Majority-rule-based preference aggregation on multi-attribute domains with CP-nets

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ABSTRACT

This paper studies the problem of majority-rule-based collective decision-making where the agents' preferences are represented by CP-nets (Conditional Preference Networks). As there are exponentially many alternatives, it is impractical to reason about the individual full rankings over the alternative space and apply majority rule directly. Most existing works either do not consider computational requirements, or depend on a strong assumption that the agents have acyclic CP-nets that are compatible with a common order on the variables. To this end, this paper proposes an efficient SAT-based approach, called `MajCP` (Majority-rule-based collective decision-making with CP-nets), to compute the majority winning alternatives. Our proposed approach only requires that each agent submit a CP-net; the CP-net can be cyclic, and it does not need to be any common structures among the agents' CP-nets. The experimental results presented in this paper demonstrate that the proposed approach is computationally efficient. It offers several orders of magnitude improvement in performance over a Brute-force algorithm for large numbers of variables.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems

General Terms

Algorithms, Design

Keywords

CP-nets; Voting; Preference aggregation; Majority rule

1. INTRODUCTION

Group decision making where a collective decision needs to be derived from individual preferences has been an active area of research [1]. In particular, various aggregation rules and voting procedures have been developed as group decision-making mechanisms [9]. However, the decision-making process tends to become much more complex when the attributes of the domain are interdependent. As an example, a research group plan to order several PCs and the group members need to decide on a standard group PC configuration. The decisions are not independent, because, perhaps, the preferred operating systems may depend on the given processor type. For instance, "I prefer to choose WinXP operating system rather than Linux if an Intel processor is given." Hence, we cannot decide on the issues separately. Moreover, in many real world decision-making problems, the number of alternatives is exponential in the number of domain variables. The prohibitive size of such combinatorial domain makes it impractical to represent preference relations explicitly.

In this paper, we investigate the theory of CP-nets as a formal model for representing and reasoning with the agents' preferences. There are some preference relations can not be modelized by CP-nets and its variants. For instance, Domshlak *et al.* [5] compare compare the expressive power of soft constraints and CP-nets and study several examples in which the preference relations can not be represented by CP-nets. However, CP-nets are quite commonly used and to some extent, representative of a variety of languages. Moreover, CP-nets and its variants can be used to specify individual preference relations in a relatively compact, intuitive, and structured manner, making it easier to encode human preferences and supports the decision-making systems in real world applications.

In this paper, given that the individual preferences have been elicited and represented as CP-nets, the problem of majority-rule-based preference aggregation will be addressed. Recent work on the complexity of computing dominance relations shows that dominance testing[1] for an arbitrary CP-net is PSPACE-complete [6]. However, computing the majority winning alternatives with multiple agents' CP-nets may furthermore require dominance testing on each pair of alternatives on each individual CP-net. For example, having 10 binary variables, each involved agent would need to compare $\binom{2^{10}}{2} = 523776$ pairs of alternatives. This problem is likely to be even harder than NP or coNP problems. The problem of computing aggregation rules from a collection of CP-nets has been studied in the literature, e.g.,[8, 10]. In particular, Lang and Xia [8] consider decomposition with voting rules assuming that the agents' preferences can be represented with acyclic CP-nets being compatible with a common order on the variables. However, such an assumption is unlikely to be applicable in most real world applications [11]. Xia *et al.* [12] partially addressed this shortcoming by introducing an *order-independent* sequential composition of voting rules. In their framework, the profile is still required to be compatible with *some* order on the variables, but this order is not specified in the definition of the rule. Nevertheless, the domain restriction by this order-independent sequential composition of voting rules is still severe: there must exist some (unspecified) directed acyclic graph that the profile is compatible with. Xia *et al.* [11] generalize the earlier, more restrictive method, proposing an aggregation methodology that does not require any relationship among the

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[1]A dominance testing, given an individual CP-net and two alternatives o and o', tests whether o is preferred to o' according to the preferences induced by that CP-net.

agents' CP-net structures. However, the performance of their algorithm also depends on the consistency among the structures of the agents' CP-nets.

To this end, our paper addresses the above drawbacks, proposing an efficient SAT-based approach, called `MajCP` (Majority-rule-based collective decision-making with CP-nets), to compute the majority winning alternatives. The proposed approach allows the agents to have different preferential independence structures, and enables us to aggregate preferences when the agents' CP-nets are cyclic. With multiple agents' CP-nets as input, it first reduces the problem into an extended SAT (Boolean satisfiability problem) for cardinality constraints, such that the set of possible winners can be obtained by computing the models of the corresponding SAT. Then the set of majority winners is the subset of the possible winners after filtering out those that are majority-dominated by some alternative. The proposed approach reduces the search space and is computationally efficient. According to the experimental evaluation, it offers several orders of magnitude improvement in performance over a brute-force algorithm for large numbers of variables.

The paper is structured as follows. We provide background information about CP-nets and majority rule in Section 2. In Section 3, we study a hypercube-wise composition of majority rule and analyze its incompatibility with the original majority preferences by several examples. After that, we present our proposed approach for computing the winning alternatives in Section 4 and the experimental results in Section 5. Finally, we discuss about the concluding remarks in Section 6.

2. BACKGROUND

2.1 CP-nets overview

Let $\mathbf{V} = \{X_1, \ldots, X_n\}$ be a set of n variables. For each $X_i \in \mathbf{V}$, $D(X_i)$ is the *value domain* of X_i. A variable X_i is *binary* if $D(X_i) = \{x_i, \bar{x}_i\}$. If $\{x_i, \bar{x}_i\}$ is the binary domain of X_i, then $x_i = \neg\bar{x}_i$; $\bar{x}_i = \neg x_i$. If $\mathbf{X} = \left\{X_{i_1}, \ldots, X_{i_p}\right\} \subseteq \mathbf{V}$, with $i_1 < \cdots < i_p$, then $D(\mathbf{X})$ denotes $D(X_{i_1}) \times \cdots \times D\left(X_{i_p}\right)$. The assignments of variable values to $\mathbf{X}$ are denoted by $\mathbf{x}$, $\mathbf{x}'$ etc., and represented by concatenating the values of the variables. For instance, if $\mathbf{X} = \{X_1, X_2, X_3\}$, an assignment $\mathbf{x} = x_1\bar{x}_2x_3$ assigns x_1 to X_1, $\bar{x}_2$ to X_2 and x_3 to X_3. If $\mathbf{X} = \mathbf{V}$, $\mathbf{x}$ is a *complete assignment*; otherwise $\mathbf{x}$ is called a *partial assignment*. For an assignment $\mathbf{x}$, we denote by $\mathbf{x}[X_i]$ the value $x_i \in D(X_i)$ assigned to variable X_i by that assignment; and $\mathbf{x}[\mathbf{W}]$ denotes the assignment of the variable values $\mathbf{w} \in D(\mathbf{W})$ assigned to the set of variables $\mathbf{W} \subseteq \mathbf{X}$ by that assignment. We also allow logical operations between the value assignments to binary variables. For instance, $x_1\bar{x}_2 = x_1 \wedge \bar{x}_2 = (X_1 = x_1) \wedge (X_2 = \bar{x}_2)$. That is, x_1 is $True$ and x_2 is $False$. If $\mathbf{p} = x_1\bar{x}_2$ and $\mathbf{q} = x_3$, then $\mathbf{p} \vee \mathbf{q} = (x_1\bar{x}_2) \vee x_3 = ((X_1 = x_1) \wedge (X_2 = \bar{x}_2)) \vee (X_3 = x_3)$.

Let $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$ be a partition of the set of variables $\mathbf{V}$ and $\succ$ a preference relation over $D(\mathbf{V})$. $\mathbf{X}$ is *conditionally preferentially independent* of $\mathbf{Y}$ given $\mathbf{Z}$ if and only if, for all $\mathbf{x}, \mathbf{x}' \in D(\mathbf{X})$, $\mathbf{y}, \mathbf{y}' \in D(\mathbf{Y})$ and $\mathbf{z} \in D(\mathbf{Z})$:

$$\mathbf{xyz} \succ \mathbf{x'yz} \text{ iff } \mathbf{xy'z} \succ \mathbf{x'y'z}$$

A CP-net $\mathcal{N}$ [3] over a set of variables $\mathbf{V} = \{X_1, \ldots, X_n\}$ is an annotated directed graph G over $X_1, \ldots, X_n$, in which nodes stand for the problem variables. Each node X_i is annotated with a conditional preference table $CPT(X_i)$, which associates a total order $\succ^{X_i|\mathbf{u}}$ with each instantiation $\mathbf{u}$ of X_i's parents $Pa(X_i)$. For instance, let $\mathbf{V} = \{X_1, X_2, X_3\}$, all three variables are binary-valued. Assume that the preference of a given agent over $2^{\mathbf{V}}$ can be defined by a CP-net, whose structural part is the directed graph $G = \{(X_1, X_2), (X_2, X_3), (X_1, X_3)\}$. Then the agent's preference over the values of X_1 is unconditional, preference over the values of X_2 (resp. X_3) is conditioned on the value of X_1 (resp. the context of X_1 and X_2). The conditional preference statements contained in the CPTs are written with the following notation, e.g. $x_1\bar{x}_2 : x_3 \succ \bar{x}_3$ means that if x_1 is $True$ and x_2 is $False$, then the agent prefers $X_3 = x_3$ to $X_3 = \bar{x}_3$.

In this paper, we assume that each agent A_j's preference is captured by a binary-valued (possibly cyclic) CP-net $\mathcal{N}_j$ and the ordering $\succ_{A_j}^{X_i|\mathbf{u}}$, $\mathbf{u} \in D(Pa_j(X_i))$, expressed in the CPTs of the network is total. As such, conditional expressions of indifference are not allowed, and an agent will not be indifferent between two alternatives. However, as the preference relation induced from a CP-net is generally not complete, two alternatives can be incomparable for an agent.

2.2 Majority rule

In classical social choice theory, majority rule is one of the most well known aggregation rule for collective decision-making. It is a binary decision rule that selects one of two alternatives, based on which has more than half of the votes. The semantics of majority voting in the context of CP-nets has been provided by Rossi *et al.* [10]:

DEFINITION 1 (MAJORITY SEMANTICS). *Given two alternatives o and o', let $\boldsymbol{S}_{\succ}$, $\boldsymbol{S}_{\prec}$, $\boldsymbol{S}_{\bowtie}$ be the sets of agents who say, respectively, that $o \succ o'$, $o \prec o'$, and $o \bowtie o'$ (incomparable). We say that o majority-dominates o' (written as $o \succ_{maj} o'$) if and only if there is a majority of agents who prefer o to o' (i.e., $|\boldsymbol{S}_{\succ}| > |\boldsymbol{S}_{\prec}| + |\boldsymbol{S}_{\bowtie}|$). Two alternatives o and o' are majority-incomparable (written as $o \bowtie_{maj} o'$) if they are not ordered in either way.*

In order to determine the winning alternatives according to majority rule, the Condorcet method has usually been used [2]. The following definitions of the Condorcet winner and weak Condorcet winner are adapted from the standard social choice literature [1]:

DEFINITION 2 (CONDORCET WINNER). *An alternative o is a Condorcet winner if and only if it majority-dominates every other alternative in a pair-wise matchup: $\forall o' \in O$ and $o' \neq o$, $o \succ_{maj} o'$.*

DEFINITION 3 (WEAK CONDORCET WINNER). *An alternative o is a* weak *Condorcet winner if and only if it majority-dominates or is incomparable to every other alternative in a pair-wise matchup: $\forall o' \in O$ and $o' \neq o$, $o \succ_{maj} o'$ or $o \bowtie_{maj} o'$.*

When the Condorcet winner exists, it is unique. A Condorcet winner is also a weak Condorcet winner, while the reverse does not hold: a weak Condorcet winner is not necessarily a Condorcet winner. In majority-rule based group decision-making, it is possible for a paradox to form, in which collective preferences can be cyclic (i.e. not transitive), even if the preferences of individual agents are not. For instance, it is possible that there are alternatives o_1, o_2, and o_3 such that a majority prefers o_1 to o_2, another majority prefers o_2 to o_3, and yet another majority prefers o_3 to o_1. The requirement of majority rule then provides no Condorcet winner. Consequently, the set of majority winning alternatives can be empty. Also, there can be more than one weak Condorcet winner when the number of agents is even or the individual preferences are incomplete (i.e.

[2] There are also some other aggregation methods which do not comply with the Condorcet criterion, e.g., approval voting, Borda count, plurality voting, etc..

partial order). Note that the set of weak Condorcet winners are majority-incomparable to each other.

Rossi *et al.* [10] study the computational complexity of a brute-force algorithm for aggregating preference based on majority rule. Suppose that there are a set of m agents making decisions over a set of n binary variables. To test whether an alternative is a winner we need to compare the given alternative with all other alternatives (2^n) in all CP-nets (m). Recall that computing the majority-dominance relation between a pair of alternatives require individual dominance testing on each agent's CP-net, which is PSPACE-complete. Moreover, finding the set of majority winners is even more challenging. We need to compare all alternatives (2^n) to all other alternatives (2^n) in all CP-nets(m). Consequently, it is impractical to use pair-wise comparison over the alternative space directly.

3. H-COMPOSITION OF MAJORITY RULES

Instead of applying voting directly over the alternative space, Xia *et al.* [11] propose a *hypercube-wise composition* (*H-composition*) of local voting rules. An H-composition of local rules is defined as the following two steps. First, the set of all possible alternatives are represented as a hypercube, and alternatives that differ on only one variable are neighbours on this hypercube as discussed in [4]. Then an induced graph is generated by applying local rules to each pair of neighbours on this hypercube. In the second step, a *choice set* is selected based on the induced graph as the set of winners. According to the representation in [11], we apply majority rule between each pair of neighbours and obtain the following majority induced graph:

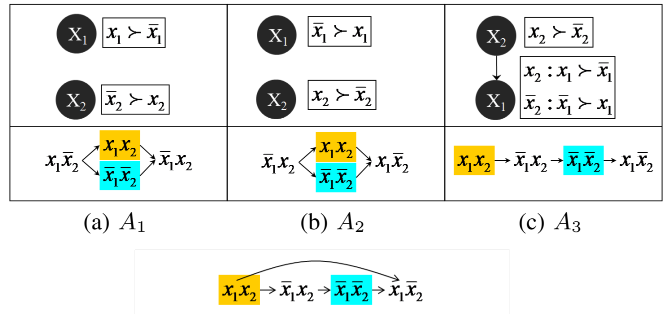
DEFINITION 4 (MAJORITY INDUCED GRAPH). *Given a collection of CP-nets* $\mathbf{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$*, the majority induced graph, denoted by* $\mathcal{G} = (O, E)$*, is defined by the following edges between alternatives. For each variable* X_i*, any two alternatives* $o,\ o' \in O$ *that differ only on the value of* X_i*, let there be a directed edge* $o \rightarrow o'$ *if a majority of agents prefer* o *to* o'*; there be a directed edge* $o' \rightarrow o$ *if a majority of agents prefer* o' *to* o*. If* o *and* o' *are majority-incomparable,* $\mathcal{G}$ *does not contain any edge between* o *and* o'*.*

For any two alternative $o,\ o' \in O$ that differ only on the value of X_i, $o[X_i] = x_i$ and $o'[X_i] = \bar{x}_i$, let $\mathbf{W} = \mathbf{V} - \{X_i\}$ and $\mathbf{w} = o[\mathbf{W}]\,(= o'[\mathbf{W}])$. Whether or not there is a directed edge $o \rightarrow o'$ (resp. $o' \rightarrow o$) can be computed directly from the conditional preference table $CPT_j(X_i)$ of each agent A_j's CP-net $\mathcal{N}_j$. Because for each agent A_j, $\mathcal{N}_j \models o \succ o'$ (resp. $\mathcal{N}_j \models o' \succ o$) if and only if $x_i \succ_{A_j}^{X_i|\mathbf{w}} \bar{x}_i$ (resp. $\bar{x}_i \succ_{A_j}^{X_i|\mathbf{w}} x_i$). Note that a pair of neighbours o and o' are incomparable if and only if the number of agents is even and the number of agents who prefer o to o' is equal to the number of agents who prefer o' to o.

The dominance relations in $\mathcal{G}$ are then induced by the directed paths between alternatives [11]:

DEFINITION 5 (GRAPH DOMINANCE). *Given a collection of CP-nets* $\mathbf{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$*, let* $\mathcal{G} = (O, E)$ *be the majority induced graph. For any* $o, o' \in O$*, we say that* o *dominates* o' *in* $\mathcal{G}$*, denoted by* $o \succ_{\mathcal{G}} o'$ *if and only if: i) there is a directed path from* o *to* o'*, and ii) there is no directed path from* o' *to* o*.*

According to Xia *et al.* [11], the *transitive closure* $\succeq_{\mathcal{G}}$ of E specifies the minimum preorder such that if there is a directed path from o to o' in $\mathcal{G}$ then $o \succeq_{\mathcal{G}} o'$. $\succ_{\mathcal{G}}$ is the strict order induced by $\succeq_{\mathcal{G}}$: $o \succ_{\mathcal{G}} o'$ if and only if $o \succeq_{\mathcal{G}} o'$ and $o' \not\succeq_{\mathcal{G}} o$. Based on the induced graph, a choice set function is then defined, which always chooses the following alternatives as winners.



(a) A_1 (b) A_2 (c) A_3

(d) Majority induced graph

Figure 1: Illustration for Proposition 1

DEFINITION 6 (GRAPH WINNER). *Let* $\mathcal{G} = (O, E)$ *be the majority induced graph for a collection of CP-nets* $\mathbf{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$*, we say,*

- *an alternative is a* global Condorcet winner *(GCW), if it dominates all other alternatives in* $\mathcal{G}$*;*
- *an alternative is a* local Condorcet winner *(LCW), if it dominates all its neighbours in* $\mathcal{G}$*;*
- *an alternative is a* weak local Condorcet winner *(wLCW), if it dominates or is incomparable to all its neighbours in* $\mathcal{G}$*.*

When the global Condorcet winner (GCW) exists, it is unique. A GCW is also a local Condorcet winner (LCW), while the reverse does not hold: a LCW is not necessarily a GCW. Similarly, a LCW is also a weak local Condorcet winner (wLCW), while a wLCW is not necessarily a LCW.

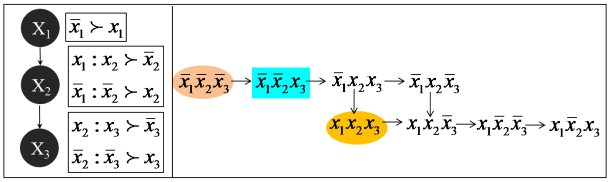
However, we emphise here that GCW, LCW and wLCW in $\mathcal{G}$ are different from the meaning of a (weak) Condorcet winner (Definition 2 and 3), which refers to a majority winner in pair-wise election. In the following section, we analyze the relation between the preferences derived from a majority induced graph $\mathcal{G}$ and the original majority preferences among the agents.

PROPOSITION 1. *Majority-domination* $\succ_{maj}$ *does not follow from graph domination* $\succ_{\mathcal{G}}$*.*

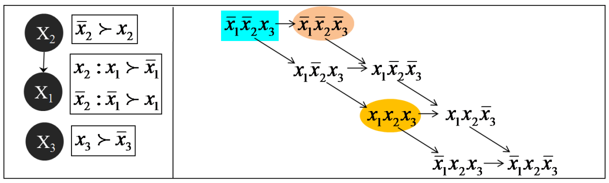
PROOF. To prove this proposition, we need to prove that given a collection of CP-nets $\mathbf{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$, the majority induced graph $\mathcal{G} = (O, E)$ and a pair of alternatives $o, o' \in O$ and $o \neq o'$, it may be the case that $o \succ_{\mathcal{G}} o'$ but $o \not\succ_{maj} o'$. Consider an example of 3 agents making decision over 2 binary domain variables. The agents' CP-nets, their partial order over the alternative space and the majority induced graph are depicted in Figure 1. According to the majority induced graph (see Figure 1(d)), there is a directed path from outcome x_1x_2 to $\bar{x}_1\bar{x}_2$ and no directed path from $\bar{x}_1\bar{x}_2$ to x_1x_2, i.e. $x_1x_2 \succ_{\mathcal{G}} \bar{x}_1\bar{x}_2$. However, for both A_1 and A_2, these two alternatives are incomparable (see Figure 1(a) and Figure 1(b)), and thus, $\bar{x}_1\bar{x}_2$ and x_1x_2 are majority-incomparable, i.e. $\bar{x}_1\bar{x}_2 \bowtie_{maj} x_1x_2$. Consequently, in this example, $x_1x_2 \succ_{\mathcal{G}} \bar{x}_1\bar{x}_2$ but $x_1x_2 \not\succ_{maj} \bar{x}_1\bar{x}_2$. □

PROPOSITION 2. *The preference relation* $\succ_{\mathcal{G}}$ *derived from the majority induced graph does not preserve the strict majority preference relation* $\succ_{maj}$*.*

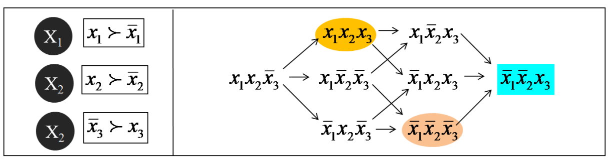
PROOF. To prove this proposition, we need to prove that given a collection of CP-nets $\mathbf{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$, the majority induced



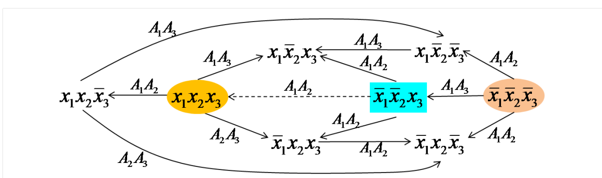
(a) A_1



(b) A_2

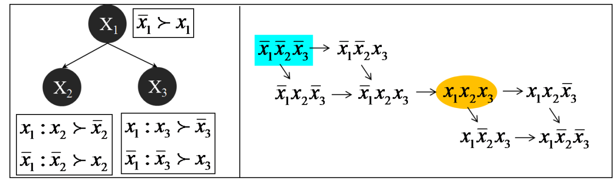


(c) A_3

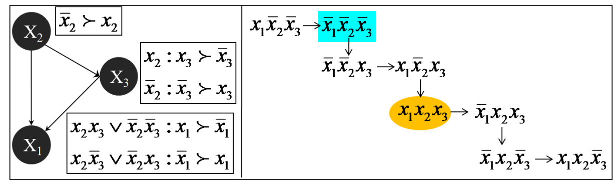


(d) Majority induced graph

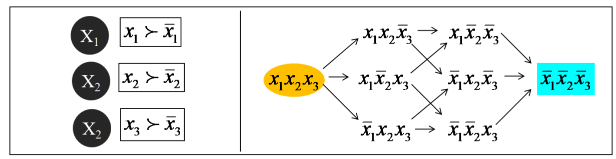
Figure 2: Illustration for Propositions 2 and 3



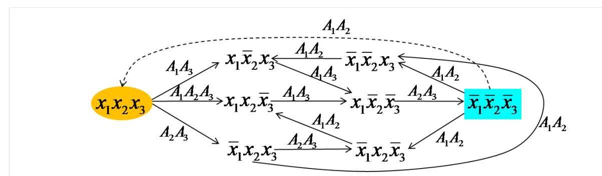
(a) A_1



(b) A_2



(c) A_3



(d) Majority induced graph

Figure 3: Illustration for Proposition 4

graph $\mathcal{G} = (O, E)$ and a pair of alternatives $o, o' \in O$ and $o \neq o'$, it may be the case that $o \succ_{maj} o'$ but $o \not\succ_{\mathcal{G}} o'$. Consider the agents' CP-nets, their partial order over the alternative space and the corresponding majority induced graph depicted in Figure 2. According to the majority induced graph Figure. 2(d), there is no directed path from alternative $\bar{x}_1\bar{x}_2x_3$ to alternative $x_1x_2x_3$, i.e. $\bar{x}_1\bar{x}_2x_3 \not\succ_{\mathcal{G}} x_1x_2x_3$. However, both A_1 and A_2 preferred $\bar{x}_1\bar{x}_2x_3$ to $x_1x_2x_3$ (see Figure 2(a) and Figure 2(b)), and thus $\bar{x}_1\bar{x}_2x_3 \succ_{maj} x_1x_2x_3$. Consequently, in this example, $\bar{x}_1\bar{x}_2x_3 \succ_{maj} x_1x_2x_3$ but $\bar{x}_1\bar{x}_2x_3 \not\succ_{\mathcal{G}} x_1x_2x_3$. □

As $\succ_{\mathcal{G}}$ does not preserve the strict majority preference $\succ_{maj}$, a (weak) local Condorcet winner that dominates or is incomparable to all it neighbours may still be majority-dominated by some alternative, and thus is not guaranteed to be a weak Condorcet winner.

COROLLARY 1. *A (weak) local Condorcet winner is not necessarily a weak Condorcet winner.*

Consider the example in Figure 2. Alternative $x_1x_2x_3$ is a LCW as it dominates all its neighbours ($x_1x_2\bar{x}_3$, $x_1\bar{x}_2x_3$ and $\bar{x}_1x_2x_3$) in the majority induced graph (see Figure 2(d)). However, it is majority-dominated by another alternative $\bar{x}_1\bar{x}_2x_3$ because both A_1 (Figure 2(a)) and A_2 (Figure 2(b)) preferred $\bar{x}_1\bar{x}_2x_3$ to $x_1x_2x_3$ and thus is not a (weak) Condorcet winner.

Now we are interested in whether or not the (weak) local Condorcet winners set is guaranteed to be a non-majority-dominated set, i.e. the alternatives in this set can only be majority-dominated by some alternative in this set but not by any other alternatives outside this set. Unfortunately, the following proposition gives a negative answer to this question.

PROPOSITION 3. *A (weak) local Condorcet winner can be majority-dominated by an alternative outside the set of (weak) local Condorcet winners, even though it is not majority-dominated by any other (weak) local Condorcet winner.*

PROOF. Consider the example in Figure 2, there are only two LCWs $x_1x_2x_3$ and $\bar{x}_1\bar{x}_2\bar{x}_3$ and $x_1x_2x_3 \bowtie_{maj} \bar{x}_1\bar{x}_2\bar{x}_3$: they are incomparable for both A_2 (see Figure 2(b)) and A_3 (see Figure 2(c)). However, as we mentioned before, $x_1x_2x_3$ is majority-dominated by alternative $\bar{x}_1\bar{x}_2x_3$, which is not a LCW or wLCW. □

Finally, we are interested in the following question: whether a global Condorcet winner that dominates every other alternative in the majority induced graph, is guaranteed to be non-majority-dominated, i.e. a (weak) Condorcet winner.

PROPOSITION 4. *A global Condorcet winner is not necessarily a (weak) Condorcet winner.*

PROOF. Consider the agents' CP-nets, their preference ordering over the alternative space and the corresponding majority induced graph in Figure 3. In this example, there is a unique global Condorcet winner $x_1x_2x_3$ in $\mathcal{G}$: there is a directed path from $x_1x_2x_3$ to every other alternative and no incoming edges to $x_1x_2x_3$ (see Figure 3(d)). However, this global Condorcet winner $x_1x_2x_3$ is majority-dominated by $\bar{x}_1\bar{x}_2\bar{x}_3$ ($\bar{x}_1\bar{x}_2\bar{x}_3 \succ_{maj} x_1x_2x_3$), because two agents ($A_1$ and A_2) prefer $\bar{x}_1\bar{x}_2\bar{x}_3$ to $x_1x_2x_3$ (see Figure 3(a) and Figure 3(b)). □

Proposition 4 further shows that the strict preference relation $\succ_{\mathcal{G}}$ derived from the majority induced graph might be conflicting with the original majority preference relation $\succ_{maj}$. For instance, for the example in Figure 3, $x_1x_2x_3 \succ_{\mathcal{G}} \bar{x}_1\bar{x}_2\bar{x}_3$, however, $x_1x_2x_3 \prec_{maj} \bar{x}_1\bar{x}_2\bar{x}_3$.

From the above, it become clear that the majority induced graph may not always represent the majority preferences properly. In particular, a winners in the majority induced graph, i.e., a GCW, LCW or wLCW winner is not necessarily a (weak) Condorcet winner. However, we observe that a (weak) Condorcet winner must be a wLCW.

THEOREM 1. *Let $\mathcal{G} = (O, E)$ be the majority induced graph for a collection of CP-nets $\mathbf{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$. Then a (weak) Condorcet winner is also a weak local Condorcet winner in $\mathcal{G}$.*

PROOF. Suppose a (weak) Condorcet winner o is not a wLCW, then there exist at least one neighbour o' in $\mathcal{G}$ such that $o' \rightarrow o \in E$. That means, there is a majority of agents prefers o' to o ($o' \succ_{maj} o$), contradicting the fact that o is a (weak) Condorcet winner. Thus, a (weak) Condorcet winner must also be a wLCW. □

A (weak) Condorcet winner must be a wLCW, while the reverse does not hold: a wLCW is not necessarily a (weak) Condorcet winner. Nonetheless, this relation provides us a more efficient way to compute the majority winners among a large alternative space: first compute the set of wLCWs, and then compute the set of majority winners by filtering out those that are majority-dominated by some alternative.

REMARK. Here the notions of majority induced graph coincides with the definitions in [11] when the number of agents is odd, and differ only in the presence of incomparability between neighbours when the number of agents is even. According to [11], when a pair of neighbours o and o' are majority-incomparable, $\mathcal{G}$ contains directed edges both from o to o' and o' to o. However, their definition may exclude the weak Condorcet winners when the number of agents is even. For instance, if a weak Condorcet winner is majority-incomparable to one of its neighbours, then it is not considered to be a wLCW (nor a GCW or LCW) according to their definition.

4. COMPUTE THE (WEAK) CONDORCET WINNER

In this section, we present our proposed approach, `MajCP` (majority-rule-based collective decision-making with CP-nets), for computing the majority winning alternatives. The proposed approach includes the following two steps. First, we compute the set of wLCWs via a reduction to an extended SAT (Boolean satisfiability problem) for cardinality constraints (See Algorithm 1). Then, in the second step, the set of (weak) Condorcet winners can be obtained by filtering out those that are majority-dominated by some alternative.

Assume m agents $\mathbf{A} = \{A_1, \ldots, A_m\}$ are making decisions over a set of n variables $\mathbf{V} = \{X_1, \ldots, X_n\}$. The preference of each agent A_j is captured by a (possibly cyclic) binary-valued CP-net $\mathcal{N}_j$ and let $\mathbf{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$. We first reduce the problem of computing the set of wLCWs into a corresponding SAT problem. The variables in our reduction consist of the variables in the agents' CP-nets. Firstly, we generate a set of optimality constraints that a wLCW must satisfy according to majority rule. For each variable X_i, each agent A_j's has a conditional preference table $CPT_j(X_i)$ stating the conditional preference on the values of variable X_i with each instantiation of X_i's parents $Pa_j(X_i)$. We separate these condition entries in $CPT_j(X_i)$ into the following two categories.

- The set of parent context in which agent A_j prefers x_i to $\bar{x}_i$: $\mathbf{U}_{A_j}^{x_i \succ \bar{x}_i} = \left\{\mathbf{u} \in D(Pa_j(X_i)) \mid x_i \succ_{A_j}^{X_i|\mathbf{u}} \bar{x}_i\right\}$.
- The set of parent context in which agent A_j prefers $\bar{x}_i$ to x_i: $\mathbf{U}_{A_j}^{\bar{x}_i \succ x_i} = \left\{\mathbf{u} \in D(Pa_j(X_i)) \mid \bar{x}_i \succ_{A_j}^{X_i|\mathbf{u}} x_i\right\}$.

Let $P_j^i = \bigvee_{\mathbf{u} \in \mathbf{U}_{A_j}^{x_i \succ \bar{x}_i}} \mathbf{u}$ (resp. $\bar{P}_j^i = \bigvee_{\mathbf{u} \in \mathbf{U}_{A_j}^{\bar{x}_i \succ x_i}} \mathbf{u}$), i.e., the disjunction of the condition part of the entry whose conclusion is $x_i \succ \bar{x}_i$ (resp. $\bar{x}_i \succ x_i$) in the $CPT_j(X_i)$ of agent A_j (line 14–20). Note that if agent A_j has unconditional preference over a variable X_i, $Pa_j(X_i) = \emptyset$ and $x_i \succ_{A_j}^{X_i} \bar{x}_i$ (resp. $\bar{x}_i \succ_{A_j}^{X_i} x_i$), that means the condition P_j^i (resp. $\bar{P}_j^i$) is always $True$ and $\bar{P}_j^i$ (resp. P_j^i) is always $False$ (line 7–11). Thus, $x_i \succ_{A_j}^{X_i|P_j^i} \bar{x}_i$ (resp. $\bar{x}_i \succ_{A_j}^{X_i|\bar{P}_j^i} x_i$).

For each individual agent A_j, $\mathbf{U}_{A_j}^{x_i \succ \bar{x}_i}$ and $\mathbf{U}_{A_j}^{\bar{x}_i \succ x_i}$ are complementary, and thus $P_j^i = \neg\bar{P}_j^i$ (resp. $\bar{P}_j^i = \neg P_j^i$). For any setting $\mathbf{w} = D(\mathbf{W})$ ($\mathbf{W} = \mathbf{V} - \{X_i\}$) that satisfies P_j^i (resp. $\bar{P}_j^i$), then $x_i\mathbf{w} \succ_{A_j} \bar{x}_i\mathbf{w}$ (resp. $\bar{x}_i\mathbf{w} \succ_{A_j} x_i\mathbf{w}$).

Given a directed graph $\mathcal{G} = (O, E)$, for any two alternatives $o, o' \in O$ that differ only on the value of X_i: $o[X_i] = x_i$ and $o'[X_i] = \bar{x}_i$. Let $q = (m+1)/2$ (m is the total number of agents) (line 1). There is an directed edge $o \rightarrow o'$ (resp. $o' \rightarrow o$) in $\mathcal{G}$ if and only if, for the setting $\mathbf{w} = o[\mathbf{W}] (= o'[\mathbf{W}])$ and $\mathbf{W} = \mathbf{V} - \{X_i\}$, there exist a set of at least q agents, denoted by $\mathbf{S}$ ($\mathbf{S} \subseteq \mathbf{N}$), each agent $A_j \in \mathbf{S}$ has the following conditional (unconditional) preference $x_i \succ_{A_j}^{X_i|\mathbf{w}} \bar{x}_i$ (resp. $\bar{x}_i \succ_{A_j}^{X_i|\mathbf{w}} x_i$), i.e., $\mathbf{w}$ satisfies $\bigwedge_{A_j \in \mathbf{S}} P_j^i$ (resp. $\bigwedge_{A_j \in \mathbf{S}} \bar{P}_j^i$). Furthermore, there will be a set of $\binom{m}{q}$ distinct q-subsets of agents that satisfies this majority requirement, denoted by $\mathbf{C}$. Consequently, if the setting $\mathbf{w}$ satisfies $\bigvee_{\mathbf{S} \in \mathbf{C}} \left(\bigwedge_{A_j \in \mathbf{S}} P_j^i\right)$ (resp. $\bigvee_{\mathbf{S} \in \mathbf{C}} \left(\bigwedge_{A_j \in \mathbf{S}} \bar{P}_j^i\right)$), then there is an directed edge $o \rightarrow o'$ (resp. $o' \rightarrow o$), and thus $o \succ_{\mathcal{G}} o'$ (resp. $o' \succ_{\mathcal{G}} o$). For the purpose of explanation, we reason directly with cardinality formulas, which has been widely explored in CSPs and SAT (cardinality constraints), see e.g., [2] and [7]. For each variable X_i, let F_i and F_i' be the following cardinality formula respectively (line 24):

$$F_i = [\geq q]\left(P_1^i, \ldots, P_m^i\right) \tag{1}$$

$$F_i' = [\geq q]\left(\bar{P}_1^i, \ldots, \bar{P}_m^i\right) \tag{2}$$

Such that F_i (resp. F_i') is $True$ when at least q formulas among $P_1^i, \ldots, P_m^i$ (resp. $\bar{P}_1^i, \ldots, \bar{P}_m^i$) are $True$. Note that the cardinality formula F_i (resp. F_i') is logically equivalent to the classical propositional formula $\bigvee_{\mathbf{S} \in \mathbf{C}} \left(\bigwedge_{A_j \in \mathbf{S}} P_j^i\right)$ (resp. $\bigvee_{\mathbf{S} \in \mathbf{C}} \left(\bigwedge_{A_j \in \mathbf{S}} \bar{P}_j^i\right)$).

Given an directed graph $\mathcal{G} = (O, E)$, let o, $o' \in O$ be two alternatives that differ only on the value of a variable X_i, $o[X_i] = x_i$ and $o'[X_i] = \bar{x}_i$. Let $\mathbf{w} = o[\mathbf{W}] (= o'[\mathbf{W}])$ and $\mathbf{W} = \mathbf{V} - \{X_i\}$. If the setting $\mathbf{w}$ satisfies F_i (resp. F_i'), then there is an directed edge $o \rightarrow o'$ (resp. $o' \rightarrow o$). Consequently, the wLCWs must satisfy the following optimality constraints for each variable X_i (line 25).

DEFINITION 7 (OPTIMALITY CONSTRAINTS). *Given a collection of CP-nets $\mathbf{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$, for each variable X_i, the majority-optimality constraint φ_i to the value of X_i is:*

$$\varphi_i = (F_i \Rightarrow x_i) \wedge (F_i' \Rightarrow \bar{x}_i) \tag{3}$$

Algorithm 1: `MajCP`

Input: **N**, a set of CP-nets of the agents;
Output: CW, a set of (weak) Condorcet winners

1 $q \leftarrow (m+1)/2$ where m is the total number of agents;
2 $\varphi \leftarrow True$;
3 **foreach** $X_i \in \boldsymbol{V}$ **do**
4 $\quad list, list' \leftarrow \emptyset$;
5 $\quad$ **foreach** $\mathcal{N}_j \in \boldsymbol{N}$ **do**
6 $\quad\quad$ **if** $Pa_j(X_i) = \emptyset$ **then**
7 $\quad\quad\quad$ **if** $x_i \succ^{X_i}_{A_j} \bar{x}_i$ **then**
8 $\quad\quad\quad\quad P^i_j \leftarrow True; \bar{P}^i_j \leftarrow False$;
9 $\quad\quad\quad$ **else**
10 $\quad\quad\quad\quad \bar{P}^i_j \leftarrow True; P^i_j \leftarrow False$;
11 $\quad\quad\quad$ **end**
12 $\quad\quad$ **else**
13 $\quad\quad\quad P^i_j \leftarrow False; \bar{P}^i_j \leftarrow False$;
14 $\quad\quad\quad$ **foreach** *cp-statement* $\in CPT_j(X_i)$ **do**
15 $\quad\quad\quad\quad$ **if u** $\in \mathbf{U}^{x_i \succ \bar{x}_i}_{A_j}$ **then**
16 $\quad\quad\quad\quad\quad P^i_j \leftarrow P^i_j \vee$ **u**
17 $\quad\quad\quad\quad$ **else**
18 $\quad\quad\quad\quad\quad \bar{P}^i_j \leftarrow \bar{P}^i_j \vee$ **u**
19 $\quad\quad\quad\quad$ **end**
20 $\quad\quad\quad$ **end**
21 $\quad\quad$ **end**
22 $\quad\quad$ **add** P^i_j to $list$; **add** $\bar{P}^i_j$ to $list'$;
23 $\quad$ **end**
24 $\quad F_i \leftarrow [\geq q]\, list; F'_i \leftarrow [\geq q]\, list'$;
25 $\quad \varphi_i \leftarrow (F_i \Rightarrow x_i) \wedge (F'_i \Rightarrow \bar{x}_i)$;
26 $\quad \varphi \leftarrow \varphi \wedge \varphi_i$
27 **end**
28 $graphWinners \leftarrow$ the models of φ;
29 $CW \leftarrow$ `optimalityCheck`$(graphWinners)$;
30 **return** CW;

Note that if there is an odd number of agents, $F'_i = \neg F_i$ and the above constraint φ_i can be simplified to:

$$\varphi_i = (F_i \Leftrightarrow x_i)$$

Finally, let φ be the conjunction of all φ_i (one for each variable) (line 26):

$$\varphi = \bigwedge_{X_i \in \mathbf{V}} \varphi_i \tag{4}$$

THEOREM 2. *Let $\mathcal{G} = (O, E)$ be the majority induced graph for a collection of CP-nets $\boldsymbol{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$. An alternative o is a weak local Condorcet winner if and only if it satisfies the above SAT φ.*

PROOF. (Soundness) Let o be an alternative that satisfies φ. For every neighbour o' of o that differs on the value of a single variable $X_i \in \mathbf{V}$, as o satisfies $\varphi_i = (F_i \Rightarrow x_i) \wedge (F'_i \Rightarrow \bar{x}_i)$, then either there is an directed edge $o \rightarrow o'$ or there is no edge between o and o'. According to Definition 6, o is a wLCW.
(Completeness) Assume first that there is at least one wLCW o, and suppose that o does not satisfy φ. Then there exists at least one optimality constraint $\varphi_i = (F_i \Rightarrow x_i) \wedge (F'_i \Rightarrow \bar{x}_i)$ that o does not satisfy. As F_i and F'_i are mutually exclusive, and for the sake of simplicity we assume that o does not satisfy $F_i \Rightarrow x_i$. An implication is unsatisfied only when the hypothesis is $True$ and the conclusion is $False$. That is, o satisfies F_i yet $o[X_i] = \bar{x}_i$. Let o' be a neighbour of o, $o[\mathbf{W}] = o'[\mathbf{W}]$ ($\mathbf{W} = \mathbf{V} - \{X_i\}$) and $o'[X_i] = x_i$. Then, o' satisfy $F_i \Rightarrow x_i$. There must be an edge $o' \rightarrow o$ in $\mathcal{G}$ and thus $o' \succ_{\mathcal{G}} o$, contradicting the fact that o is a wLCW. Hence, the above SAT φ must be satisfied by all the alternatives that are wLCWs. $\square$

As such, we reduce the problem of computing wLCWs into a SAT problem and the set of wLCWs can be obtained by computing the models of the corresponding SAT (line 28). Recall that a wLCW is not necessarily a weak Condorcet winner. In the second step, we need to test the majority optimality of each wLCW (i.e. a model of the corresponding SAT) by comparing it to all other alternatives and filtering out those that are majority-dominated by some alternative (line 29).

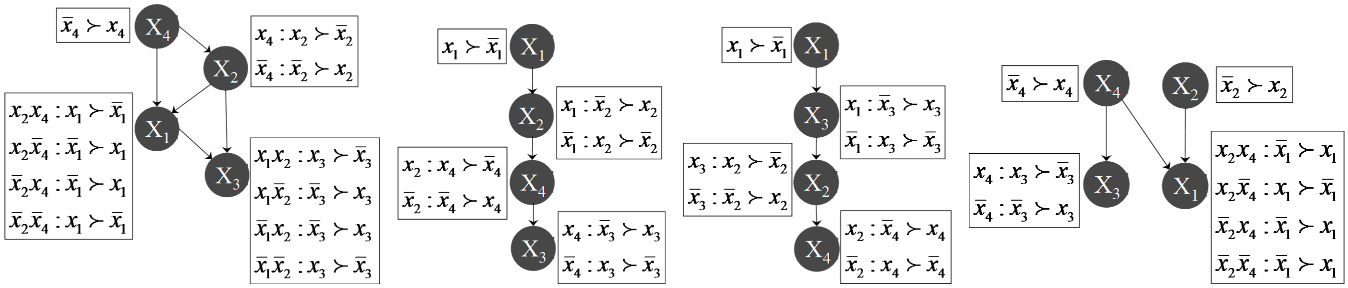
THEOREM 3 (COMPLEXITY). *Given a collection of m CP-nets $\boldsymbol{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$, if $\forall \mathcal{N}_j \in \boldsymbol{N}$, the node in-degree is bounded by a constant, then translating the problem of computing weak local Condorcet winners into a corresponding extended SAT problem for cardinality constraints is polynomial.*

PROOF. Assume there are n variables and the number of parents of a node in the dependency graph of each agent is bounded by a constant d. In order to translate the problem of computing wLCWs into the corresponding SAT problem φ, we need to generate a majority-optimality constraint φ_i for each variable X_i. For each variable X_i, we need to check each $\mathcal{N}_j$'s conditional preference table $CPT_j(X_i)$. The number of cp-statements in $CPT_j(X_i)$ is exponential in the number of parents of X_i in the dependency graph of a $\mathcal{N}_j$. Since we assume that node in-degree is bounded by a constant d, the exponential is still a constant (i.e. 2^d) and the number of variables included in the condition entry of every cp-statement is also bounded by d. Thus, the running time of translation is $O(n \cdot m \cdot 2^d \cdot d)$. $\square$

THEOREM 4 (COMPLEXITY). *Given a collection of m CP-nets $\boldsymbol{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$, if $\forall \mathcal{N}_j \in \boldsymbol{N}$, the node in-degree is bounded by a constant, then i) checking whether an alternative is a weak local Condorcet winner is polynomial; and, ii) finding the set of weak local Condorcet winners is NP-complete.*

PROOF. Based on Theorem 2, to check whether an alternative o is a wLCW we just need to check whether o is a model of the corresponding extended SAT problem φ, that is, whether o satisfies the optimality constraint $\varphi_i = (F_i \Rightarrow x_i) \wedge (F'_i \Rightarrow \bar{x}_i)$ of each variable X_i. The constraint $F_i \Rightarrow x_i$ (resp. $F'_i \Rightarrow \bar{x}_i$) is satisfied if and only if the condition F_i (resp. F'_i) is $False$ or the conclusion x_i (resp. $\bar{x}_i$) is $True$. For instance, if o assigns $\bar{x}_i$ to X_i, o satisfies $F'_i \Rightarrow \bar{x}_i$. Thus, o satisfies φ_i if and only if o also satisfies $F_i \Rightarrow x_i$. Also, as $o[X_i] = \bar{x}_i$, o satisfies $F_i \Rightarrow x_i$ if and only if F_i is evaluated to $False$. Checking the truth value of F_i can be done by counting the elements in the list of F_i that is evaluated to $True$: if there are fewer than $(m+1)/2$ formulas are evaluated to $True$ then F_i is evaluated to $False$. Suppose there are n variables and node in-degree is bounded by a constant d. Then there are m formulas listed in F_i and each formula is a disjunction of at most 2^d conjunctions of at most d literals. Consequently, the running time of checking whether an alternative is a model of φ is thus $O(n \cdot m \cdot 2^d \cdot d)$.

Regarding the problem of finding the set of wLCWs. As we already show that testing whether an alternative is a wLCW (i.e. is a model of φ) is polynomial, the problem of finding the set of



(a) CP-net of A_1 (b) CP-net of A_2 (c) CP-net of A_3 (d) CP-net of A_4

Figure 4: CP-nets of the agents

wLCWs (i.e. the models of φ) is in NP. To show hardness, we reduce 3-SAT to our problem: given a 3-CNF formula F, for each clause $(x_1 \vee x_2 \vee x_3) \in F$, we construct the optimality constraint: $\geq 2\,[\bar{x}_1, \bar{x}_2] \Rightarrow x_3$. Any satisfying assignment of the original 3-CNF formula, at lease one of x_1, x_2 and x_3 is true. If x_1 or x_2 are $True$, then the condition of the optimality constraint is not satisfied and thus the optimality constraint is satisfied. If x_1 and x_2 are both $False$, then x_3 is $True$, which satisfies the optimality constraint as this is the preferred value of a majority of agents. Hence, any model of the original 3-CNF formula is an optimal assignment of the set of optimality constraints. The argument reverses: any wLCW is also a model. $\square$

We emphise here that the above complexity is for testing or finding the wLCWs rather than the (weak) Condorcet winners in Definition 2 and 3. As we show in Section 3 (Corollary 1), a wLCW is not necessarily a weak Condorcet winner. To find out the set of weak Condorcet winners, we still need to filter out from the set of wLCWs those candidates that are majority-dominated by some alternative. This checking is required even when there exists only one wLCW. Consequently, the complexity for finding the set of weak Condorcet winner remains PSPACE complete.

EXAMPLE. Now, we demonstrate the execution of the proposed approach with an example. Assume four agents $\mathbf{A} = \{A_1, A_2, A_3, A_4\}$ making decision over a set of four Boolean variables X_1, X_2, X_3 and X_4.Consider the agents' CP-nets depicted in Figure 4. We first generate a set of majority-optimality constraints that a wLCW must satisfy. For variable X_1, we refer to each agent A_j's conditional preference table $CPT_j\,(X_1)$:
A_1: $\mathbf{U}_{A_1}^{x_1 \succ \bar{x}_1} = \{x_2 x_4, \bar{x}_2 \bar{x}_4\}$ and $\mathbf{U}_{A_1}^{\bar{x}_1 \succ x_1} = \{x_2 \bar{x}_4, \bar{x}_2 x_4\}$, thus $P_1^1 = x_2 x_4 \vee \bar{x}_2 \bar{x}_4$ and $\bar{P}_1^1 = x_2 \bar{x}_4 \vee \bar{x}_2 x_4$;
A_2: the preference over variable X_1 is unconditional, $x_1 \succ_{A_2}^{X_1} \bar{x}_1$, thus $P_2^1 = True$ and $\bar{P}_2^1 = False$;
A_3: the preference over variable X_1 is unconditional, $x_1 \succ_{A_3}^{X_1} \bar{x}_1$, thus $P_3^1 = True$ and $\bar{P}_3^1 = False$;
A_4: $\mathbf{U}_{A_4}^{x_1 \succ \bar{x}_1} = \{x_2 \bar{x}_4\}$ and $\mathbf{U}_{A_4}^{\bar{x}_1 \succ x_1} = \{x_2 x_4, \bar{x}_2 x_4, \bar{x}_2 \bar{x}_4\}$, thus $P_4^1 = x_2 \bar{x}_4$ and $\bar{P}_4^1 = x_2 x_4 \vee \bar{x}_2 x_4 \vee \bar{x}_2 \bar{x}_4$.

Consequently, $F_1 = [\geq 3]\left(P_1^1, P_2^1, P_3^1, P_4^1\right)$ and $F_1' = [\geq 3]\left(\bar{P}_1^1, \bar{P}_2^1, \bar{P}_3^1, \bar{P}_4^1\right)$. F_1 can be simplified to $(x_2 \vee \bar{x}_4)$. F_1' is unsatisfiable and evaluated to $False$, because two formulas $\bar{P}_2^1$ and $\bar{P}_3^1$ out of four in the formula list of F_i' are $False$. Hence, the winning alternative must satisfy the following optimality constraint for variable X_1: $\varphi_1 = (x_2 \vee \bar{x}_4 \Rightarrow x_1) \wedge (False \Rightarrow \bar{x}_1)$. An implication is unsatisfied only when the hypothesis is $True$ and the conclusion is $False$, thus $False \Rightarrow \bar{x}_1$ is always $True$ and φ_1 can be simplified to $\varphi_1 = x_2 \vee \bar{x}_4 \Rightarrow x_1$.

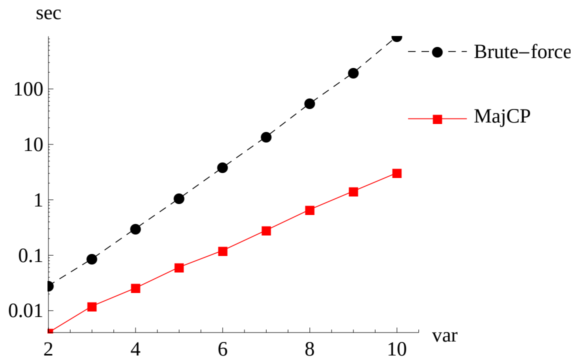
Similarly, we obtained the following optimality constraints (simplified form of the cardinality constraints) for variable X_2, X_3 and X_4:
X_2: $\varphi_2 = (\bar{x}_1 x_3 x_4 \Rightarrow x_2) \wedge (x_1 \bar{x}_3 \vee x_1 \bar{x}_4 \vee \bar{x}_3 \bar{x}_4 \Rightarrow \bar{x}_2)$;
X_3: $\varphi_3 = (\bar{x}_1 \bar{x}_2 \Rightarrow x_3) \wedge (x_1 \bar{x}_2 \Rightarrow \bar{x}_3)$;
X_3: $\varphi_4 = \bar{x}_4$;
Consequently, we obtain the following SAT:
$\varphi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 = (x_2 \vee \bar{x}_4 \Rightarrow x_1) \wedge (\bar{x}_1 x_3 x_4 \Rightarrow x_2) \wedge (x_1 \bar{x}_3 \vee x_1 \bar{x}_4 \vee \bar{x}_3 \bar{x}_4 \Rightarrow \bar{x}_2) \wedge (\bar{x}_1 \bar{x}_2 \Rightarrow x_3) \wedge (x_1 \bar{x}_2 \Rightarrow \bar{x}_3) \wedge \bar{x}_4$

The above SAT has only one satisfied assignment $x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4$. After checking the majority optimality of $x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4$, it is also a weak Condorcet winner in this example.

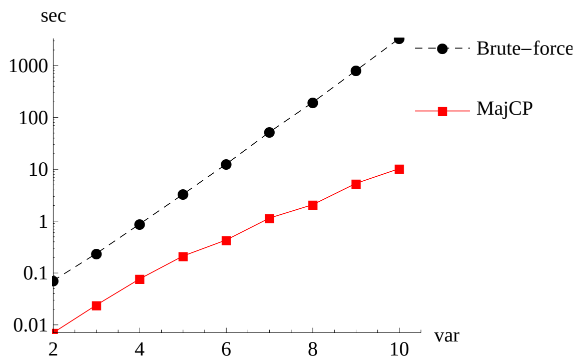
5. EXPERIMENT

In this section, we present the experimental results regarding the execution time of the proposed approach. We compare the performance of the proposed `MajCP` approach to a `Brute-force` algorithm, which runs a direct election over the alternative space. In these experiments, the numbers of agents are 5 and 15 respectively, and we vary the numbers of variables from 2 to 10. The number of parents of a variable in the agents' CP-nets is bounded by 6. For each number of agents and each number of variables, we generate 5,000 random examples of the agents' CP-nets.

The log-scale plots in Figure 5 show the average execution times of the `Brute-force` algorithm and the proposed `MajCP` approach in the case of 5 agents and 15 agents, respectively. It demonstrates that the proposed `MajCP` approach is much more efficient than the `Brute-force` algorithm. In general, for large numbers of variables, it offers several orders of magnitude improvement in performance over the `Brute-force` algorithm both for 5 agents and 15 agents. For instance, when there are 10 variables, the execution time of `MajCP` is reduced by more than three orders of magnitude as compared to `Brute-force` algorithm. We further test 100 cases for 11 variables and 5 agents (resp. 11 variables and 15 agents), which shows that the execution time of the `Brute-force` algorithm is on average more than 5000 seconds (resp. 9000 seconds). On the other hand, the proposed `MajCP` approach can produce the majority winners in about 10 seconds (resp. 15 seconds). Note that when there exist wLCWs (1 or more), the proposed `MajCP` approach still need to test the majority-optimality of the wLCWs by comparing each wLCW to all other alternatives. However, when there are no wLCWs, the proposed approach can return the result quickly by only solving the corresponding SAT problem. For instance, given 15 agents and 10 variables, when there does not exists any wLCWs, the proposed approach returns the results within 0.04 seconds. Table 1 provides the probability that there exists no wLCWs for the given agents' preferences in



(a) 5 agents



(b) 15 agents

Figure 5: Average execution time comparison (Log scale plot)

those experiments.

Table 1: The percentage of cases when there are no wLCWs

Agents	Variables				
	2	4	6	8	10
5	4.71%	12.73%	19.67%	22.11%	24.51%
15	5.44 %	15.99%	22.22%	25.42%	27.94%

6. CONCLUSION AND FUTURE WORK

In this paper, we have introduced an efficient approach to compute the set of winning alternatives from a collection of CP-nets based on majority rule. Unlike previous work where the agents' preferences are required to satisfy some restrictive conditions on the dependence graph (such as the existence of a common acyclic graph to all the agents), the proposed approach allows the agents to have different preferential independence structures and also works on cyclic CP-nets. It first computes a set of weak local Condorcet winners (wLCWs) by reduces the problem into an extended SAT (Boolean satisfiability problem) for cardinality constraints. Then the set of majority winning alternatives is a subset of wLCWs after filtering out those are majority-dominated by some alternative. The proposed approach reduces the size of search space and is computationally efficient.

Future research can extend the proposed approach to compute the winners of other aggregation rules. Another extension would be to investigate techniques to aggregate preferences that are represented by more powerful variants such as TCP-nets and UCP-nets.

7. ACKNOWLEDGEMENTS

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