least some guidelines. So, they decide to vote by majority about accepting or rejecting the following statements: "it is a case of death by natural causes at least as much as a suicide" ($\texttt{scd} \rightarrow \texttt{dnc}$); "it is a case of suicide at least as much as a case of murder" ($\texttt{mrd} \rightarrow \texttt{scd}$); and "it is a case of death by natural causes at least as much as a case of murder" ($\texttt{mrd} \rightarrow \texttt{dnc}$). If the first prosecutor accepts all three statements, the second accepts only the first one, and the third accepts only the second statement then, by majority on each statement, they end up in an instance of the following table, which is a variant of Table 1:*

$\{x,y\}$	$\{y,z\}$	$\{x,z\}$	$\{x,y\}$	$\{y,r\}$	$\{x,r\}$
$y \preceq x$	$z \preceq y$	$z \preceq x$	$\models y \rightarrow x$	$\models z \rightarrow y$	$\models z \rightarrow x$
$y \preceq x$	$y \prec z$	$x \prec z$	$\models y \rightarrow x$	$\not\models z \rightarrow y$	$\not\models z \rightarrow x$
$x \prec y$	$z \preceq y$	$x \prec z$	$\not\models y \rightarrow x$	$\models z \rightarrow y$	$\not\models z \rightarrow x$
$y \preceq x$	$z \preceq y$	$x \prec z$	$\models y \rightarrow x$	$\models z \rightarrow y$	$\not\models z \rightarrow x$

To sum up, by first reading the Condorcet's paradox in terms of ranking functions (Fact 2), and then interpreting such functions from the point of view of logical semantics (Formula 8), we can show the equivalence between a concrete PA problem and a JA one. This finding is generalized in the next section.

3.3 PA = $\mathbf{JA}^{\rightarrow}_{[0,1]}$

What we are after is to show that any total preorder can be translated to a judgment set in such a way that the "ordinal content" of the total preorder is preserved by its translation. This directly yields a translation of PA structures to JA structures.

As illustrated in the previous section, preferences can be viewed as implications in many-valued semantics. Now, consider a finite set of propositional atoms $\mathbf{P}$, and the set $im(\mathbf{P}) = \{x \rightarrow y \mid x, y \in \mathbf{P}\}$. If $\texttt{Iss}^J = im(\mathbf{P})$, then it is a set of issues consisting of implications alone. In this case we call $\mathfrak{ag}(\texttt{Iss}^J)$, i.e., the agenda built on such an $\texttt{Iss}^J$, an *implicative agenda*. The desired translation function can now be defined.

Definition 4 (Translating $\preceq$). *Define the function $\mathbb{J} : \mathfrak{P}(Iss^P) \longrightarrow 2^{\mathfrak{ag}(im(Iss^P))}$ as follows:*

$$\mathbb{J}(\preceq) := \{x \rightarrow y \mid (x, y) \in \preceq\} \cup \{\overline{x \rightarrow y} \mid (x, y) \not\in \preceq\}.$$

Informally, $\mathbb{J}$ sends total preorders to subsets of the implicative agenda built out of the PA issues, i.e., where $\texttt{Iss}^J = im(\texttt{Iss}^P)$. The point is now to show that $\mathbb{J}$ does actually better, sending total preorders exactly to judgment sets. As the following fact shows, it is so that the set of functions satisfying the constraints specified by $\mathbb{J}(\preceq)$ consists exactly of the set of functions preserving the preference $\preceq$. The fact follows directly from Definition 4 and Fact 2.

Fact 4 (Semantic content of $\mathbb{J}$). *Let $\preceq \in \mathfrak{P}(X)$ and $u : X \longrightarrow [0, 1]$ preserving $\preceq$:*

$$\widehat{\mathbb{J}(\preceq)} \quad = \quad |u|_{\langle u(X), \leq \rangle} \qquad (9)$$

As an immediate consequence we also obtain that $\widehat{\mathbb{J}(\preceq)} = \mathfrak{u}(\preceq)$ (by Fact 3) and hence that $\mathbb{J}$ turns out to be a bijection between $\mathfrak{P}(\texttt{Iss}^P)$ and $\mathfrak{J}(im(\texttt{Iss}^P))$. It is now possible to prove the correspondence result.

[7] Formula 4 sets a constraint for the semantics of implication which is satisfied by several fuzzy semantics for implication. Section 4 will introduce one of such semantics, known as Gödel implication. We refer the interested reader to [8].

$\{x,y\}$	$\{y,z\}$	$\{x,z\}$
$y \prec x$	$z \prec y$	$z \prec x$
$y \prec x$	$y \prec z$	$x \prec z$
$x \prec y$	$z \prec y$	$x \prec z$
$y \prec x$	$z \prec y$	$x \prec z$

$\{x,y\}$	$\{y,z\}$	$\{x,z\}$
$u(y) < u(x)$	$u(z) < u(y)$	$u(z) < u(x)$
$u(y) < u(x)$	$u(y) < u(z)$	$u(x) < u(z)$
$u(x) < u(y)$	$u(z) < u(y)$	$u(x) < u(z)$
$u(y) < u(x)$	$u(z) < u(y)$	$u(x) < u(z)$

$\{x,y\}$	$\{y,z\}$	$\{x,z\}$
$\not\models x \rightarrow y$	$\not\models y \rightarrow z$	$\not\models x \rightarrow z$
$\not\models x \rightarrow y$	$\not\models z \rightarrow y$	$\not\models z \rightarrow x$
$\not\models y \rightarrow x$	$\not\models y \rightarrow z$	$\not\models z \rightarrow x$
$\not\models x \rightarrow y$	$\not\models y \rightarrow z$	$\not\models z \rightarrow x$

Table 1: The many faces of Condorcet's paradox.

THEOREM 1 (CORRESPONDENCE). *Let* $\preceq \in \mathfrak{P}(\mathit{Iss}^P)$. *It holds that:*

$$x \preceq y \quad \textit{iff} \quad \widehat{\mathbb{J}(\preceq)} \models x \rightarrow y. \tag{10}$$

Leaving technicalities aside, Theorem 1 states that each total preorder can be translated to a judgment set which has exactly the same ordinal content, that is to say, which orders the alternatives in $\mathtt{Iss}^P$ in the same way. Table 2 spells out the judgments corresponding to the standard preference statements about a total preorder $\preceq$.

Theorem 1, in addition to the fact that function $\mathbb{J}$ is bijective, guarantees that the set of all PA structures can be mapped into the set of all JA structures in such a way that each PA structure corresponds exactly to one many-valued JA structure with an implicative agenda. Given a PA structure $\mathfrak{S}^P$, this can be easily done by constructing the corresponding JA structure $\mathbb{J}(\mathfrak{S}^P)$ as follows: we pose $\mathtt{Agn}^J := \mathtt{Agn}^P$; $\mathtt{Iss}^J := im(\mathtt{Iss}^P)$; $\mathtt{Prf}^J := \mathbb{J}(\mathtt{Prf}^P)$ (where $\mathbb{J}$ over sets of profiles is defined in the natural way); and finally, $\mathtt{Agg}^J(\mathbb{J}(\mathfrak{p})) := \mathbb{J}(\mathtt{Agg}^P(\mathfrak{p}))$. Let us call such JA structures *many-valued implicative* JA structures, and let us denote the JA problem they formalize as $\mathrm{JA}^{\rightarrow}_{[0,1]}$. We thus find a surjective map of PA into $\mathrm{JA}^{\rightarrow}_{[0,1]}$.

It is worth to briefly compare Theorem 1 with the correspondence between PA and JA first studied in [11] and then further investigated in [5]. In that work, a correspondence is proven which is based on the translation of PA for linear orders, into JA for first-order logic. Roughly, $x \prec y$ is translated to $P(y,x)$ where P is a binary predicate for which the axioms of linear orders apply. First of all, Theorem 1 proves a correspondence for total preorders, and not only for linear orders. Second, as will be shown in Section 5, Theorem 1 also offers the stepping stone for unifying PA and JA by generalizing Formula 8. The first-order logic translation, although somhow more straightforward, does not seem to offer similar insights.

3.4 Importing impossibilities

We have obtained a perfect match between the standard PA structures, and a specific subset of all JA structures. It becomes therefore possible to import impossibility results across the two frameworks. In this section we show how two central results for PA structures transfer directly to $\mathrm{JA}^{\rightarrow}_{[0,1]}$ structures via $\mathbb{J}$.

Preferences		**Judgments**
$x \preceq y$	iff	$\widehat{\mathbb{J}(\preceq)} \models x \rightarrow y$
$x \prec y$	iff	$\widehat{\mathbb{J}(\preceq)} \not\models y \rightarrow x$
$x \approx y$	iff	$\widehat{\mathbb{J}(\preceq)} \models x \rightarrow y, \widehat{\mathbb{J}(\preceq)} \models y \rightarrow x$

Table 2: Preferences and judgments.

To see how results from PA can be imported to many-valued JA on implicative agendas it suffices to notice that function $\mathbb{J}$ (Definition 4) yields corresponding JA versions of the PA aggregation conditions (see Section 2.1) in the natural way. We denote the translation of a condition by prefixing $\mathbb{J}$ to the name of the condition. For example, $\mathbb{J}(\mathbf{U})$ denotes the following condition: $(\forall x, y \in im(\mathtt{Iss}^P))(\forall \mathfrak{j} \in \mathbb{J}(\mathtt{Prf}^P)[[(\forall i \in \mathtt{Agn}^P)[\widehat{J}_i \not\models x \rightarrow y]] \Rightarrow [J \not\models x \rightarrow y]]$.[8] As an example of the results that can be imported we provide the $\mathrm{JA}^{\rightarrow}_{[0,1]}$ formulation of Arrow's theorem, which follows from Arrow's theorem [1] by Proposition 1 and Definition 4.

COROLLARY 1 (ARROW IN JA). *For any* $\mathfrak{S}^J$ *in* $\mathrm{JA}^{\rightarrow}_{[0,1]}$, *there exists no aggregation function which satisfies* $\mathbb{J}(\mathbf{U})$, $\mathbb{J}(\mathbf{I})$ *and* $\mathbb{J}(\mathbf{NoDict})$.

A preference aggregation theorem which acquires an interesting interpretation in the JA setting is the so-called impossibility of a Paretian liberal [13]. Such theorem makes use of the following PA aggregation condition:

Minimal liberalism (**ML**). There are at least two agents who always dictate the ordering of at least one pair of issues each: $(\exists i \neq j \in \mathtt{Agn}^P)(\exists x, y, w, z \in \mathtt{Iss}^P)(\forall \mathfrak{p} \in \mathtt{Prf}^P)[[y \prec_i x \Rightarrow y \prec x] \;\&\; [z \prec_j w \Rightarrow z \prec w]]$.

The JA version of the theorem follows by Proposition 1 and Definition 4.

COROLLARY 2 (PARETIAN LIBERAL IN JA). *For any* $\mathfrak{S}^J$ *in* $\mathrm{JA}^{\rightarrow}_{[0,1]}$, *there exists no aggregation function which satisfies* $\mathbb{J}(\mathbf{U})$ *and* $\mathbb{J}(\mathbf{ML})$.

That is to say, there is no way of aggregating the judgments of different agents preserving unanimity if there are at least two agents who have the authority to impose the acceptance/rejection of at least one implication each. The impossibility can be illustrated by a simple example, which expands on Example 1.

EXAMPLE 2 (CONFLICTS OF EXPERTISE). *The prosecutors decide to ask two eminent lawyers—Prof.A and Prof.B—for help. Prof.A is a celebrated expert on the legislation concerning murder and suicide and so, they think, his opinion should settle the question whether the case at hand is a case of murder rather than of suicide or vice versa. Similarly, Prof.B is an expert on the legislation concerning death by natural causes and suicide and, they think, his opinion will also settle the question whether the case is a case of death by natural causes rather than a case of suicide or vice versa.*

[8] Recall Table 2.

Finally, what the two experts agree upon will also be taken as settled. So they let Prof.A and Prof.B cast their opinions:[9]

$$\widehat{Prof.A} \not\models \texttt{dnc} \rightarrow \texttt{mrd}, \not\models \texttt{mrd} \rightarrow \texttt{scd}, \not\models \texttt{dnc} \rightarrow \texttt{scd}$$

$$\widehat{Prof.B} \not\models \texttt{dnc} \rightarrow \texttt{mrd}, \not\models \texttt{scd} \rightarrow \texttt{mrd}, \not\models \texttt{scd} \rightarrow \texttt{dnc}$$

The following judgment set Prof.AB would result:

$$\widehat{Prof.AB} \not\models \texttt{dnc} \rightarrow \texttt{mrd}, \not\models \texttt{mrd} \rightarrow \texttt{scd}, \not\models \texttt{scd} \rightarrow \texttt{dnc}$$

which, however, no ranking function u can satisfy since u should be such that: $u(\texttt{mrd}) < u(\texttt{dnc})$, $u(\texttt{scd}) < u(\texttt{mrd})$ and $u(\texttt{dnc}) < u(\texttt{scd})$, which is impossible.[10]

This section has illustrated how our analysis enables the transfer of impossibility results from PA to a JA setting. In the next section we move on to the extension of the PA framework by incorporating JA-like features.

4. RANKINGS AS TRUTH VALUES

The striking difference between JA and PA is that, in JA, issues display logical form. Is there a consistent way to talk about complex issues in PA obtained by performing logical operations on atomic issues? The present section shows how to answer this question by generalizing Formula 8.

4.1 Ranking logically complex issues

In the previous section we have seen that, once we consider the set of issues $\texttt{Iss}^P$ of a preference aggregation problem $\mathfrak{S}^P$ to be a finite set of propositional atoms, any ranking function u can be viewed as an interpretation function of those atoms on the real interval $[0, 1]$. The natural question follows: how to inductively extend a function u in order to interpret issues in $\texttt{Iss}^P$ which consist of propositional formulae, and not just atoms? This question takes us into the realm of many-valued logics, and several possibilities are available. However, given Formula 8, we are looking for something in particular. We want an implication to be satisfied exactly when the antecedent is ranked at most as high as the consequent. More precisely, let us denote with $\breve{u}$ the inductive extension of the ranking function u. What we are looking for is a many-valued logic such that the following holds for any ranking function u and formulae ϕ, ψ:

$$\phi \preceq \psi \ \text{ iff } \ \breve{u}(\phi) \leq \breve{u}(\psi) \ \text{ iff } \ \breve{u}(\phi \rightarrow \psi) = 1 \qquad (11)$$

That is to say, the desired logic should be able to encode in the language the total order $\leq$ of rankings, so that $\breve{u}$ assigns the maximum ranking 1 to $\phi \rightarrow \psi$ (i.e., $\phi \rightarrow \psi$ is satisfied by u) iff the value assigned by $\breve{u}$ to ϕ is at most the same value assigned by $\breve{u}$ to ψ. The intuition behind Formula 11 consists in viewing the maximum ranking as the designated value for expressing the truth of compound formulae, and in particular of those formulae which express preferences between other formulae. Notice that Formula 11 is just a notational variant of Formula 8.

The property expressed in Formula 11 turns out to be a typical property of the family of t-norm many-valued logics, or logics based on triangular norms [8]. In such logics the connective $\rightarrow$ denotes the algebraic residuum operation on truth degrees (i.e., rankings). Residua come always in pairs with t-norm operations, so what is going to characterize the logic we are looking for is the t-norm we choose to be paired with the residuum denoted by $\rightarrow$. In the light of the JA framework introduced in Section 2, the most straightforward candidate is the algebraic infimum, denoted by the standard logic conjunction $\wedge$. To sum up, we want that the following holds for any ranking function u and formulae ϕ, ψ, ξ:

$$\breve{u}(\phi \wedge \xi) \leq \breve{u}(\psi) \quad \text{iff} \quad \breve{u}(\xi) \leq \breve{u}(\phi \rightarrow \psi) \qquad (12)$$

Then, if we assume the rest of the connectives $\neg$ and $\vee$ to denote, as usual, the algebraic complementation and, respectively, the algebraic supremum, the many-valued logic satisfying Formulae 11 and 12 is the logic known as Gödel-Dummett logic (**GD** in short).[11]

4.2 Gödel-Dummett logic

Propositional Logic and **GD** have the same language. As to the axiomatization, **GD** is axiomatized by any axiom system for propositional intuitionistic logic, plus the linearity axiom schema $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$ [8]. This axiomatization is sound and complete with respect to the concrete linearly ordered Heyting Algebra on the $[0, 1]$ interval—also known as Gödel algebra: $\mathcal{G} = \langle [0, 1], \sqcap, \sqcup, \trianglelefteq, 0, 1 \rangle$, where $\sqcap$ and $\sqcup$ are the $\texttt{min}$, respectively, $\texttt{max}$ operations, 0 and 1 are the designated elements, and $\trianglelefteq$ is the residuum operation forming an adjoint pair with $\sqcap$ [6, 10].

5. COMPLEX PREFERENCES

By using the semantics of **GD** (i.e., the Gödel algebra) it is possible to extend the PA framework in order to incorporate preferences between issues represented as logical formulae. The present section investigates this idea.

5.1 Gödel-Dummett preferences

Take a finite set Φ of propositional formulae. A **GD**-valuation of Φ yields a total preorder on Φ. We call such total preorders *Gödel-Dummett* (GD in short) *preferences*.

DEFINITION 5. *(GD preferences) A GD preference is a total preorder on a set of formulae Φ which can be mapped to the total preorder $\langle [0, 1], \leq \rangle$ by a function $u^* : \Phi \longrightarrow [0, 1]$ such that: i) u^* is a homomorphism from Φ to $\mathcal{G}$;*[12] *ii) $\forall x, y \in \Phi$, $x \preceq y$ iff $u^*(x) \leq u^*(y)$.*

A few comments are in order. Consider a set of issues $\Phi := \{p, q, p \wedge q\}$. The set of GD preferences over Φ is only a subset of all the total preorders over Φ, and they are precisely those total preorders over Φ which rank the issue $p \wedge q$ in such a way that the meaning of $\wedge$—according to $\mathcal{G}$—is preserved. In other words, aggregating GD preferences boils down to doing PA on a specifically restricted domain.

It follows that PA on logically complex preferences is, in a sense, a subset of the standard PA problem. However,

[9] We do not represent the whole judgment sets, but just their salient parts.

[10] Notice that this is nothing but a reformulation of Sen's famous example about *Lady Chatterly's Lover* [13].

[11] It is instructive to notice that, as a result, we get for $\rightarrow$ the semantics of the so-called Gödel implication [8]:

$$u^*(\phi \rightarrow \psi) = \begin{cases} 1 & \text{if } u^*(\phi) \leq u^*(\psi) \\ u^*(\psi) & \text{if } u^*(\psi) < u^*(\phi) \end{cases}$$

[12] Technically, u^* is the restriction to Φ of the homomorphism from the smallest term algebra including Φ and $\mathcal{G}$.

the standard PA problem can on the other hand be easily translated to the aggregation of GD preferences where the issues are all atomic, e.g., $\{p, q, r\}$. And in this sense, PA on logically complex preferences is a generalization of standard PA. It becomes possible to express preferences such as "the least preferred between issues x and y should be ranked at most as high as z" ($x \wedge y \preceq z$), or "the most preferred between issues x and y should be strictly less preferred than z" ($x \vee y \prec z$). The following example shows in some more detail what kind of expressivity is gained by such a generalization.

Example 3 (Voting on aggregation constraints). *Being it impossible to reach an agreement between the two experts Prof.A and Prof.B (Example 2), the three prosecutors decide to vote in order to decide which of the two experts to believe. They decide to proceed like this: each of them will cast an "approval" or "rejection" vote for both Prof.A and Prof.B but, at the same time, they will cast an "approval" or "rejection" vote also about whether to approve at least one of the two, since they want to make sure that if the pole ends up with the rejection of both professors, then they should have collectively approved such outcome. Again, they soon discover that voting by pairwise majority could lead them to the following paradoxical situation:*[13]

$\{\top, A\}$	$\{\top, B\}$	$\{\top, A \vee B\}$
$A \prec \top$	$B \prec \top$	$A \vee B \prec \top$
$\top \preceq A$	$B \prec \top$	$\top \preceq A \vee B$
$A \prec \top$	$\top \preceq B$	$\top \preceq A \vee B$
$A \prec \top$	$B \prec \top$	$\top \preceq A \vee B$

The example illustrates how, in GD preferences, the set of issues (e.g., $\{A, B, A \vee B\}$) is backed by the logical constant $\top$, which always obtains the maximal ranking. This allows us to represent a well-behaved form of voting known as approval voting [2]. In fact, the faulty procedure devised by the three prosecutors is an application of approval voting extended to issues displaying logical form.

The interesting aspect of Example 3 is that the agents cast votes on a constraint of the aggregation process itself, namely $A \vee B$. They vote whether to accept or not that at least one expert gets elected. While this is a characteristic feature of JA, it is not proper of standard PA. This shows what kind of expressivity the domain of GD preferences allows within the PA setting, enhancing the aggregation of preferences with features which are typical of the aggregation of judgments.

5.2 Dictatorship for GD preferences

We discuss here an impossibility holding for PA on logically complex preferences. Before stating it formally, let us introduce it in simple terms.

Suppose you want to aggregate a number of GD preferences, like in Example 3. A way to proceed would be to look at each pair of issues (x, y), and build a table by checking whether the statement that (x, y) belongs—or not—to the preferences of each agent is true or false. That is to say, for every pair (x, y) write 'Yes' if it is so that $x \preceq y$ and 'No' otherwise, and write 'Yes' if it is so that $y \prec x$ and 'No' otherwise. Applying this idea to Example 3 yields Table 3.[14] Notice that the first three columns of this table are a notational variant of the table in Example 3. Now, suppose you want to devise your aggregation function for GD to work by only looking at such Yes-No sequences (i.e., the columns), in such a way that if two columns are the same in two different tables (i.e., two different profiles), then the result of the aggregation is also the same. The majority rule clearly satisfies such requirement, but Example 3 has shown that such rule would lead to inconsistent collective GD preferences. The theorem proves that the dictatorship rule for such tables is, in fact, the only rule satisfying the requirement.

$\top \preceq A$	$\top \preceq B$	$\top \preceq A \vee B$	$A \prec \top$	$B \prec \top$	$A \vee B \prec \top$
No	*No*	*No*	*Yes*	*Yes*	*Yes*
Yes	*No*	*Yes*	*No*	*Yes*	*No*
No	*Yes*	*Yes*	*Yes*	*No*	*No*

Table 3: Judgments about GD preferences.

Theorem 2. *(Impossibility for GD preferences) Let $\mathfrak{S}^P$ contain a set of issues $\mathtt{Iss}^P$ s.t. $\{p, q, p \wedge q\} \subseteq \mathtt{Iss}^P$ (where $\wedge$ can be substituted by $\vee$ or $\rightarrow$) and $\mathtt{Prf}^P$ is the set of GD preference profiles on $\mathtt{Iss}^P$. An aggregation function for $\mathbb{J}(\mathfrak{S}^P)$ satisfies $\mathbf{Sys}^{\models}$ iff it does not satisfy $\mathbf{NoDict}^{\models}$.*

It should be stressed that the systematicity and dictatorship assumed in the theorem are JA conditions, which are applied to the aggregation of GD preferences via the translation $\mathbb{J}$ (Definition 4). It is worth noticing, in particular, that $\mathbf{NoDict}^{\models}$ is a weaker form of $\mathbb{J}(\mathbf{NoDict})$ since the type of dictatorship involved in $\mathbf{NoDict}^{\models}$ presupposes the possibility of dictating ties and not only strict preferences.

Theorem 2 is not just an impossibility result but also a possibility one. Dropping $\mathbf{Sys}^{\models}$ would guarantee the inexistence of an agent able to dictate ties as well as strict preferences. We surmise that several other (im)possibility results can be proven along the same lines.

6. CONCLUSIONS

By exploiting insights borrowed from the semantics of many-valued logics, the paper has investigated a framework for aggregation which unifies features proper of PA with features proper of JA. The framework allows to express preferences over logically complex issues.

Concretely, the paper has proven a correspondence result (Theorem 1) between standard PA and a specific subclass of JA problems which has been the stepping stone for the further extension of the PA framework with JA features. In addition, a characterization of dictatorship (Theorem 2) has been provided for the unified framework.

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8. REFERENCES

[1] K. Arrow. *Social Choice and Individual Values.* John Wiley, New York, 2nd edition, 1963.

[13] We provide only the salient part of the table.

[14] Only part of the table is provided. It is worth noticing that we are here applying a typical JA method to a PA setting.

[2] S. J. Brams and P. C. Fishburn. Approval voting. *The American Political Science Review*, 72(3):831–847, 1978.

[3] Y. Chevaleyre, U. Endriss, J. Lang, and N. Maudet. A short introduction to computational social choice. In *Proceedings of SOFSEM-2007*, volume 4362 of *LNCS*. Springer, January 2007.

[4] G. Debreu. Representation of a preference ordering by a numerical function. In R. M. Thrall, C. H. Coombs, and R. L. Davis, editors, *Decision Processes*. John Wiley, 1954.

[5] F. Dietrich and C. List. Arrow's theorem in judgment aggregation. *Social Choice and Welfare*, 29(19–33), 2007.

[6] M. Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 24(2):97–106, 1959.

[7] W. Gaertner. *A Primer in Social Choice Theory*. Oxford University Press, 2006.

[8] R. Hähnle. Advanced many-valued logics. In D. M. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic, 2nd Edition*, volume 2, pages 297–395. Kluwer, 2001.

[9] P. Hájek. Making fuzzy description logics more general. *Fuzzy Sets and Systems*, 154(1):1–15, 2005.

[10] A. Horn. Logic with truth values in a linearly ordered Heyting algebra. *Journal of Symbolic Logic*, 34(3):395–408, 1969.

[11] C. List and P. Pettit. Aggregating sets of judgments: Two impossibility results compared. *Synthese*, 140:207–235, 2004.

[12] C. List and C. Puppe. Judgment aggregation: A survey. In *Oxford Handbook of Rational and Social Choice*. Oxford University Press, 2009.

[13] A. K. Sen. The impossibility of a Paretian liberal. *The Journal of Political Economy*, 78:152–157, 1970.

APPENDIX

Proof of Theorem 1. From Definition 3 we have that: for all x, y in X, it holds that $\forall u \in \mathfrak{u}(\preceq), x \preceq y$ iff $u(x) \leq u(y)$ and hence that if $x \preceq y$ then $\forall u \in \mathfrak{u}(\preceq), u(x) \leq u(y)$. From Fact 2 we also know that $\mathfrak{u}(\preceq) \neq \emptyset$, that is, $\exists u \in \mathfrak{u}(\preceq), x \preceq y$ iff $u(x) \leq u(y)$. It therefore follows that if $\forall u \in \mathfrak{u}(\preceq), u(x) \leq u(y)$ then $x \preceq y$. We thus obtain that, for all $x, y \in X$, $\forall u \in \mathfrak{u}(\preceq), u(x) \leq u(y)$ iff $x \preceq y$. Now, by Fact 3 and Fact 4 we have that: $\widehat{\mathbb{J}(\preceq)} = \mathfrak{u}(\preceq)$. □

Proof of Theorem 2. [Right to left] It is easy to see that a dictatorship is always systematic. [Left to right] A set $V \subseteq \mathtt{Agn}^P$ is *almost decisive* for x over y (in symbols, $AD_V(x, y)$) iff: $(\forall \mathfrak{p} \in \mathfrak{P}(\mathtt{Iss}^P))[[[[(\forall i \notin V), x \prec_V y] \ \& \ [(\forall i \in V), y \preceq_V x]] \Rightarrow [y \preceq x]] \ \& \ [[[(\forall i \notin V), x \preceq_V y] \ \& \ [(\forall i \in V), y \prec_V x]] \Rightarrow [y \prec x]]]$. A set $V \subseteq \mathtt{Agn}^P$ is *decisive* for x over y (in symbols, $D_V(x, y)$) iff: $(\forall \mathfrak{p} \in \mathfrak{P}(\mathtt{Iss}^P))[[(\forall i \in V), y \preceq_V x] \Leftrightarrow [y \preceq x]]$. We need two lemmata.

Lemma 1 (Contagion property). *Let $\mathfrak{S}^P$ be so that: $\{p, q, p \wedge q\} \subseteq \mathtt{Iss}^P$, and $\mathtt{Prf}^P$ is the set of GD preferences on $\mathtt{Iss}^P$. If there exists an individual $i \in \mathtt{Agn}^P$ such that $AD_V(x, y)$ for some pair (x, y) in $\mathbb{J}(\mathfrak{S}^P)$ then, under the condition $\mathbf{Sys}^{\vDash}$, i is decisive for any pairs, that is, i is a dictator.*

Proof (sketch). Let $x := p, y := q, z := p \wedge q$ and let I denote $\mathtt{Agn}^P - \{i\}$. We show that if i is almost decisive for any of the propositions in $\{p, q, p \wedge q\}$ then it is decisive for all of them. Although we provide the proof for conjunction only, the argument holds for the other connectives as well.

For the properties of GD preferences, we have $D_V(p, p \wedge q)$ and hence, a fortiori, $AD_V(x, y) \Rightarrow D_V(p, p \wedge q)$ for any x, y. The same holds for q. We prove the following claim.

Claim: $AD_i(p, q) \Rightarrow D_i(p \wedge q, p)$. We prove that if $AD_i(p, q)$ and $p \preceq_i p \wedge q$ then $p \preceq p \wedge q$, and similarly for $p \wedge q \prec_i p$. The proof of the claim has the following logical structure: $((\neg A \vee B) \wedge C) \rightarrow D$, so we have to prove two subclaims.

(1) Assume the antecedent of $AD_i(p, q)$ to hold. We have to assume either $p \preceq_i p \wedge q$ or $p \wedge q \prec_i p$. Consider the following class of profiles:

$$\begin{array}{lll} p \prec_i q & p \preceq_i p \wedge q & \dots \\ q \preceq_I p & \dots & \dots \end{array}$$

By $AD_i(p, q)$ we conclude $p \prec q$ and therefore $p \preceq p \wedge q$ since $p \wedge q \prec p$ is not consistent with the properties of GD preferences. By $\mathbf{Sys}^{\vDash}$ it follows that, whenever $p \preceq_i p \wedge q$ we can conclude $p \preceq p \wedge q$, and whenever $p \wedge q \prec_i p$, we can conclude $p \wedge q \prec p$, for all profiles satisfying the antecedent of $AD_i(p, q)$.

(2) Assume that the antecedent of $AD_i(p, q)$ does not hold. We have to assume either $p \preceq_i p \wedge q$ or $p \wedge q \prec_i p$. From (1), by $\mathbf{Sys}^{\vDash}$, it follows that, whenever $p \preceq_i p \wedge q$, we can conclude $p \preceq p \wedge q$ and, whenever $p \wedge q \prec_i p$, we can conclude $p \wedge q \prec p$, for all profiles satisfying the antecedent of $AD_i(p, q)$. This completes the proof of the claim.

The other claims $AD_i(p, q) \Rightarrow D_i(p \wedge q, q)$, $AD_i(q, p) \Rightarrow D_i(p \wedge q, q) \ \& \ D_i(p \wedge q, p)$, $AD_i(p \wedge q, p) \Rightarrow D_i(q, p)$, $AD_i(p \wedge q, q) \Rightarrow D_i(p, q)$, can be proven by application of the same sort of argument. It is now easy to see that by assuming the almost-decisiveness on a pair we can infer the decisiveness on all other pairs, which completes the proof of the lemma. □

Lemma 2 (Existence of almost decisive voters). *Let $\mathfrak{S}^P$ be such that $\{p, q, p \wedge q\} \subseteq \mathtt{Iss}^P$, and $\mathtt{Prf}^P$ is the set of GD preferences on $\mathtt{Iss}^P$. If the aggregation function for $\mathbb{J}(\mathfrak{S}^P)$ satisfies $\mathbf{Sys}^{\vDash}$, then there exist $x, y \in \mathtt{Iss}^P$ and an agent $i \in \mathtt{Agn}^P$ such that $AD_i(x, y)$.*

Proof (sketch). Because of $\mathbf{Sys}^{\vDash}$, and since all logical truths unanimously obtain the maximal ranking in GD preferences, there always exists for each pair of issues a set which is decisive for that pair, that is, $\mathtt{Agn}^P$. Let us proceed per absurdum assuming that there is no almost decisive voter. That means that for any pair of issues (x, y) there exists a set V such that, for any profile, $AD_V(x, y)$ and $1 < |V|$. Let V be the smallest (possibly not unique) of such sets, and let $J := V - \{i\}$. Take $x := p, y := q$. Assume now $AD_V(p, q)$. With an argument analogous to that one used in the proof of Lemma 1 we can prove that $D_V(p \wedge q, q)$. Now, consider the class of profiles where $q \preceq_V p \wedge q$ and $q \preceq_J p$. By $D_V(p \wedge q, q)$ we conclude $q \preceq p \wedge q$ and, since $p \prec q$ would contradict the properties of GD preferences, we conclude $q \preceq p$. From $\mathbf{Sys}^{\vDash}$ it follows that $D_J(p, q)$ and therefore $AD_J(p, q)$, against our hypothesis since $J \subset V$.

The very same argument applies under the assumption of almost-decisiveness with respect to the other pairs of issues. This completes the proof of the lemma. □

The theorem follows from Lemmata 1 and 2.