On the Complexity of Undominated Core and Farsighted Solution Concepts in Coalitional Games

(Extended Abstract)

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ABSTRACT

In this paper, we study the computational complexity of solution concepts in the context of coalitional games. Firstly, we distinguish two different kinds of core, the undominated core and excess core, and investigate the difference and relationship between them. Secondly, we thoroughly investigate the computational complexity of *undominated core* and three farsighted solution concepts—*farsighted core*, f*arsighted stable set* and *largest consistent set*.

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; I.2.11 [**Distributed Artificial Intelligence**]: Multiagent systems; J.4 [**Computer Applications**]: Social and Behavioral Sciences - Economics

General Terms

Economics, Theory

Keywords

Game theory (cooperative and non-cooperative), teamwork, coalition formation, coordination, computational complexity

1. FARSIGHTED COALITIONAL GAMES

Recent years, coalitional games or cooperative games are more and more influential among multiagent system community due to their ability to capture the cooperative behaviors among players or agents. A *coalitional game* is defined by a set of players $N = \{1, \ldots, n\}$ and a characteristic function $v : 2^N \to \mathbb{R}$ where $v(\emptyset) = 0$ and an *imputation* for a coalitional game $\mathcal{G}$ is an n-vector $x \in \mathbb{R}^n$ satisfying (a) $\sum_{i\in N} x_i = v(N)$, (b) $x_i \geq v(\{i\})$ for all $i \in N$. And $I(\mathcal{G})$ denotes the set of imputations of game $\mathcal{G}$. A fundamental problem for coalitional games is to characterize certain subsets of imputations, which are payoff distributions among players, in terms of several interesting properties. Traditionally, these concepts have been studied in economics and mathematics to shed light on the explanation of various economical intuition of fairness and stability, and also to provide a strict and profound mathematical analysis. The well-known solution concepts include the Shapley value, the core, the stable set and so on.

However, the (undominated) core and the stable set have been criticized by Harsanyi[5] and Chwe[2]. They pointed out that these two solution concepts are too myopic to capture the farsight behaviors of players or coalitions in real economical environment. To overcome the myopia in the core and the stable set, Harsanyi proposed a model with a chairman to control the coalition formation process. Chwe proposed a similar model and formalized a binary relation called indirect dominance, which is slightly different from the one discussed by Harsanyi, to capture the farsightedness of coalitions. When indirect dominance is incorporated into the undominated core and stable set, we get new solution concepts. These new solution concepts are called the *farsighted core* and *(vN-M) farsighted stable set*, respectively. Furthermore, Chwe defined another solution concept–*largest consistent set* to conclude more outcomes. Béal *et al.*[1] systematically studied these solution concepts in coalitional games. Their results lay the foundation of our work.

2. THE CORES

There are two kinds of cores in the literature. The undominated core, is defined by (direct) dominance(for $x, y \in I(\mathcal{G})$. And let $S \subseteq N$ be a coalition, x *directly dominates* y via S, denoted as $y \prec_S x$, if (a) $x(S) \leq v(S)$, (b) $y_i < x_i$ for all $i \in S$), was originally introduced by Gillies[4]. The undominated core can be expressed as $\{x \in I(\mathcal{G}) | \forall S \subseteq N, \nexists y \in I(\mathcal{G}) \text{ s.t. } x \prec_S y\}$. However, the excess core, which is defined by *excess*, i.e., $e(S, x) = v(S) - x(S)$, is the common one. Also it can be characterized by $\{x \in I(\mathcal{G}) | e(S, x) = v(S) - x(S) \leq 0, \forall S \subseteq N\}$. We illustrate the difference and relationship between the undominated core and the excess core. Owen[6] proved that if the coalitional game is superadditive, then the undominated core coincides with the excess core. We show that in the general setting, the undominated core is strictly larger than the excess one. Figure 1 summarizes our major results.

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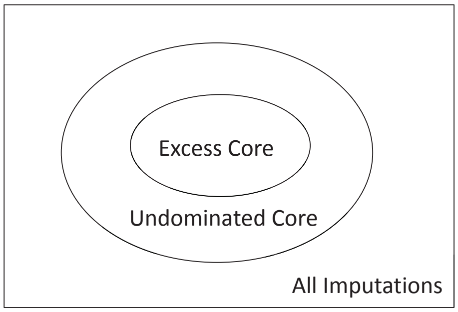

Figure 1: The undominated core and excess core

Moreover, to provide an polynomial certificate for the Turing machine, we prove a necessary and sufficient condition to determine that whether an imputation is *not* in the undominated core in the full version of the paper. With this condition, we could explicitly exam the stability of an imputation without considering the complicated direct dominance relation.

3. FARSIGHTED SOLUTION CONCEPTS

Chwe[2] extended the direct dominance into undirect dominance. For $x, y \in I(\mathcal{G})$, let $S \subseteq N$ be a coalition, we say that x *indirectly dominates* y, denoted as $y \ll x$, if there exist a finite sequence of imputations $y = x^1, \ldots, x^m = x$ and a finite sequence of coalitions $S^1, \ldots, S^{m-1}$ such that (a)$x^{j+1}(S^j) \leq v(S^j)$, (b)$x_i^j < x_i$ for all $i \in S^j$, where $j = 1, \ldots, m-1$. Based on this binary relation, the farsighted core can be defined as $\{x \in I(\mathcal{G}) | \forall S \subseteq N, \nexists y \in I(\mathcal{G})$ s.t. $x \ll y\}$, the farsighted stable sets can be characterized by $\{K \subseteq I(\mathcal{G}) | \forall x, y \in K$, neither $x \ll y$ nor $y \ll x, \forall y \in I(\mathcal{G}) \setminus K, \nexists x \in K$ s.t. $y \ll x\}$.

Furthermore, Chwe[2] considered the farsighted stable set is too exclusive, since it may exclude some outcomes that may not be consistent, but rational and foresight. To overcame this, Chew suggested a new solution concept to capture more inclusive farsighted stability: the consistent set.

Definition 1. A set $K \subseteq I(\mathcal{G})$ is consistent if:

1. $\forall x \in K$, $\forall z \in I(\mathcal{G})$ and $\forall S \in 2^N \setminus \emptyset$ such that $z(S) \leq v(S)$, there exists $y \in K$, where $y = z$ or $z \ll y$, such that $x \not<_S y$;

2. $\forall x \in I(\mathcal{G}) \setminus K$, these exist $z \in I(\mathcal{G})$, $S \in 2^N \setminus \emptyset$ with $z(S) \leq v(S)$, such that $\forall y \in K$, where $y = z$ or $z \ll y$, and it holds that $x_i < y_i$ for all $i \in S$.

Although the consistent set may not be unique, Chew([2]) proved that there exists a unique largest consistent set, i.e., it contains all the tother consistent sets. Both Chew[2] and Béal[1] showed that the largest consistent set is not empty.

4. COMPUTATIONAL MODEL

To study the computational complexity of solution concepts, we adopt the *oracle* setting to represent the coalitional games. In this setting, we consider the characteristic function v as an oracle, i.e., given a coalition, it outputs the value in polynomial time w.r.t. the size of the game representation. Moreover, we assume that players are included in the game representation, i.e., the size of the representation of a coalition S is less or equal to the size of the whole game representation.

Table 1: Summary of results

Solution Concept \ Problem	Membership	Emptiness
Undominated Core	co-**NP**-c	**NP**-hard[1]
Farsighted Core	co-**NP**-c	**NP**-c
Farsighted Stable Set	**DP**-c	Nonempty
Largest Consistent Set	Undecidable	Nonempty

[1]The emptiness problem for games, whose corresponding (0,1)-reduced games $v(S) \leq 1$, is **NP**-complete. For the general games, we conjecture that it is in Σ_2^P.

5. COMPUTATIONAL RESULTS

Finally, based on the work of Béal et al.[1] and *oracle* representation scheme in coalitional games, we thoroughly investigate the computational issues of undominated core, farsighted core, (vN-M) farsighted stable set and largest consistent set. Moreover, in the literature, researchers mostly discuss the following two problems[3]: (1) Given a game and an imputation, deciding whether the imputation belongs to a certain solution concept; (2) Given a game, deciding whether a specific solution concept is empty. We call the first and second the membership problem and emptiness problem, respectively. We study the membership problems of the undominated core, farsighted core, farsighted stable sets and largest consistent set. For the emptiness problems, we only investigate the undominated core and farsighted core since the farsighted stable sets and largest consistent set are never empty in general coalitional games[1]. Table 1 presented the summary of our major results in this paper.

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