The Cost of Principles: Analyzing Power in Compatibility Weighted Voting Games

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ABSTRACT

We propose *Compatibility Weighted Voting Games*, a variant of Weighted Voting Games in which some pairs of agents are compatible and some are not. In a Weighted Voting Game each agent has a weight, and a set of agents can form a winning coalition if the sum of their weights is at least a given quota. Whereas the original Weighted Voting Game model assumes that all agents are compatible, we consider a model in which the agents' compatibility is described by a *compatibility graph*. We consider power indices, which measure the power of each agent to affect the outcome of the game, and show that their computation is tractable under certain restrictions (chiefly that the agents' compatibilities have *spectral* structure). Through simulations we investigate the effect an agent's compatibility restrictions has on its power.

Categories and Subject Descriptors

I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multiagent Systems*; F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity

General Terms

Theory, Algorithms

Keywords

cooperative game theory; coalition formation; power indices

1. INTRODUCTION

Selfish agents interacting in a system may have conflicting interests. In domains where the agents must cooperate to achieve goals, we are interested in quantifying an agent's ability to bring about an outcome it desires, or to negotiate effectively with other agents. Such notions of *power* play a key role in cooperative game theory and its applications to multiagent systems. The distribution of power among the agents is typically measured by power indices. However, the popular power indices have been criticized for their assumption that all agents are willing to cooperate [9, 14, 15, 21]. This is unrealistic in many settings, e.g. businesses that are fierce competitors, or political parties with conflicting ideologies that are not willing to form a ruling coalition.

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Indeed the distribution of power may vary dramatically depending on whether certain agents are willing to cooperate. Consider an election in which three political parties, Parties A, B and C, each win 100 seats in a parliament of 300 seats, and where a majority of 150 seats is required to form a government. This is a Weighted Voting Game [11] (WVG), a well-studied coalitional game in which agents have a number of votes each and a coalition of agents voting together succeed if they possess enough votes to meet some quota. Now suppose Party A and Party B have opposing principles and will never form a coalition government, but Party C can cooperate with both other parties. The popular measures of power, that have no notion of *compatibility* between the agents, would say that all three parties have equal power. However we would intuitively expect Party C to have more power in negotiations to form a government.

Our contribution: We generalize prominent power indices used in cooperative games to account for incompatibility between the agents. We augment WVGs with a *compatibility graph* whose edges represent compatible pairs of agents, and consider a coalition to be feasible iff it is a clique in the graph. We thus introduce Compatibility Weighted Voting Games (CWVGs). Our analysis concentrates on WVGs, but any cooperative game can be augmented in this way. We give natural generalizations of the Banzhaf index [8] and the Shapley-Shubik index [26]. We show that using a *fixed-dimensional spectral* model for compatibility allows (1) exact computation of both power indices if the agent weights are polynomially bounded, and (2) approximation of the Banzhaf index for general weights. Finally, we present results of simulations that demonstrate typical relationships between compatibility and power. In particular we find that decreasing one's compatibility sharply decreases one's power, i.e. that the cost of principles is great.

2. RELATED WORK

There are several existing approaches to incorporating the concept of compatibility in cooperative games. The model closest to ours is Myerson's [21], which has a 'cooperation graph' on the agents, and says a coalition is feasible iff it is a connected subgraph. By contrast in our model a coalition is feasible iff it is a clique. Our model is therefore more expressive in settings where compatibility is not transitive, as we may specify that agents A and B are compatible, B and C are compatible, but A and C are not (this is not possible in the Myerson model). Myerson's model has been studied for WVGs from an axiomatic [22] and computational [24] angle. Our computational results apply to the restricted case of spectral CWVGs, which has the advantage of naturality, compared to the restrictions often applied to the Myerson model for the purposes of computation (e.g. requiring that the cooperation graph is a tree as in [17]).

Other approaches include those of Edelman [14] and Bilbao [9], which require that the set of feasible coalitions is a *convex geometry* or *partition system* respectively. Faigle and Kern [15] introduce a *precedence relation* on the agents and require that feasible coalitions 'obey' the relation. The Shapley-Owen value [23] places the agents in an ideological space, then an 'issue vector', representing issues to be voted on, induces a probability distribution on the voting order of the agents. Our model has a comparatively simple definition and intuitive interpretation, and to our knowledge has not been proposed before. Previous work on agent failures is also somewhat reminiscent of our work [1, 3, 6], but those models examine *independent agent failures*, whereas in our model a coalition with 'contradicting' agents cannot arise.

We are principally concerned with computing power indices for CWVGs. Computing power indices is challenging even for WVGs, and has attracted much research [2, 4, 5, 7, 11, 19]. We build on the dynamic programming method for WVGs described in [11, 19], extending it to *spectral* CWVGs. We also extend the Monte Carlo method in [2] to CWVGs.

3. WEIGHTED VOTING GAMES (WVGS)

A *Weighted Voting Game* (WVG) consists of a set of agents $N = \{1, \dots, n\}$, a set of weights $w_1, \dots, w_n$ and a quota q. We assume that the weights and quota are all positive integers, and that the sum of all the weights is at least the quota. A *coalition* is a subset of N. The *weight function* $w : 2^N \to \mathbb{Z}_+$ gives the weight of each coalition, defined to be the sum of the weights of its members: $w(S) = \sum_{i \in S} w_i$. The *value function* $v : 2^N \to \{0, 1\}$ gives the value of each coalition, which is either 0 (*losing*) or 1 (*winning*) as follows:

$$v(S) = \begin{cases} 1 & \text{if } w(S) \geq q \\ 0 & \text{o/w} \end{cases} \tag{1}$$

3.1 Power Indices for WVGs

Here we recall the definitions of the *Banzhaf power index* [8, 13] and the *Shapley-Shubik power index* [25, 26] for Weighted Voting Games.

Definition 1. For a WVG with value function v, the *raw Banzhaf power index* of agent i is

$$\eta_i(v) = \sum_{S \subset N : i \in S} v(S) - v(S \setminus \{i\}). \tag{2}$$

An agent is called *critical* in a coalition if it can turn the coalition from winning to losing by leaving it. The raw Banzhaf power index $\eta_i(v)$ is thus the number of coalitions in which agent i is critical. There are the related concepts of the *normalized* and the *probabilistic* Banzhaf index:

Definition 2. For a WVG with value function v, the *normalized Banzhaf power index* of agent i is

$$\beta_i(v) = \eta_i(v) / \sum_{j \in N} \eta_j(v). \tag{3}$$

Definition 3. For a WVG with value function v, the *probabilistic Banzhaf power index* of agent i is

$$\beta'_i(v) = \eta_i(v)/2^{n-1}. \tag{4}$$

The normalized Banzhaf index $\beta_i(v)$ is agent i's share of the power. The probabilistic Banzhaf index $\beta'_i(v)$ is agent i's probability of being critical, if all coalitions are equally probable.

We now define the Shapley-Shubik power index, which was introduced in [26] as a special case of the Shapley value [25] when restricted to WVGs. Let Π be the set of all permutations of N. Given a permutation $\pi \in \Pi$ and an agent i, let $S_\pi(i)$ be the coalition consisting of all agents who appear in π before i (including i).

Definition 4. For a WVG with value function v, the *Shapley-Shubik power index* of agent i is

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} v(S_\pi(i)) - v(S_\pi(i) \setminus \{i\}). \tag{5}$$

A permutation of the agents represents the agents joining the coalition one by one. Given that the sum of all the weights is at least the quota, every permutation has a unique critical agent who turns the coalition from losing to winning by joining. Therefore the Shapley-Shubik power index $\phi_i(v)$ is the probability that agent i is the critical agent, if the agents join the coalition in a random order. Note $\sum_{j \in N} \phi_j(v) = 1$, so $\phi_i(v)$ also represents i's share of the power.

4. COMPATIBILITY WVGS (CWVGS)

We are interested in introducing a notion of *compatibility*, describing which agents are able to work together. Therefore we define a *Compatibility Weighted Voting Game* (CWVG) which consists of a set of agents $N = \{1, \dots, n\}$, a set of weights $w_1, \dots, w_n$, a quota q, and a *compatibility graph* G. N forms the set of vertices of G, and an edge between two agents indicates that they are compatible. We call a coalition *feasible* if all pairs of members are compatible, or equivalently, if the coalition forms a clique in G. Let F be the set of feasible coalitions of G. The weight function $w : 2^N \to \mathbb{Z}_+$ is defined as in Section 3. For CWVGs, we define the value function v only on feasible coalitions, as we regard infeasible coalitions as 'impossible' and thus they do not have a value.[1] The value function is otherwise unchanged: $v : F \to \{0, 1\}$ satisfies equation (1) for all feasible $S \subseteq N$.

4.1 Power Indices for CWVGs

We now redefine the power indices from Section 3.1 for CWVGs, replacing the assumption that all coalitions are equally likely with the assumption that all *feasible* coalitions are equally likely (in the case of the Banzhaf index).

Definition 5. For a CWVG with compatibility graph G and value function v, agent i's *raw Banzhaf power index* is

$$\eta_i(G, v) = \sum_{S \in F : i \in S} v(S) - v(S \setminus \{i\}). \tag{6}$$

[1] Alternatively we could give infeasible coalitions value 0, and use the resulting value function $v' : 2^N \to \{0, 1\}$ to calculate the power indices given in Section 3.1. However some agents would have negative indices, and an agent with index 0 would not necessarily be a null agent (one who is never able to change the outcome).

Definition 5 differs from Definition 1 in that we sum over all feasible coalitions containing i instead of all coalitions containing i. So $\eta_i(G,v)$ is simply the number of *feasible* coalitions in which agent i is critical.

Definition 6. For a CWVG with compatibility graph G and value function v, agent i's *normalized Banzhaf power index* is[2]

$$\beta_i(G,v) = \eta_i(G,v)/\sum_{j\in N}\eta_j(G,v). \quad (7)$$

As in Definition 2, $\beta_i(G,v)$ is agent i's share of the power.

Definition 7. For a CWVG with compatibility graph G and value function v, agent i's *probabilistic Banzhaf power index* is[3]

$$\beta_i'(G,v) = 2\eta_i(G,v)/|F|. \quad (8)$$

The probabilistic Banzhaf power index $\beta_i'(G,v)$ is the probability that, when a random *feasible* coalition is selected, agent i can change the outcome by changing its vote. (That is: a feasible coalition $S \in F$ is chosen at random, then i is critical if either $i \in S$ and i can turn S from winning to losing by leaving, or $i \notin S$ and i can feasibly join S, turning it from losing to winning). To see why Definition 7 is a natural generalization of Definition 3, note that Definition 3 can be written $2\eta_i(v)/2^n$ and interpreted as the probability that, when a random coalition $S \in 2^N$ is selected, i can change the outcome by changing its vote.

We now define the Shapley-Shubik index for CWVGs. First note that while in a WVG every permutation of the agents has a unique critical agent, the same is not true in a CWVG, as it is possible that while adding agents to the coalition in the order given by the permutation, the coalition becomes infeasible before it achieves the quota. This motivates the following definition.

Definition 8. For a permutation $\pi = (\pi_1,\ldots,\pi_n)$ of N, let $\text{feas}(\pi) = \{\pi_1,\ldots,\pi_k\}$ where k is maximal such that $\{\pi_1,\ldots,\pi_k\}$ is feasible. Call π *effective* if feas(π) is winning.

Denote the set of effective permutations by E. In a CWVG, every effective permutation has a unique critical agent.

Definition 9. For a CWVG with compatibility graph G and value function v, the *Shapley-Shubik power index* of agent i is[4]

$$\phi_i(G,v) = \frac{1}{|E|}\sum_{\pi\in E: i\in \text{feas}(\pi)} v(S_\pi(i)) - v(S_\pi(i)\setminus\{i\}). \quad (9)$$

The Shapley-Shubik power index $\phi_i(G,v)$ is the probability that agent i is the unique critical agent, if all *effective* permutations are equally likely. The original Shapley-Shubik power index is known [16] to satisfy the four axioms of *efficiency* (the sum of the indices is 1), *anonymity* (two agents who make the same marginal contribution to each coalition have the same power), *null player property* (null players, who are never critical, have power 0) and *transfer* (an additivity-like property for simple games). If we generalize these four axioms to CWVGs in the natural way, we find that Definition 9 satisfies all except transfer (essentially because normalizing over $|E|$ prevents the additivity from working). If we divide by $n!$ instead of $|E|$ in equation (9), we gain transfer but lose efficiency. We prioritize efficiency.

Note that if the compatibility graph G is a complete graph (i.e. all agent pairs are compatible) then all the power index definitions in this section simplify to those of Section 3.1.

An Example: For the example given in Section 1, the raw Banzhaf indices are $\eta_A(G,v) = \eta_B(G,v) = 1$ and $\eta_C(G,v) = 2$ because Parties A and B are critical in only one coalition each ($\{A,C\}$ and $\{B,C\}$ respectively) and C is critical in both those coalitions. Therefore the normalized Banzhaf indices are $\beta_A(G,v) = \beta_B(G,v) = 0.25$ and $\beta_C(G,v) = 0.5$. The set E of effective permutations contains the permutations CAB, CBA, ACB, BCA. A is the critical agent in CAB, B the critical agent in CBA, and C the critical agent in ACB and BCA. Therefore the Shapley-Shubik indices are also $\phi_A(G,v) = \phi_B(G,v) = 0.25$ and $\phi_C(G,v) = 0.5$.

5. CALCULATING POWER INDICES

It has been shown [20] that computing $\eta_i(v)$ or $\phi_i(v)$ for a WVG cannot be done in polynomial time unless P=NP, via a reduction of the Partition Problem. Therefore the same holds for computing $\eta_i(G,v)$ or $\phi_i(G,v)$ for a CWVG (as a CWVG simplifies to a WVG when the compatibility graph is complete). However, there exist algorithms based on Dynamic Programming techniques [19] to compute $\eta_i(v)$ or $\phi_i(v)$ for a WVG in $O(n^2 w_{\max})$ or $O(n^3 w_{\max})$ time respectively, where $w_{\max} = \max_{i\in N} w_i$. We reproduce the algorithm, using the notation of [11].

5.1 Dynamic Programming method for WVGs

Take a WVG with n agents and quota q. For any $1 \le j \le n$, $0 \le W \le q-1$, $0 \le s \le n$, define $X(j,W,s)$ to be the number of size-s subsets of $\{1,\ldots,j\}$ of weight exactly W. This quantity obeys the following recurrence relation on j. For all $2 \le j \le n$, $0 \le W \le q-1$, $1 \le s \le n$,

$$X(j,W,s) = \begin{cases} X(j-1,W,s) & \\ \quad + X(j-1,W-w_j,s-1) & \text{if } w_j \le W \\ X(j-1,W,s) & \text{if } w_j > W \end{cases} \quad (10)$$

with initial values:

$$X(1,W,s) = \begin{cases} 1 & \text{if } (W,s) = (w_1,1) \text{ or } (0,0) \\ 0 & \text{o/w} \end{cases} \quad (11)$$

for all $0 \le W \le q-1$, $0 \le s \le n$, and $X(j,W,0) = \mathbb{1}_{W=0}$ for all $1 \le j \le n$, $0 \le W \le q-1$.

THEOREM 1 (MATSUI MATSUI [19]). *For any WVG with n agents and maximum weight w_{max}, we can compute $X(j,W,s)$ $\forall\ 1 \le j \le n$, $0 \le W \le q-1$, $0 \le s \le n$ in $O(n^3 w_{max})$ time.*

PROOF. By using the recurrence relation and initial values above, we compute $O(n^2 q) \le O(n^2\sum_{i\in N} w_i) \le O(n^3 w_{\max})$ values of X, each in constant time. □

COROLLARY 1. *For any WVG with n agents and maximum weight w_{max}, and any $1 \le i \le n$, the Shapley-Shubik power index $\phi_i(v)$ can be computed in $O(n^3 w_{max})$ time.*

PROOF. Define a new WVG by removing agent i. The new game has $n' = n-1$ agents. For the new WVG, compute

[2] If all raw indices are 0, then all $\beta_i(G,v) = 0$.
[3] Note $|F| > 0$ always, as $\emptyset \in F$
[4] If $|E| = 0$ i.e. there are no feasible winning coalitions, then all agents have index 0

$X(j, W, s)$ for all $1 \le j \le n'$, $0 \le W \le q-1$, $0 \le s \le n'$ as in Theorem 1. This takes $O(n^3 w_{\max})$ time. Now note that $\phi_i(v) = \frac{1}{n!} \sum_{s=0}^{n-1} s!(n-s-1)! N_s$ where N_s is the number of size-s coalitions $S \subseteq N \setminus \{i\}$ of weight $q - w_i \le w(S) \le q-1$. For all $0 \le s \le n-1$, $N_s = \sum_{W=\max(0,q-w_i)}^{q-1} X(n', W, s)$. □

Corollary 2. *For any WVG with n agents and maximum weight w_{max}, and any $1 \le i \le n$, the raw Banzhaf power index $\eta_i(v)$ can be computed in $O(n^2 w_{max})$ time.*

Proof. Define a new WVG by removing agent i. The new game has $n' = n - 1$ agents. For the new WVG, calculate the value $X(j, W) =$ the number of subsets of $\{1, \dots, j\}$ of weight exactly W, for all $1 \le j \le n'$ and $0 \le W \le q-1$. This can be done in $O(n^2 w_{\max})$ time exactly as in Theorem 1 by omitting the variable s. Then $\eta_i(v) = \sum_{W=\max(0,q-w_i)}^{q-1} X(n', W)$. □

5.2 Hardness for general CWVGs

We have seen that for WVGs, the power indices depend only on the number of coalitions of each weight (and size, for the Shapley-Shubik index). For CWVGs, the power indices can be shown to depend only on the number of *feasible* coalitions of each weight (and size). We may hope to calculate these quantities as in Section 5.1, and obtain a method that is polynomial in n when the weights are bounded by a polynomial in n. However we show that even if the weights are bounded by a constant, power indices cannot be computed in polynomial time unless P=NP.

Definition 10. Let $M \ge 1$. RawBz(M) is the decision problem given by:
Input: A CWVG $(G; q; w_1, \dots, w_n)$ with $w_{max} \le M$.
Output: Determine whether $\eta_n(G, v) > 0$.

Similarly define NormBz(M), ProbBz(M) and SS(M) as in Definition 10, replacing $\eta_n(G, v)$ with $\beta_n(G, v)$, $\beta'_n(G, v)$, and $\phi_n(G, v)$ respectively. We show NP-completeness of RawBz(M) for all M via a reduction from the Clique Decision Problem (Clique), a well-known NP-complete problem.

Definition 11. Clique is the decision problem given by:
Input: An undirected graph $G = (V, E)$ and a number k.
Output: Determine whether G contains a clique of size k.

Theorem 2. *For all $M \ge 1$, RawBz(M) is NP-complete.*

Proof. RawBz(M) is in NP, as the witness to a 'yes' answer is a feasible coalition in which n is critical. To show NP-completeness, we show a polynomial-time reduction from Clique. Given any instance (G, k) of Clique, construct the following CWVG. Let the set of agents be the vertices $\{1, \dots, n\}$ of G plus an extra agent $n+1$. Let all agents have weight 1, and the quota be $k+1$. Define the compatibility graph G' like so: let a pair of agents which are vertices of G be compatible iff they have an edge in G, and let agent $n+1$ be compatible with all other agents. Then $\eta_{n+1}(G', v) > 0$ iff $\exists$ a clique of size k in G. □

Note that as $\eta_n(G, v) > 0 \Leftrightarrow \beta_n(G, v) > 0 \Leftrightarrow \beta'_n(G, v) > 0 \Leftrightarrow \phi_n(G, v) > 0$, NormBz(M), ProbBz(M) and SS(M) are all NP-complete $\forall M \ge 1$. This motivates our restriction to CWVGs whose compatibility graphs have *spectral* structure.

5.3 Spectral CWVGs

We define the following model of *spectral* compatibility, which induces a compatibility graph. Each agent i has a *position* $\mathbf{p_i} \in [n]^m = \{1, \dots, n\}^m$ in an m dimensional spectrum. In a political setting, this could represent m political issues, and for each issue the parties are ordered from most enthusiastic to least enthusiastic, or leftwing to rightwing. Each agent i has a *tolerance hyperrectangle* $[a_i^1, b_i^1] \times \dots \times [a_i^m, b_i^m]$ containing the agents it is willing to cooperate with. (Every agent is contained within its own tolerance hyperrectangle). Then two agents are compatible iff each is contained in the other's tolerance hyperrectangle. That is, on each issue each agent has an interval of attitudes it will accept, and two agents are compatible iff they both accept each other on every issue.[5] We say that a CWVG is *spectral* if its compatibility graph is induced by a spectral model, and *m-spectral* if its compatibility graph is induced by an m-dimensional spectral model. Note that all CWVGs are trivially spectral for large enough dimension (e.g. allow one dimension for each incompatible pair of agents).

We extend the algorithm in Section 5.1 to calculate the power indices for spectral CWVGs. First we present the algorithm for 1-spectral CWVGs for ease of understanding, and then we present the generalization to m dimensions.

5.4 Method for 1-spectral CWVGs

In the 1-dimensional case, we streamline the algorithm by numbering the agents according to their left to right position on the spectrum. That is, agent i has position $i \in \{1, \dots, n\}$. Each agent i has a tolerance interval $[a_i, b_i]$.

Definition 12. Take a 1-spectral CWVG with n agents. For all $1 \le j, s, l, t \le n$ and $1 \le W \le w(N)$ let $X(j, W, s, l, t)$ be the number of size-s *feasible* coalitions $S \subseteq \{1, \dots, j\}$ of weight W, such that $l = \min_{i \in S} i$ and $t = \min_{i \in S} b_i$.

Remark 1. l is the most leftwing member of S and t is the most rightwing agent tolerated by all members of S. As we only define $X(j, W, s, l, t)$ for $s \ge 1$ and $W \ge 1$, (i.e. not the empty set) l and t are well-defined. Note also that unless $l \le t$, $X(j, W, s, l, t) = 0$.

Theorem 3. *For all $2 \le j, s \le n$, $1 \le l, t \le n$ and $1 \le W \le w(N)$, $X(j, W, s, l, t) = X(j-1, W, s, l, t) + A$ where*

$$A = \begin{cases} X(j-1, W - w_j, s-1, l, t) & \text{if } b_j > t \\ \sum_{t'=t}^{n} X(j-1, W - w_j, s-1, l, t') & \text{if } b_j = t \end{cases} \quad (12)$$

if $w_j \le W, a_j \le l, j \le t$, and $b_j \ge t$, and $A = 0$ otherwise.

Proof. Note we shall often write '$S \in X(j, W, s, l, t)$' to mean 'S is counted by $X(j, W, s, l, t)$'. First observe that the coalitions counted by $X(j, W, s, l, t)$ are divided into those that do not contain j and those that do. There are $X(j-1, W, s, l, t)$ of the former, and let A denote the number of the latter. We determine A.

If $w_j > W$ or $a_j > l$ or $j > t$ or $b_j < t$ then any coalition $S \in X(j, W, s, l, t)$ cannot contain j, so $A = 0$. If $w_j \le W$ and $a_j \le l$ and $j \le t$ and $b_j \ge t$, there are two cases:

[5] This is similar but not the same as an *interval graph* [10], where the vertices are intervals and there are edges between intervals that intersect.

If $\mathbf{b_j} > \mathbf{t}$, there is a bijection between $\{S \in X(j, W, s, l, t) : j \in S\}$ and $X(j-1, W-w_j, s-1, l, t)$:

If $S \in X(j, W, s, l, t)$ and $j \in S$, let $S' = S \setminus \{j\}$. $S' \subseteq \{1, \ldots, j-1\}$ is feasible, $w(S') = W - w_j$, $|S'| = s-1$, $\min_{i \in S'}(i) = l$ (as $s \geq 2$) and $\min_{i \in S'}(b_i) = t$ (as $b_j > t$), so $S' \in X(j-1, W-w_j, s-1, l, t)$.

Conversely if $S' \in X(j-1, W-w_j, s-1, l, t)$ let $S = S' \cup \{j\}$. S is feasible, as $a_j \leq l$ and $j \leq t$. $S \subseteq \{1, \ldots, j\}, w(S) = W, |S| = s$, $\min_{i \in S}(i) = l$ (as $s \geq 2$) and $\min_{i \in S}(b_i) = t$ (as $b_j > t$), so $S \in X(j, W, s, l, t)$ and $j \in S$.

If $\mathbf{b_j} = \mathbf{t}$, there is a bijection between $\{S \in X(j, W, s, l, t) : j \in S\}$ and $\bigcup_{t'=t}^{n} X(j-1, W-w_j, s-1, l, t')$:

If $S \in X(j, W, s, l, t)$ and $j \in S$, let $S' = S \setminus \{j\}$. $S' \subseteq \{1, \ldots, j-1\}$ is feasible, $w(S') = W - w_j$, $|S'| = s-1$, $\min_{i \in S'}(i) = l$ (as $s \geq 2$) and $\min_{i \in S'}(b_i) \geq t$ (as $b_j = t$), so $S' \in \bigcup_{t'=t}^{n} X(j-1, W-w_j, s-1, l, t')$.

Conversely if $S' \in X(j-1, W-w_j, s-1, l, t')$ for some $t' \geq t$ let $S = S' \cup \{j\}$. S is feasible, as $a_j \leq l$ and $j \leq t \leq t'$. $S \subseteq \{1, \ldots, j\}, w(S) = W, |S| = s$, $\min_{i \in S}(i) = l$ (as $s \geq 2$) and $\min_{i \in S}(b_i) = t$ (as $b_j = t \leq t'$), so $S \in X(j, W, s, l, t)$ and $j \in S$. □

THEOREM 4. *For any 1-spectral CWVG with n agents and maximum weight w_{max}, we can calculate $X(j, W, s, l, t)$ $\forall\ 1 \leq j, s, l, t \leq n$ and $1 \leq W \leq w(N)$ in $O(n^5 w_{max})$ time.*

PROOF. We have the initial values

$$X(j, W, 1, l, t) = \begin{cases} 1 & \text{if } \exists i \leq j : W = w_i, l = i, t = b_i \\ 0 & \text{o/w} \end{cases} \quad (13)$$

for all $1 \leq j, l, t \leq n$, $1 \leq W \leq w(N)$ and

$$X(1, W, s, l, t) = \begin{cases} 1 & \text{if } W = w_1, s = 1, l = 1, t = b_1 \\ 0 & \text{o/w} \end{cases} \quad (14)$$

for all $1 \leq W \leq w(N)$, $1 \leq s, l, t \leq n$. Using the initial values above and the recurrence relation in Theorem 3, we compute $O(n^4 w(N)) \leq O(n^5 w_{\max})$ values of $X(j, W, s, l, t)$. Each value is computed in constant time unless $w_j \leq W$, $a_j \leq l$, $j \leq t$ and $b_j = t$, in which case $X(j, W, s, l, t)$ is computed in $O(n)$ time. In the worst case, for each $1 \leq j \leq n$, for each $w_j \leq W \leq nw_{max}$, $1 \leq s \leq n$ and $a_j \leq l \leq n$ we must calculate $X(j, W, s, l, b_j)$ in $O(n)$ time. Therefore we perform $O(n^4 w_{\max})$ many $O(n)$ computations, and $O(n^5 w_{\max})$ many $O(1)$ computations. □

COROLLARY 3. *For any 1-spectral CWVG with n agents and maximum weight w_{max}, and any $1 \leq i \leq n$, we can calculate the Shapley-Shubik index $\phi_i(G, v)$ in $O(n^6 w_{max})$ time.*

PROOF. Let compat(i) be the set of agents with whom i is compatible (not including i). Form a new CWVG consisting only of the agents in compat(i). The new game has $n' \leq n-1$ agents. For the new game, compute $X(j, W, s, l, t)$ for all $1 \leq j, s, l, t \leq n'$ and $1 \leq W \leq w(\text{compat}(i))$ as described in Theorem 4. Now (recalling that E is the set of effective permutations) note that $|E|\phi_i(G, v) = \sum_{s=0}^{n'} s!(n-s-1)!N_s$ where N_s is the number of size-s feasible coalitions $S \subseteq \text{compat}(i)$ of weight $q - w_i \leq w(S) \leq q-1$. For all $1 \leq s \leq n'$

$$N_s = \sum_{\substack{\max(q-w_i, 1) \leq W \leq q-1 \\ 1 \leq l, t \leq n'}} X(n', W, s, l, t) \quad (15)$$

and $N_0 = \mathbb{1}_{w_i \geq q}$. Thus we can compute a single $\phi_i(G, v).|E|$ in $O(n^5 w_{\max})$ time. Then as $\sum_{j \in N} \phi_j(G, v) = 1$, $\phi_i(G, v) = \phi_i(G, v).|E| / \sum_{j \in N} \phi_j(G, v).|E|$ so we may compute a single $\phi_i(G, v)$ by computing $\phi_j(G, v).|E|$ for all $1 \leq j \leq n$, in $O(n^6 w_{\max})$ time. □

COROLLARY 4. *For any 1-spectral CWVG with n agents and maximum weight w_{max}, and any $1 \leq i \leq n$, we can calculate the raw, normalized and probabilistic Banzhaf indices of agent i in $O(n^4 w_{max})$, $O(n^5 w_{max})$ and $O(n^4 w_{max})$ time respectively.*

PROOF. To compute $\eta_i(G, v)$, form a new CWVG consisting only of the agents in compat(i). This game has $n' \leq n-1$ agents. For the new game, we calculate $X(j, W, l, t)$ (the number of feasible subsets of $\{1, \ldots, j\}$ of weight W, with leftmost member l and rightmost agent tolerated t) for all $1 \leq j, l, t \leq n'$ and $1 \leq W \leq w(\text{compat}(i))$. This is done in $O(n^4 w_{\max})$ time as in Theorem 4, by omitting the variable s. Then[6]

$$\eta_i(G, v) = \sum_{\substack{\max(q-w_i, 1) \leq W \leq q-1 \\ 1 \leq l, t \leq n'}} X(n', W, l, t) + \mathbb{1}_{w_i \geq q}. \quad (16)$$

To compute a single normalized Banzhaf index, we compute all n raw Banzhaf indices. To compute $\beta'_i(G, v) = \eta_i(G, v)/|F|$, first we compute $\eta_i(G, v)$ in $O(n^4 w_{\max})$ time as above. To compute $|F|$, for the original game compute $X(j, W, l, t)$ for all $1 \leq j, l, t \leq n$ and $1 \leq W \leq w(N)$ as in Theorem 4 by omitting the variable s. This takes $O(n^4 w_{\max})$ time. Then[7] $|F| = \sum_{\substack{1 \leq W \leq w(N) \\ 1 \leq l, t \leq n}} X(n, W, l, t) + 1$. □

5.5 Method for m-spectral CWVGs

We present the generalized method for m-spectral CWVGs.

Definition 13. Take an m-spectral CWVG with n agents. For any $1 \leq j, s \leq n$, $1 \leq W \leq w(N)$, and $\mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t} \in [n]^m$ let $X(j, W, s, \mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t})$ be the number of size-s *feasible* coalitions $S \subseteq \{1, \ldots, j\}$ of weight W, such that $\forall 1 \leq k \leq m$, $\mathbf{u}^k = \max_{i \in S} a_i^k$, $\mathbf{l}^k = \min_{i \in S} p_i^k$, $\mathbf{r}^k = \max_{i \in S} p_i^k$, and $\mathbf{t}^k = \min_{i \in S} b_i^k$.

Remark 2. $\mathbf{l}^k$ (or $\mathbf{r}^k$) is the k-dimension position of the most k-leftwing (resp. k-rightwing) member of S. $\mathbf{u}^k$ (or $\mathbf{t}^k$) is the k-dimension position of the most k-leftwing (resp.k-rightwing) agent tolerated by all members of S. Note that unless $\mathbf{u} \leq \mathbf{l} \leq \mathbf{r} \leq \mathbf{t}$, $X(j, W, s, \mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t}) = 0$.

Definition 13 satisfies the following recurrence relation.

THEOREM 5. *For all $2 \leq j, s \leq n$, $1 \leq W \leq w(N)$ and $\mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t} \in [n]^m$ s.t. $\mathbf{u} \leq \mathbf{l} \leq \mathbf{r} \leq \mathbf{t}$, $X(j, W, s, \mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t})$ satisfies the following recurrence relation over j:*
If $w_j \leq W$ and $\mathbf{l} \leq \mathbf{p_j} \leq \mathbf{r}$ and $\mathbf{a_j} \leq \mathbf{u}$ and $\mathbf{t} \leq \mathbf{b_j}$ then let

$$\begin{aligned} U &= \{\mathbf{u'} \in [n]^m : \forall k, \mathbf{u'}^k \leq \mathbf{u}^k \text{ with equality if } \mathbf{u}^k > a_j^k\} \\ L &= \{\mathbf{l'} \in [n]^m : \forall k, \mathbf{l'}^k \geq \mathbf{l}^k \text{ with equality if } \mathbf{l}^k < p_j^k\} \\ R &= \{\mathbf{r'} \in [n]^m : \forall k, \mathbf{r'}^k \leq \mathbf{r}^k \text{ with equality if } \mathbf{r}^k > p_j^k\} \\ T &= \{\mathbf{t'} \in [n]^m : \forall k, \mathbf{t'}^k \geq \mathbf{t}^k \text{ with equality if } \mathbf{t}^k < b_j^k\}, \end{aligned} \quad (17)$$

[6] As $X(n, W, l, t)$ is only defined for $W \geq 1$, it does not count the empty coalition. $\mathbb{1}_{w_i \geq q}$ counts whether agent i is critical when added to the empty coalition.

[7] The +1 counts the empty coalition, which is feasible.

then

$$X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) = X(j-1,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) + \sum_{\substack{\mathbf{u}'\in U,\mathbf{l}'\in L,\\ \mathbf{r}'\in R,\mathbf{t}'\in T}} X(j-1,W-w_j,s-1,\mathbf{u}',\mathbf{l}',\mathbf{r}',\mathbf{t}'). \quad (18)$$

Otherwise $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) = X(j-1,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$.

Remark 3. This recurrence relation is essentially similar to that in Theorem 3. The condition '$\mathbf{l} \le \mathbf{p_j} \le \mathbf{r}$ and $\mathbf{a_j} \le \mathbf{u}$ and $\mathbf{t} \le \mathbf{b_j}$' is necessary for the existence of coalitions in $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ that contain j. Just as in Theorem 3 it is important whether $t > b_j$ or $t = b_j$ to determine what values of t' we need to sum over, here we list the values of $\mathbf{u}',\mathbf{l}',\mathbf{r}',\mathbf{t}'$ to sum over in U,L,R,T. For example if all of the inequalities '$\mathbf{l} \le \mathbf{p_j} \le \mathbf{r}$ and $\mathbf{a_j} \le \mathbf{u}$ and $\mathbf{t} \le \mathbf{b_j}$' are strict, then $U = \{\mathbf{u}\}$, $L = \{\mathbf{l}\}$, $R = \{\mathbf{r}\}$, $T = \{\mathbf{t}\}$ and (18) becomes $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) = X(j-1,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) + X(j-1,W-w_j,s-1,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$. Complication only arises when one or more of the inequalities is an equality.

Proof. In this proof we will write '$S \in X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$' to mean 'S is counted by $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$'. First note that $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ is divided into coalitions that do not contain j (there are $X(j-1,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ of these) and those that do. If $w_j > W$ or $\mathbf{l} \not\le \mathbf{p_j}$ or $\mathbf{p_j} \not\le \mathbf{r}$ or $\mathbf{a_j} \not\le \mathbf{u}$ or $\mathbf{t} \not\le \mathbf{b_j}$, then $\forall S \in X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$, $j \notin S$. So $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) = X(j-1,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$. If $w_j \le W$ and $\mathbf{l} \le \mathbf{p_j} \le \mathbf{r}$ and $\mathbf{a_j} \le \mathbf{u}$ and $\mathbf{t} \le \mathbf{b_j}$ then define the function

$$f : \{S \in X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) : j \in S\} \to \bigcup_{\substack{\mathbf{u}'\in U,\mathbf{l}'\in L,\\ \mathbf{r}'\in R,\mathbf{t}'\in T}} X(j-1,W-w_j,s-1,\mathbf{u}',\mathbf{l}',\mathbf{r}',\mathbf{t}') \quad (19)$$

by $f(S) = S \setminus \{j\}$. We show this is a bijection. First, it is well-defined: $\forall S \in X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ s.t. $j \in S$, $S \setminus \{j\} \subseteq \{1,\dots,j-1\}$ is feasible with weight $W - w_j$ and size $s-1$. We have $\mathbf{a_j} \le \mathbf{u}$ so $\forall k$ either ${\mathbf{a_j}}^k < \mathbf{u}^k \Rightarrow \max_{i\in S\setminus\{j\}} {\mathbf{a_i}}^k = \mathbf{u}^k$, or ${\mathbf{a_j}}^k = \mathbf{u}^k \Rightarrow \max_{i\in S\setminus\{j\}} {\mathbf{a_i}}^k \le \mathbf{u}^k$. Therefore $\max_{i\in S\setminus\{j\}} \mathbf{a_i} \in U$. Similarly $\min_{i\in S\setminus\{j\}} \mathbf{b_i} \in T$, $\min_{i\in S\setminus\{j\}} \mathbf{p_i} \in L$ and $\max_{i\in S\setminus\{j\}} \mathbf{p_i} \in R$. Therefore $f(S)$ is in the codomain of f given in equation (19).

Secondly, f is clearly injective. Finally, we show f is surjective: for any $S' \in X(j-1,W-w_j,s-1,\mathbf{u}',\mathbf{l}',\mathbf{r}',\mathbf{t}')$ for some $\mathbf{u}' \in U, \mathbf{l}' \in L, \mathbf{r}' \in R, \mathbf{t}' \in T$, let $S = S' \cup \{j\}$. Every member of S' tolerates j ($\mathbf{u}' \le \mathbf{u} \le \mathbf{l} \le \mathbf{p_j} \le \mathbf{r} \le \mathbf{t} \le \mathbf{t}'$) and j tolerates every member of S' ($\mathbf{a_j} \le \mathbf{u} \le \mathbf{l} \le \mathbf{l}' \le \mathbf{r}' \le \mathbf{r} \le \mathbf{t} \le \mathbf{b_j}$) so S is feasible. $S \subseteq \{1,\dots,j\}$ is of weight W and size s. Now, $\max_{i\in S} {\mathbf{a_i}}^k = \max({\mathbf{u}'}^k, {\mathbf{a_j}}^k)$. As $\mathbf{u}' \in U$, $\mathbf{u}' \le \mathbf{u}$ and $\forall k$ either ${\mathbf{u}'}^k < \mathbf{u}^k \Rightarrow \mathbf{u}^k = {\mathbf{a_j}}^k$ so $\max({\mathbf{u}'}^k, {\mathbf{a_j}}^k) = \mathbf{u}^k$, or ${\mathbf{u}'}^k = \mathbf{u}^k$ so as ${\mathbf{a_j}}^k \le \mathbf{u}^k$, $\max({\mathbf{u}'}^k, {\mathbf{a_j}}^k) = \mathbf{u}^k$. So $\max_{i\in S} \mathbf{a_i} = \mathbf{u}$. Similarly $\min_{i\in S} \mathbf{b_i} = \mathbf{t}$, $\min_{i\in S} \mathbf{p_i} = \mathbf{l}$, and $\max_{i\in S} \mathbf{p_i} = \mathbf{r}$. So $S = S' \cup \{j\} \in X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ and $f(S) = S'$. This completes the proof that f is bijective.

To obtain equation (18), it remains to observe that the size of the codomain of f given in equation (19) is equal to the sum in equation (18). □

Theorem 6. *For any m-spectral CWVG with n agents, maximum weight w_{max}, we can calculate $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ for all $1 \le j,s \le n$, $1 \le W \le w(N)$ and $\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t} \in [n]^m$ in $O(2^{4m}n^{4m+3}w_{max})$ time.*

Proof. We have initial values

$$X(1,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) = \mathbb{1}_{(W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})=(w_1,1,\mathbf{a_1},\mathbf{p_1},\mathbf{p_1},\mathbf{b_1})} \quad (20)$$

for all $1 \le W \le w(N)$, $1 \le s \le n$ and $\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t} \in [n]^m$, and

$$X(j,W,1,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t}) = \mathbb{1}_{\exists i\le j:(W,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})=(w_i,\mathbf{a_i},\mathbf{p_i},\mathbf{p_i},\mathbf{b_i})} \quad (21)$$

for all $1 \le j \le n$, $1 \le W \le w(N)$, and $\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t} \in [n]^m$. We use these and the recurrence relation in Theorem 5 to compute the required values of $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$.

For any particular values of $(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$, if $w_j > W$ or $\mathbf{l} \not\le \mathbf{p_j}$ or $\mathbf{p_j} \not\le \mathbf{r}$ or $\mathbf{a_j} \not\le \mathbf{u}$ or $\mathbf{t} \not\le \mathbf{b_j}$ then $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ is calculated in $O(1)$ time. Otherwise, $X(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ is calculated in $O(n^\alpha)$ time, where $0 \le \alpha \le 4m$ is the number of equalities in '$\mathbf{l} \le \mathbf{p_j} \le \mathbf{r}$ and $\mathbf{a_j} \le \mathbf{u}$ and $\mathbf{t} \le \mathbf{b_j}$'. Now, $\forall 0 \le \alpha \le 4m$, the number of values of $(j,W,s,\mathbf{u},\mathbf{l},\mathbf{r},\mathbf{t})$ such that there are α equalities is $O(\binom{4m}{\alpha}n^{4m+3-\alpha}w_{\max})$. So $\forall 0 \le \alpha \le 4m$ we compute $O(\binom{4m}{\alpha}n^{4m+3-\alpha}w_{\max})$ values of X in $O(n^\alpha)$ time each. So the total complexity is $O\left(\sum_{\alpha=0}^{4m}\binom{4m}{\alpha}n^{4m+3}w_{\max}\right) = O(2^{4m}n^{4m+3}w_{\max})$. □

Corollary 5. *For any m-spectral CWVG with n agents, for any $1 \le i \le n$ we can calculate the raw Banzhaf, normalized Banzhaf, probabilistic Banzhaf, and Shapley-Shubik indices of agent i in $O(2^{4m}n^{4m+2}w_{max})$, $O(2^{4m}n^{4m+3}w_{max})$, $O(2^{4m}n^{4m+2}w_{max})$ and $O(2^{4m}n^{4m+4}w_{max})$ time respectively.*

Proof. The proof is the same as for Corollaries 3 and 4, by using Theorem 6 and omitting the variable s when calculating the Banzhaf indices. □

The complexities in Corollary 5 do not equal those in Corollaries 3 and 4 when $m = 1$, as in the 1-dimensional case we streamline the algorithm by ordering the agents from left to right, which can only be done for at most one dimension in the m-dimensional case (and results in a messy algorithm). Corollary 5 shows that for m-spectral CWVGs, we can calculate each of the power indices in $O(p(n)w_{\max})$ time where $p(n)$ is a polynomial that depends on m. We anticipate that most applications would only require a few dimensions, (for example a political model would probably need no more than five) and so consider Corollary 5 to be very useful.

6. APPROXIMATING POWER INDICES

The results in the previous section are useful only for sufficiently small weights, so we also examine *approximating* power indices. For WVGs they have been approximated by Monte Carlo methods [2, 19]. To approximate the probabilistic Banzhaf index $\beta_i'(v)$, samples are taken from the space of coalitions $S \subseteq N \setminus \{i\}$ and we test whether i is critical. Similarly, to approximate the Shapley-Shubik index $\phi_i(v)$ samples are taken from the space of permutations Π and we test whether i is critical. [2] determines how many samples are needed to construct a confidence interval of desired accuracy and confidence level. Unfortunately it seems that in order to approximate the probabilistic Banzhaf index $\beta_i'(G,v) = 2\eta_i(G,v)/|F|$ for CWVGs, we must sample uniformly from the space F of feasible coalitions. This is hard: for arbitrary compatibility graphs, uniformly generating feasible coalitions (cliques) cannot be done in polynomial time unless RP=NP (Theorem 1.17 of [27]). However we show that for spectral CWVGs, feasible coalitions *can* be generated efficiently. We do this by transforming our counting algorithm from the previous section into a uniform generation algorithm, using a technique from [18] (Theorem 3.3).

THEOREM 7. *For any m-spectral CWVG with n agents, there exists an algorithm to uniformly generate feasible coalitions of the game, which has complexity $O(2^{4m}n^{4m+1})$ for the pre-procedure, and then $O(n^{4m+1})$ per sample.*

PROOF. We wish to count feasible coalitions of any weight and size, so we may simply consider $X(j, \mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t})$ = the number of non-empty feasible subsets of $\{1, \ldots, j\}$ such that $\forall 1 \le k \le m$, $\mathbf{u}^k = \max_{i \in S} a_i^k$, $\mathbf{l}^k = \min_{i \in S} p_i^k$, $\mathbf{r}^k = \max_{i \in S} p_i^k$, and $\mathbf{t}^k = \min_{i \in S} b_i^k$. By Theorem 6, we can calculate $X(j, \mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t})$ for all $1 \le j \le n$, $\mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t} \in [n]^m$ in $O(2^{4m}n^{4m+1})$ time by omitting the variables W and s.

Having pre-computed these values, we can construct an algorithm **Count**, that takes as input a function $f : \{x+1, \ldots, n\} \to \{0, 1\}$, for some $1 \le x \le n-1$, such that $f^{-1}(1)$ is a feasible coalition, and returns the number of feasible coalitions $S \subseteq N$ such that S is an extension of f (that is, $\forall i \in \{x+1, \ldots, n\}, i \in S \Leftrightarrow f(i) = 1$). This is equal to the number of feasible coalitions $S \subseteq \{1, \ldots, x\}$ such that $S \cup f^{-1}(1)$ is feasible. So, letting $\mathbf{u} = \min_{i \in f^{-1}(1)} \mathbf{p_i}$, $\mathbf{l} = \max_{i \in f^{-1}(1)} \mathbf{a_i}$, $\mathbf{r} = \min_{i \in f^{-1}(1)} \mathbf{b_i}$, and $\mathbf{t} = \max_{i \in f^{-1}(1)} \mathbf{p_i}$ (the minima and maxima are taken coordinatewise), we have[8]

$$\mathbf{Count}(f) = \sum_{\substack{\mathbf{u}' \le \mathbf{u}, \mathbf{l}' \ge \mathbf{l}, \\ \mathbf{r}' \le \mathbf{r}, \mathbf{t}' \ge \mathbf{t}}} X(x, \mathbf{u}', \mathbf{l}', \mathbf{r}', \mathbf{t}') + 1. \qquad (22)$$

Having pre-computed the $X(j, \mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t})$ values, computing **Count** has complexity $O(n^{4m})$. Next, we use **Count** to construct a recursive algorithm **UGen** that takes as input a function $f : \{x+1, \ldots, n\} \to \{0, 1\}$, for some $0 \le x \le n$, (with $x = n$ meaning $f = \emptyset$) such that $f^{-1}(1)$ is a feasible coalition, and *uniformly generates* a feasible coalition $S \subseteq N$ such that S is an extension of f. Having pre-computed the

Algorithm UGen(f)

```
if domain(f) = N then
  return f^{-1}(1)
else
  f_0 ← (x, 0) ∪ f; f_1 ← (x, 1) ∪ f
  if f_1^{-1}(1) is feasible then
    N_0 ← Count(f_0); N_1 ← Count(f_1)
    return { UGen(f_0),  with probability N_0/(N_0 + N_1)
           { UGen(f_1),  with probability N_1/(N_0 + N_1)
  else
    return UGen(f_0)
  end if
end if
```

$X(j, \mathbf{u}, \mathbf{l}, \mathbf{r}, \mathbf{t})$, calling **UGen**($\emptyset$) (which uniformly generates a member of F) has time complexity $O(n^{4m+1})$. $\square$

We may now give a result for constructing a confidence interval for the probabilistic Banzhaf power index that corresponds to a similar result for WVGs (Theorem 3, [2]).

THEOREM 8. *For any m-spectral CWVG with n agents, any $1 \le i \le n$, required accuracy $\epsilon > 0$ and confidence level $1-\delta$, we can construct a confidence interval with width 2ϵ of the form $[\hat{\beta}_i' - \epsilon, \hat{\beta}_i' + \epsilon]$ which contains the correct probabilistic Banzhaf index $\beta_i'(G, v)$ with probability at least $1 - \delta$. The required number of samples is $k = \ln \frac{2}{\delta} / 2\epsilon^2$. After a pre-procedure with complexity $O(2^{4m}n^{4m+1})$, each sample can be generated in $O(n^{4m+1})$ time.*

PROOF. For each sample $S \in F$, we check whether agent i is critical. Let $X_1, \ldots, X_k$ be the k Bernoulli trials corresponding to the k samples, so $X_j = 1$ iff i is critical in the j^{th} trial, and 0 otherwise. Let $\hat{\beta}_i' = (\sum_{j=1}^k X_j)/k$. To show that we achieve the required accuracy and confidence, we follow the argument given in [2], which uses *Hoeffding's inequality*. We find that if $k \ge \ln \frac{2}{\delta}/2\epsilon^2$ then $\mathbb{P}(|\hat{\beta}_i' - \beta_i'(G, v)| \ge \epsilon) \le \delta$. The complexity results follow from Theorem 7. $\square$

Theorem 8 shows that for m-spectral CWVGs we can build a confidence interval for $\beta_i'(G, v)$ with accuracy ϵ and confidence $1 - \delta$ in time $O(p(n)\ln(1/\delta)/\epsilon^2)$, where $p(n)$ is a polynomial that depends on m. Note that Theorem 8 does not require bounds on the weights (unlike our results for computation). Unfortunately this technique cannot easily be applied to approximating the Shapley-Shubik index for CWVGs: to do so we must sample uniformly from E (i.e. count E); whereas F can be counted efficiently using Theorem 6 and omitting the variable W, to count E we do need information about weight, and so this cannot be done.

7. SIMULATIONS

We now present the results of simulations demonstrating the relationship between compatibility and power in some typical settings. Figure 1 shows the relationship between an agent's power and its degree in the compatibility graph. It was produced by 10,000 tests. For each test, a random Erdős-Rényi graph with 10 vertices and edge probability 0.8 was generated, each agent's weight was independently generated uniformly between 1 and 20, and the quota was taken to be $w(N)/2$. The normalized Banzhaf index was computed for all agents, and categorized by degree. Figure 1 shows, for each degree, the mean normalized Banzhaf index for agents of that degree. We see that for agents of high degree, power decreases sharply with each collaborator lost. The plot for the Shapley-Shubik index is similar. Figures 2 and 3 show how an agent's power is affected by its position and the width of its tolerance interval,in a 1-spectral CWVG with 11 agents. They were produced by 10,000 tests. For each test, each agent's weight was independently generated uniformly between 1 and 20 then the quota was taken to be $w(N)/5$. For each agent, the width of the tolerance interval $b_i - a_i$ was independently generated uniformly between 0 and 10,then the positioning of the interval was chosen uniformly from the possible choices.The normalized Banzhaf index was computed for all agents, and categorized by both position and width of tolerance interval. Figures 2 and 3 show the mean normalized Banzhaf index for agents of given position or tolerance interval width. Figure 2 confirms that central agents have more power, and indicates that small shifts in position near the centre do not greatly affect power. Figure 3 shows that increasing one's tolerance interval increases one's power, and that the increase is greater for smaller widths.

8. CONCLUSIONS AND FURTHER WORK

Compatibility Weighted Voting Games are a natural generalization of Weighted Voting Games, for which we have defined power indices that are computationally tractable under certain well-motivated restrictions. See the rightmost

[8] '+1' counts the empty coalition

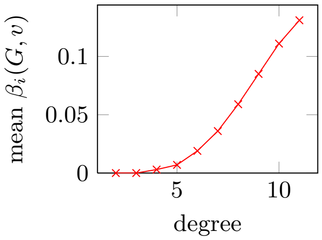

Figure 1 Degree vs power

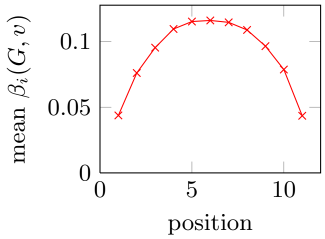

Figure 2 Position vs power

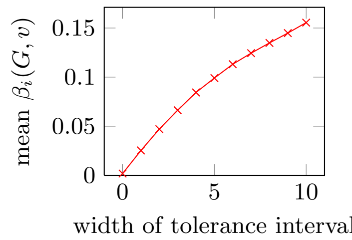

Figure 3 Tol. interval width vs power

Table 1 Summary of complexities for computing the Banzhaf and Shapley-Shubik power indices

Restrictions	WVGs	CWVGs
None	not POLY(n) unless P=NP	not POLY(n) unless P=NP
Weights are poly. in n	POLY(n)	not POLY(n) unless P=NP
Weights are poly. in n **and** game is m-spectral, fixed m	N/A	POLY(n)

column of Table 1 for a summary of our results regarding computation, and note that unlike those results, our approximation result for the Banzhaf index does *not* rely on bounds on the weights. We have shown that these power indices are strongly affected by restrictions on compatibility, i.e. that the cost of principles is great. Further work could take several directions. Although we have devised an efficient approximation scheme for the probabilistic Banzhaf index, it remains to be seen whether a similar scheme for the Shapley-Shubik index exists. Other power indices such as the Deegan-Packel index [12] could be generalized for CWVGs and their properties analyzed. Lastly our concept of compatibility and the associated power indices could easily be applied to all simple monotonic games.

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