The Control Complexity of r-Approval: from the Single-Peaked Case to the General Case

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ABSTRACT
A natural generalization of the single-peaked elections is the \( k \)-peaked elections, where at most \( k \) peaks are allowed in each vote. Motivated by \( \mathcal{NP} \)-hardness in general and polynomial-time solvability in single-peaked elections, we aim at establishing a complexity dichotomy of several control problems for \( r \)-approval voting in \( k \)-peaked elections with respect to \( k \). It turns out that most \( \mathcal{NP} \)-completeness results in general also hold in \( k \)-peaked elections, even for \( k = 2, 3 \). On the other hand, we derive polynomial-time algorithms for certain control problems for \( k = 2 \). In addition, we also study the problems from the viewpoint of parameterized complexity and achieve both \( \mathcal{FPT} \) and \( \mathcal{W} \)-hardness results. Several of our results apply to approval voting and sincere-strategy preference-based approval voting as well.

Categories and Subject Descriptors
F.2 [Theory of Computation]: Analysis of Algorithms and Problem Complexity; G.2.1 [Combinatorics]: Combinatorial algorithms; J.4 [Computer Applications]: Social Choice and Behavioral Sciences

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single-peaked; approval voting; control; parameterized complexity; kernel

1. INTRODUCTION
Voting is a common method for preference aggregation and collective decision-making, and has applications in political elections, multi-agent systems, web spam reduction, etc. However, by the Arrow’s impossibility theorem [1], there is no voting system which satisfies a certain set of desirable criteria (see [1] for the details) when more than two candidates are involved. One possible way to bypass the Arrow’s impossibility theorem is to restrict the domain of the preferences, for instance, the single-peaked domain introduced by Black [4]. Intuitively, in a single-peaked election, one can order the candidates from left to right such that every voter’s preferences increase first and then decrease after some point as the candidates are considered from left to right.

Recently, the complexity of various voting problems in single-peaked elections has been attracting attention of many researchers from both theoretical computer science and social choice communities [6, 13, 15, 16, 23]. It turned out that many voting problems being \( \mathcal{NP} \)-hard in general become polynomial-time solvable when restricted to single-peaked elections. However, most elections in practice are not purely single-peaked, which motivates researchers to study more general models of elections. We refer readers to [7, 8, 9, 11, 14] for some variations of single-peaked model. In this paper, we consider a natural generalization of single-peaked elections, where more than one peak may occur in each vote. This generalization might be relevant for many real-world applications. For example, consider a group of people who are willing to select a special day for an event. In this setting, each voter may have several special days which he/she prefers for some reason, and the longer the other days away from these favorite days, the less they are preferred by the voter. We call this generalization \( k \)-peaked elections.

In this paper, we mainly study control problems for \( r \)-approval voting restricted to \( k \)-peaked elections. In a control attack, there is an external agent (e.g., the chairman in an election) who is willing to influence the results of the election by doing some tricks. There could be two goals that the external agent wants to reach. One goal is to make some distinguished candidate win the election. The other goal could be to make someone lose the election. The former case is called a constructive control and the latter case is called a destructive control. Moreover, the tricks involved in a control attack include adding some new, unregistered votes to the registered votes, deleting votes from the registered votes, adding new candidates to the election or deleting candidates from the election. We refer readers to [12, 18, 19] for further information on control attacks.

Approval voting is one of the most famous voting systems and has been extensively studied both in theory and in practice. In an approval voting, we are given a set \( \mathcal{C} \) of candidates and a set \( \mathcal{V} \) of voters. Each voter approves or disapproves every candidate \( c \in \mathcal{C} \). The system selects a candidate who is approved by the most voters as a winner. An \( r \)-approval voting is a variant of approval voting, where each voter \( v \) casts a vote \( \pi_v \), defined as a linear order (a transitive, antisymmetric, and total binary relationship) over the candidates, and approves the candidates ordered in the top \( r \) positions in \( \pi_v \) and disapproves all other candidates. Particularly, 1-approval is also called plurality.

Another prominent variant related to approval voting is the sincere-strategy preference-based approval voting (SP-AV for...
short), proposed by Brams and Sanver [5]. In an SP-AV election, each voter provides both a linear order of the candidates and a subset $C$ of candidates such that the candidates are approved according to $C$, and the “admissible” and “sincere” properties should be fulfilled (see [5, 12] for the precise definition).

In the following, we consider only constructive control. Hemaspaandra et al. [18] proved that the control problems by adding/deleting votes for approval voting are $NP$-complete. The proofs can be adapted to show the $NP$-completeness of control by adding/deleting votes in $SP-AV$ [12]. Lin [21] proved that control by adding votes in 4-approval and control by deleting votes in 3-approval are both $NP$-complete, while control by adding votes in 3-approval and control by deleting votes in 2-approval are polynomial-time solvable. As for the control by modification of candidates, approval voting turned out to be immune\(^1\) to control by adding candidates and polynomial-time solvable for control by deleting candidates [18]. However, the control problems by adding/deleting candidates are $NP$-complete for $r$-approval, even when degenerated to 1-approval [19]. The $NP$-completeness also holds for control by adding/deleting candidates in $SP-AV$ [12]. Recently, control in approval voting and $r$-approval voting have also been considered with respect to single-peaked elections. Faliszewski et al. [15] proved that the control problems by adding/deleting votes in approval are polynomial-time solvable in single-peaked elections\(^2\). Moreover, the control problems by adding/deleting candidates for 1-approval are polynomial-time solvable in single-peaked elections [15].

Motivated by the $NP$-completeness in the general case and the polynomial-time solvability in the single-peaked case, we study the complexity of control problems for $r$-approval voting in $k$-peaked elections with respect to various values of $k$, aiming at exploring the complexity border for these control problems. Faliszewski et al. [14] studied a nearly single-peaked model which is called Swoon-SP and can be considered as a special case of 2-peaked elections. They proved that the control problems by adding/deleting candidates for 1-approval are $NP$-complete when restricted to Swoon-SP elections, implying the $NP$-completeness of these problems in 2-peaked elections. We complement their results by studying the adding/deleting votes case. Our findings are summarized in Table 1. In particular, we show that, control by adding votes in $r$-approval with $r$ being a constant is polynomial-time solvable in 2-peaked elections, but $NP$-complete in $k$-peaked elections for $k \geq 3$. Meanwhile, if $r$ is not a constant, then control by adding votes in $r$-approval in 2-peaked elections becomes $NP$-complete. Moreover, the deleting votes case turns out to be $NP$-complete for $k$-peaked elections with $k \geq 2$, even for $r$ being a constant.

In addition, we present some results for $r$-approval control problems with respect to their parameterized complexity. Recently, many voting problems have been studied from the perspective of parameterized complexity. See [2] for an overview. A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where $\Sigma$ is a finite alphabet. The first component $I \in \Sigma$ is called the main part of the problem while the second component $\kappa \in \mathbb{N}$ is called the parameter. Downey and Fellows [10] established the parameterized complexity theory, where the class $FPT$ (stands for fixed-parameter tractable) includes all parameterized problems which admit $O(f(\kappa) \cdot |I|^{O(1)})$-time algorithms. Here $f(\kappa)$ is a computable function. Another important parameterized complexity class is $W[1]$ which is the basic class for showing fixed-parameter intractability results. A problem is $W[1]$-hard if all problems in $W[1]$ are $FPT$-reducible to the problem. We can show a problem being $W[1]$-hard by giving an $FPT$-reduction from another $W[1]$-hard problem.

Given two parameterized problems $Q$ and $Q'$, an $FPT$-reduction from $Q$ to $Q'$ is an algorithm that takes as input an instance $(I, \kappa)$ of $Q$ and outputs an instance $(I', \kappa')$ of $Q'$ such that

1. the algorithm runs in $f(\kappa) \cdot |I|^{O(1)}$ time, where $f$ is a computable function in $\kappa$;
2. $(I, \kappa) \in Q$ if and only if $(I', \kappa') \in Q'$; and
3. $\kappa' \leq g(\kappa)$, where $g$ is a computable function in $\kappa$.

Liu et al. [22] proved that control by adding votes in approval voting is $W[1]$-hard and control by deleting votes in approval voting is $W[2]$-hard\(^3\), with the numbers of added and deleted votes as parameters, respectively. In addition, they proved that control by adding candidates in 1-approval is $W[2]$-hard, with the number of added candidates as the parameter. Betzler and Uhlmann [3] complemented the results in [22] by proving that control by deleting candidates in 1-approval is $W[2]$-hard, with the number of deleted candidates as the parameter.

We extend the above results to $k$-peaked elections by showing that control by deleting candidates in 1-approval restricted to 3-peaked elections is $W[1]$-hard with the number of deleted candidates as the parameter. For the general elections, we present two $FPT$ results for control by adding/deleting votes in $r$-approval voting, with $r$ being a constant. Here, the parameter is the number of votes added/deleted.

Remarks: All our results apply to both unique-winner and nonunique-winner models. For the sake of clarity, our proofs and algorithms are solely based on the unique-winner model. The corresponding results for the nonunique-winner model can be derived in the similar way. Our $NP$-completeness results hold for approval voting and SP-AV as well. For $NP$-completeness results, we present only the $NP$-hardness proofs, since all problems studied are clearly in $NP$.

2. PRELIMINARIES

Multisets. A multiset $S = \{s_1, s_2, \ldots, s_{|S|}\}$ is a generalization of a set where objects are allowed to appear more than once, that is, $s_i = s_j$ is allowed for $i \neq j$. An element of $S$ is one copy of some object. We use $s \in S$ to denote that $s$ is an element of $S$. The size of $S$, denoted by $|S|$, is the number of elements in $S$. For two multisets $A$ and $B$, we use $A \uplus B$ to denote the multiset containing all elements from $A$ and $B$, and use $A \ominus B$ to denote the multiset containing, for each object $s$, $\max\{0, n_1 - n_2\}$ copies of $s$, where $n_1$ and $n_2$ denote the numbers of copies of $s$ in $A$ and $B$, respectively. For example, for $A = \{1, 1, 1, 2, 3, 3, 4\}$ and $B = \{1, 2, 3\}$.

\(^1\)A voting system is immune to a control behavior if one cannot make a candidate who is not a winner become a final winner by imposing the strategic behavior on the election.

\(^2\)In [15], for the approval voting, an election is single-peaked if there is an order of the candidates such that each voter’s approved candidates are contiguous within the order.

Table 1: A summary of the complexity of r-approval control problems. Our new results are highlighted with underlines. “Thm. #” means that the result follows from Theorem # in this paper. The two FPT results are with respect to the number of added/deleted votes as parameters, respectively. Results marked by ♦ are from [21], by ♣ from [19], by ♠ from [15] and by ∆ from [14].

A ∪ B = \{1, 1, 1, 1, 2, 2, 3, 3, 4\} and A ∩ B = \{1, 1, 3, 4\}. A multiset B is a submultiset of a multiset A, if, for every object s that occurs n times in A, B contains at least n copies of s. We use B ⊆ A to denote that B is a submultiset of A.

r-Approval. An r-approval election can be specified by a set Π of candidates, a set V of voters where every v ∈ V casts a vote πv which is defined as a linear order over the candidates Π. Each voter v contributes 1 point to each of the candidates ordered in the top r positions in πv, and contributes 0 points to the other candidates. For convenience, we also use >v to denote πv. In a vote >v and a candidate c, the position of c in >v is defined as |{c’ ∈ Π | c’ >v c}| + 1, where >v ⊆ Π means that c is ordered before c in >v. The multiset of votes casted by V is denoted by ΠV. A winner is a candidate who gets the highest total score. If there is only one winner, we call it a unique winner.

A linear order π in Π is single-peaked if there is a linear order L of Π such that for every a > b in ΠL and every three candidates a, b, c ∈ Π with a > b > c or c > b > a, where a > b means a is ordered before b in L. The candidate ordered in the first position of >v is the peak of >v with respect to L.

For an order L = (c1, c2, ..., cm) of Π and a vote πv, we say πv is k-peaked with respect to L if there is a k-partition L1 = (c1, c2, ..., cki), L2 = (ck+i+1, ck+i+2, ..., cki+j), ..., Lk = (c1, c1+i, ..., cm) of Π such that πv(C(Li)) is single-peaked with respect to Li for all 1 ≤ i ≤ k, where C(Li) is the set of candidates appearing in Li. See Fig. 1 for an example.

An election is k-peaked if there is an order L of Π such that every vote in the election is k-peaked with respect to L. Here L is called a k-harmonious order.

Problem definitions. The problems studied in this paper are defined as follows. Throughout this paper, we fix p as the distinguished candidate.

r-Approval Control by Adding Votes in k-Peaked Elections (r-AV-k)
Input: An r-approval election (∏ ∪ {p}, ΠV) with a multiset ΠT of unregistered votes, where both ΠV and ΠT are k-peaked with respect to a given k-harmonious order L, and an integer 0 ≤ R ≤ |Πr|.
Question: Are there at most R votes ΠT in ΠV such that p is the unique winner/a winner in (∏ ∪ {p}, ΠV ⊖ ΠT)?

3. 2-Peaked Elections
In this section, we study the control problems for r-approval in 2-peaked elections. The following three theorems summarize our findings.

Theorem 1. r-AV-2 is polynomial-time solvable for every constant r.
Recall that r-AV is \( \mathcal{NP} \)-complete for every constant \( r \geq 4 \) but polynomial-time solvable when restricted to single-peaked elections [15]. Theorem 1 shows that the polynomial-time solvability of r-AV remains when extending from single-peaked elections to 2-peaked elections, for \( r \) being a constant. This bound is tight as indicated by the following theorem. More precisely, if \( r \) is not a constant, r-AV becomes \( \mathcal{NP} \)-complete in 2-peaked elections, in contrast to the polynomial-time solvability in the single-peaked case [15].

**Theorem 2.** r-AV-2 is \( \mathcal{NP} \)-complete, if \( r \) is not a constant.

The control problem by deleting votes for r-approval is polynomial-time solvable in single-peaked elections for even non-constant \( r \) [15]. The following theorem shows that by increasing the number of peaks only by one, this problem becomes \( \mathcal{NP} \)-complete.

**Theorem 3.** r-DV-2 is \( \mathcal{NP} \)-complete for every constant \( r \geq 3 \).

### 3.1 Proof of Theorem 1

We prove Theorem 1 by giving a polynomial-time algorithm based on dynamic programming.

Let \( ((C \cup \{p\}, \Pi_v), \Pi_r, L, R) \) be an instance of r-AV-2. For \( c \in C \), let \( \overline{c}(1) \) be the candidate lying immediately before \( c \) in \( L \) and \( \overline{c}(i) \) be the candidate lying immediately before \( \overline{c}(i-1) \) in \( L \). Similarly, we use \( \overline{c}(1) \) and \( \overline{c}(i) \) to denote the candidates lying immediately after \( c \) and \( \overline{c}(i-1) \), respectively. For example, if \( L = (a, b, c, d, e, f, g, h) \), then \( \overline{d}(1) = e, \overline{d}(4) = h, \overline{d}(6) = c \) and \( \overline{d}(3) = a \).

For a vote \( \pi_v \), let \( \Pi_v(\pi_v) \) denote the set of candidates who get 1 point and 0(v) denote the set of candidates who get 0 points, from \( \pi_v \). For a candidate \( c \), let \( SC_V(c) \) be the total score of \( c \) from \( \Pi_v \), that is, \( SC_V(c) = \{|\pi_v \in \Pi_v | e \in \Pi_v(\pi_v)\}\).

Given an order \( A = (a_1, a_2, \ldots, a_n) \), a discrete interval \( I \) over \( A \) is a consecutive sub-order \((a_i, a_{i+1}, \ldots, a_{i+k})\) of \( A \). We denote the first element of \( \Pi_v(\pi_v) \) by \( l(I) \) and the last element of \( \Pi_v(\pi_v) \) by \( r(I) \). We also use \( A(l(I), r(I)) \) to denote \( I \).

Let \( S(I) \) denote the set of elements appearing in \( I \) and set \( |I| := |S(I)| \). For example, for a discrete interval \( I = A(3, 6) \) over the order \( A = (2, 5, 3, 10, 4, 6, 0) \), \( S(I) = \{3, 4, 6\} \). A k-discrete interval over an order \( A \) is a collection of \( k \) disjoint discrete intervals over \( A \), where “disjoint” means no element in \( A \) appears in more than one discrete interval. For a k-discrete interval \( I \), let \( S(I) = \bigcup_{t \in T} S(I) \).

**Observation 1.** For each k-peaked election \( (C, \Pi_v) \) associated with a k-harmonious order \( L \) over \( C \), and each vote \( \pi_v \in \Pi_v \), there is a k'-discrete interval \( I \) over \( L \) such that \( 0 < k' \leq k \) and \( I(\pi_v) \in S(I) \).

By Observation 1, for every vote \( \pi_v \) in a 2-peaked election associated with \( L \) as a 2-harmonious order, \( 1(v) \) can be represented by a 2-discrete interval or a 1-discrete interval over \( L \). See Fig. 2 for an example.

We first derive a polynomial-time algorithm for 4-AV-2. It is easy to generalize the algorithm to r-AV-2 with \( r \) being a constant. The following observation is trivial.

**Observation 2.** Every true-instance of r-AV has a solution where each added vote approves \( p \).

Due to Observation 2, we can safely assume that for each \( \pi_v \in \Pi_r \), \( p \in 1(v) \). By Observation 1, for every vote \( \pi_v \in \Pi_r \), \( 1(v) \) can be represented by a 2-discrete interval or a 1-discrete interval over \( L \).

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**Figure 2:** This figure shows two votes \( \pi_v = (c_3, c_4, c_7, c_6, c_8, c_9, c_5, c_2, c_1) \) and \( \pi_v = (c_7, c_6, c_5, c_8, c_9, c_{10}, c_1, c_4, c_3, c_2) \). Each vote gives one point to its top four ordered candidates. \( 1(v) \) is represented by a 2-discrete interval \( \{I_p = (c_3, c_4, c_7, c_6)\} \) and \( 1(v) \) is represented by a 1-discrete interval \( \{I_p = (c_3, c_6, c_7, c_8)\} \).

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**Figure 3:** Illustration of (4) and (5) in the definition of \( DT \).

It is easy to see that the given instance is a true-instance if there is a \( DT(n, R', s, s_1, s_2, \ldots, s_6, s_0) \), \( 0 \leq k' \leq s \) for all \( s \in \{s_1, s_2, \ldots, s_6, s_0\} \). Therefore, to solve the problem we need to calculate the values of \( DT(i, j, s_1, s_2, \ldots, s_6, s_0) \) for all \( 1 \leq j \leq R' \), \( j \leq s \leq k' \leq |I||I| - 1 \) and \( s' \leq s \) for all \( s' \in \{s_1, s_2, \ldots, s_6, s_0\} \). Thus, we have at most \( |I||I| \cdot R'(\sqrt{|I|} + R')^{10} \) entries to calculate.

We use the following iterative recurrence to update the table. \( DT(i, j, s, s_1, s_2, \ldots, s_6, s_0) \) is calculated as follows:

- For each vote \( \pi_v \in \Pi_r \), \( p \in 1(v) \) can be represented by a 2-discrete interval \( I_p = (c_3, c_4, c_7, c_6) \) or a 1-discrete interval \( I_p = (c_3, c_6) \) with \( p \in S(1(v)) \). Let \( S \) be the set of all votes \( \pi_v \in \Pi_r \) where \( 1(v) \) is represented by a 1-discrete interval over \( L \). We say two votes have the same type if they approve the same candidates. Since every vote approves exactly four candidates, \( S \) has at most four different types:
- Type 1: votes approving \( (3, 2, 1, 7) \) \( \{p, p, p, p\} \) (and \( p \) votes approving \( (2, 1, 7, 2) \) \( \{p, p, p, p\} \);
- Type 2: votes approving \( (1, 7, 2, 2) \) \( \{p, p, p, p\} \);
- Type 3: votes approving \( (7, 2, 1, 2) \) \( \{p, p, p, p\} \);
- Type 4: votes approving \( (7, 2, 2, 2) \) \( \{p, p, p, p\} \).

We then can all possibilities of how many votes in the solution are from each of the four types of votes in \( S \). This reduces the original instance to at most \( 4^4 \) sub-instances. Thus, in the following, we assume that every vote in \( \Pi_r \) is represented by a 2-discrete interval.

**Dynamic Programming Algorithm:**

To solve the problem for a true-instance, we construct a 4-dimensional table \( DT(i, j, s_1, s_2, \ldots, s_6, s_0) \) for all \( 1 \leq i \leq j \leq R' \).

**Dynamic Programming Algorithm:**

Our dynamic programming algorithm uses a binary dynamic table \( DT(i, j, s_1, s_2, \ldots, s_6, s_0) \), where we set \( DT(i, j, s_1, s_2, \ldots, s_6, s_0) = 1 \) if there is a submultiset \( \Pi_{r,s} \subseteq \{\pi_{v_1}, \pi_{v_2}, \ldots, \pi_{v_t}\} \) satisfying

1. \( \Pi_{r,s} = \{j\} \);
2. \( \pi_{v_1} \in \Pi_r \); 3. \( \max\{SC_{V,T}(c) | c \in C\} = s \);
4. \( SC_{V,T}(c_1) \leq s_1 \) for all \( 1 \leq t \leq 6 \), where \( c_3 = c = c_5 = c_6 = c_7 = c_8 = 0 \);
5. \( SC_{V,T}(c_1) \leq s_1 \) for all \( t \in \{1, 2, 3\} \), where \( c_1 = \frac{c_5}{c_7} = \frac{c_6}{c_8} \) and \( c_3 = \frac{c_5}{c_8} \).

(See Fig. 3 for an illustration of (4) and (5)).
Case 1. $\exists DT(i_1, j - 1, s, s_1, s_2, \ldots, s_6, s_{i_1, 1}, s_{i_1, 2}, s_{i_1, 3}) = 1$ such that conditions (1)-(4) hold.

Case 2. $\exists s' \in \{s_1, s_2, \ldots, s_6, s_{i_1, 1}, s_{i_1, 2}, s_{i_1, 3}\}$ with $s' = s$ and $\exists DT(i_1, j - 1, s, s_1, s_2, \ldots, s_6, s_{i_1, 1}, s_{i_1, 2}, s_{i_1, 3}) = 1$ such that conditions (1)-(4) hold.

The four conditions are:
1. $j - 1 \leq i_1 \leq i - 1$.
2. $s_1 = s' + SC_{\{v_j\}}(c_i)$ for all $1 \leq t \leq 6$.
3. $s_{i_1, 1} = s_{i_1, 2} + SC_{\{v_j\}}(c_{i_1})$ for all $c_{i_1, 1} = c_{i_1, 2}$, and
4. $s_{i_1, t} = SC_{\{v_{j-1}\}}(c_{i_1})$ for all $c_{i_1, i} \in \{(r_{j-1}^d), (r_{j-1}^c)(1), (r_{j-1}^c)(2)\}$.

The algorithm is easy to be generalized for $r \geq 4$ by using a bigger but still polynomial-sized dynamic table.

3.2 Proof of Theorem 2

We prove Theorem 2 by a reduction from a variant of Independent Set (IS) that is \textit{NP}-complete [20].

Let $\{\}$ denote an empty order containing no element.

For a linear order $A = (a_1, a_2, \ldots, a_n)$, let $A[a_1, a_2]$ (resp. $A[a_1, a_2, a_3]$) with $i \leq j$ be the sub-order $(a_i, a_{i+1}, \ldots, a_j)$ (resp. $(a_i, a_{i+1}, \ldots, a_j)$) if $i < j$ and ( ) if $i = j$, $(a_i, a_{i+1}, \ldots, a_j)$ if $i < j$ and ( ) if $i = j$, and $(a_i, a_{i+1}, \ldots, a_j)$ if $i < j$ and ( ) if $j \geq i - 1$, and let $A[a_1, a_2]$ (resp. $A[a_1, a_2, a_3]$) and $A[a_1, a_2]$ (resp. $A[a_1, a_2]$) be the reverse order of $A[a_1, a_2]$ (resp. $A[a_1, a_2]$), $A[a_1, a_2]$ and $A[a_1, a_2]$.

For two linear orders $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_m)$ with $A \cap B = \emptyset$, denote by $(B, A)$ the linear order $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$. Let $[n]$ be the set $\{1, 2, \ldots, n\}$.

A Variant of Independent Set (VIS)

\textbf{Input:} A multiset $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ where each $T_i \subseteq \mathcal{T}$ is a set of discrete intervals of size 4 over $\{1, 2, \ldots, 2n\}$ and $|T_i| \leq 3$ for all $T_i \in \mathcal{T}$.

\textbf{Question:} Is there a set $S \subseteq \bigcup_{T_i \in \mathcal{T}} T$ of discrete intervals such that $|S| = n$, $|S \cap T_i| = 1$ for every $T_i \in \mathcal{T}$ and no two discrete intervals in $S$ intersect?

Given an instance $\mathcal{E} = (\mathcal{T} = \{T_1, T_2, \ldots, T_n\})$ of VIS, we construct an instance $\mathcal{E}' = ((C \cup \{p\}, \Pi, \Pi', \mathcal{L}, R, n)$ for $r$-$\text{AV}\text{-}2$ as follows.

Let $\mathcal{I} = \bigcup_{T_i \in \mathcal{T}} T$. For each discrete interval $I \in \mathcal{I}$, let $l(I)$ be its left endpoint and $r(I)$ be its right endpoint. Let $\Gamma$ be the set of all elements appearing in some discrete interval of $\mathcal{I}$, i.e., $\Gamma = \{|S|: |S| \leq 2n\}$. Let $\Gamma = \{x_1, x_2, \ldots, x_{|\Gamma|}\}$ be an order of $\Gamma$ where $x_i \in x_{i+1}$ for all $i \in [\Gamma]$.

\textbf{Candidates:} We create three kinds of candidates, $C$, $D$, and $E$:
1. $(C) = \Gamma$; 
2. $D$ contains exactly $2n - 1$ candidates $d_1, d_2, \ldots, d_n, d_{n+1}, \ldots, d_{2n-1}$; 
3. $E$ contains exactly $(n + 3) \cdot (|C| + |D| - 1)$ dummy candidates $x_1, x_2, \ldots, x_{|\Gamma|+n+3}, d_1, d_2, \ldots, d_{n+3}$, $(\Gamma+1)$, which will never be winners. The distinguished candidate is $d_{2n}$, that is, $p = d_{2n}$. Moreover, $r = n+4$.

2-\text{Hamurapi Order:} $\mathcal{L} = (\Gamma, D, \bar{E})$ where $D = (d_1, d_2, \ldots, d_{2n-1})$ and $\bar{E} = (x_1, x_2, \ldots, x_{|\Gamma|+n+3}, d_1, \ldots, d_{n+3}, (\Gamma+1))$.

\textbf{Registered Votes} $\Pi':$ (1) For each $x \in C$, create $n - 2$ votes defined as $(x, L_c[x_1, x_2, x_3, x_4, x_5])$, $\mathcal{L}(x_i, x_{i+1})$, $L_c[x_1, x_2, x_3, x_4, x_5, d_{n+3} - 1, \ldots, d_{n+3} - n - 2, d_{n+3} - n - 1]$.

\textbf{Unregistered Votes} $\Pi_r$: For each $I_j \in T_i$, create a corresponding unregistered vote which is defined as $(\mathcal{L}(l(I_j)), r(I_j))$, $(\mathcal{L}(d_i, d_{n+3} - i - 1), \mathcal{L}(l(I_j)), x_1, r(I_j) (d_i, d_{n+3} - i - 1))$.

Clearly, this vote approves exactly all four candidates lying between $l(I_j)$ and $r(I_j)$ (including $l(I_j)$ and $r(I_j)$) in $\mathcal{L}$ and all candidates lying between $d_i$ and $d_{i+n-1}$ (including $d_i$ and $d_{i+n-1}$) in $\mathcal{L}$. Thus, every unregistered vote approves $d_i$.

It is clear that all votes are 2-peaked with respect to $\mathcal{L}$. Due to the construction, it is easy to see that $SCV(c) = n - 2$ for all $c \in C$, $SCV(d_i) = i - n - 1$ for all $d_i \in D$ with $1 \leq i \leq n - 1$, $SCV(d_i) = i - n - 1$ for all $d_i \in D$ with $1 \leq i \leq n + 1, n + 2, \ldots, 2n - 1$, and $SCV(c) \leq n - 2$ for all $c \in E$ and $SCV(d_i) = 0$.

\begin{align*}
\Rightarrow & \Rightarrow \text{ Suppose that } \mathcal{E}' \text{ is a true-instance and let } S \text{ be a solution for } \mathcal{E}. \text{ Let } S = (I_1, I_2, \ldots, I_n) \text{ be an order of } S \text{ where } I_i = S \cap T_i \text{ for all } i \in [n]. \text{ Then, we can make } d_n \text{ the unique winner by adding votes from } \Pi_r \text{ according to } S. \text{ More specifically, for each } I_i \in S \text{ we select its corresponding vote constructed as adding votes and add it to the election. Clearly, the final score of } d_n \text{ is } n. \text{ Due to the construction, no two added votes } \pi_i, \pi_j \text{ which correspond to two different intervals } I_i, I_j, \text{ respectively, approve a common candidate from } C. \text{ Thus, after adding these votes to the election, no candidate in } C \text{ has a higher score than that of } d_n. \text{ To analyze the score of } d_i \in D \text{ with } j \in [n - 1], \text{ we observe that for any } i > j \text{ the vote corresponding to } I_i \text{ does not approve } d_j. \text{ Since } SCV(d_i) = n - j - 1 \text{ and } |S \cap T_i| = 1 \text{ for all } i \in [j], \text{ we know that the final score of } d_i \text{ is less than } n. \text{ Similarly, to analyze the score of } d_i \in D \text{ with } j \in [n + 1, n + 2, \ldots, 2n - 1], \text{ we observe that for any } i \leq j \text{ the vote corresponding to } I_i \text{ does not approve } d_j. \text{ Since } SCV(d_j) = j - n - 1 \text{ and } |S \cap T_i| = 1 \text{ for all } i \in [j - n + 1, j - n + 2, \ldots, n], \text{ we know that the final score of } d_i \text{ is less than } n. \text{ The final score of each } c \in E \text{ is clearly less than } n - 2 \text{ since no unregistered vote approves } c. \text{ Summarize the above analysis, we conclude that the distinguished candidate } d_n \text{ becomes the unique winner after adding the selected votes to the election.}
\end{align*}
Thus, at least one of \(d_u\) and \(d_{v+1}\) has final score at least \(n\), contradicting that \(d_u\) is the unique winner. The claim is true. It is now easy to see that the set of discrete intervals corresponding to \(S'\) forms a solution for \(E\).

### 3.3 Proof of Theorem 3

We first prove that 3-DV-2 is \(\mathcal{NP}\)-hard by a reduction from VERTEX COVER on bounded degree-3 graphs which is \(\mathcal{NP}\)-complete [17]. Then, we will show that the proof applies to \(r\)-DV-2 for \(r \geq 4\) with a slight modification.

An undirected graph is a tuple \(G = (V, E)\) where \(V\) is the set of vertices and \(E\) is the set of edges. We also use \(V(G)\) to denote the vertex set of \(G\). For a vertex \(u \in V\), \(N_G(u)\) denotes the set of its neighbors in \(G\), that is, \(N_G(u) = \{w \mid (w, u) \in E\}\). The degree of a vertex \(u\) is the number of its neighbors. A graph is a bounded degree-3 graph if it contains at least one degree-3 vertex but no vertex having degree greater than 3. A vertex cover for a graph \(G = (V, E)\) is a subset \(S \subseteq V\) such that every edge in \(E\) has at least one of its endpoints in \(S\).

**Vertex Cover on Bounded Degree-3 Graphs (VC3)**

**Input:** A bounded degree-3 graph \(G = (V, E)\) and a positive integer \(\kappa\).

**Question:** Does \(G\) have a vertex cover of size at most \(\kappa\)?

To prove the \(\mathcal{NP}\)-hardness of 3-DV-2, we first introduce a property for bounded degree-3 graphs. This property may be of independent interest since many graph problems are \(\mathcal{NP}\)-hard when restricted to graphs with bounded degree 3.

An interval over the real line is a closed set \([a, b]\) = \{\(x \in \mathbb{R} \mid a \leq x \leq b\)\} where \(a\) and \(b\) are real numbers. An interval is trivial if \(a = b\). For an interval \(I\), denote by \(l(I)\) and \(r(I)\) its left-point and right-point, respectively. A 2-interval is a set of \(t\) intervals over the real line. A graph \(G = (V, E)\) is a \(t\)-interval graph if there is a set \(T_G\) of \(t\)-intervals and a bijection \(f : V \to T_G\) such that for every \(u, v \in V\), \((u, v) \in E\) if and only if \(f(u)\) and \(f(v)\) intersect. Here, \(T_G\) is called a \(t\)-interval representation of \(G\). For simplicity, we use \(I_u = (I_{u,1}, I_{u,2}, ..., I_{u,t})\) to denote \(f(u)\). For two real numbers \(a\) and \(b\) with \(a \leq b\), we define \((a, b)\) = \{\(x \in \mathbb{R} \mid a < x < b\)\}.

The following lemma states that every bounded degree-3 graph has a 2-interval representation such that every vertex is represented by a 2-interval with one interval is trivial, and two 2-intervals only intersect at the endpoints.

**Lemma 4.** For every bounded degree-3 graph \(G\) there is a 2-interval representation for \(G\) such that for every \(u \in V(G)\), \(I_u = (I_{u,1}, I_{u,2})\) satisfies one of the following:

1. \(I_{u,1} = [x_1, x_2], I_{u,2} = [x_2, x_3], x_1 < x_2 < x_3\) and \(I_u^i \in V(G) \setminus \{u\}\) such that \(r(I_u^i) \in (x_2, x_3)\) or \(l(I_u^i) \in (x_2, x_3)\);

2. \(I_{u,1} = [x_1, x_2], I_{u,2} = [x_3, x_4], x_1 < x_2 < x_3 < x_4\) and \(I_u^i \in V(G) \setminus \{u\}\) such that \(r(I_u^i) \in (x_1, x_2)\) or \(l(I_u^i) \in (x_1, x_2)\),

for each \(I(u') \in \{I_{u,1}^i, I_{u,2}^i\}\). Moreover, such a 2-interval representation can be found in polynomial time. See Fig. 4 for an example.

We now show the reduction. Let \(E = (G, \kappa)\) be an instance of VC3 and \(\mathcal{I}(G)\) be a 2-interval representation of \(G\) satisfying all conditions in Lemma 4. For every \(I_u = \{I_{u,1}, I_{u,2}\}\), let \(D(u)\) be the endpoints of \(I_{u,1}\) and \(I_{u,2}\) (due to
4. 3-Peaked Elections

In Section 2, we proved that control by adding votes in r-approval is polynomial-time solvable when restricted to 2-peaked elections and r being a constant. In this section, we show that the tractability of the problem does not hold when extended to 3-peaked elections.

**Theorem 5.** r-AV-3 is $\mathcal{NP}$-complete for every constant $r \geq 4$.

**Proof.** We first prove the $\mathcal{NP}$-hardness of 4-AV-3 by a reduction from INDEPENDENT SET on bounded degree-3 graphs which is $\mathcal{NP}$-complete [17]. An independent set in a graph $G = (V, E)$ is a subset $S \subseteq V$ such that every edge in $E$ has at most one of its endpoints in $S$.

**Independent Set on Bounded Degree-3 Graphs (IS3)**

**Input:** A bounded degree-3 graph $G = (V, E)$ and a positive integer $k$.

**Question:** Does $G$ have an independent set containing exactly $k$ vertices?

For an instance $E = (G, k)$ of IS3, let $\mathcal{I}(G)$ be a 2-interval representation of $G$ which satisfies all conditions in Lemma 4. Let $D(u, \Gamma)$ and $\Gamma$ be defined as in Subsect. 3.3. We construct an instance $E' = ((C \cup \{p\}, \Pi_v), \Pi_r, \mathcal{L}, R = k)$ of 4-AV-3 as follows.

**Candidates:** $C = \Gamma \cup \{p, c_1, c_2, c_3\}$.

**3-Harmonious Order:** $L = (\tilde{\Gamma}, p, c_1, c_2, c_3)$.

**Registered Votes** $\Pi_v$: The role of registered votes is to guarantee that all candidates of $\Gamma$ have the same score $k - 2$. To this end, we first create $k - 2$ votes defined as $(L \{x_1, x_1+1\}, L \{x_1, x_1\}, L \{x_1+1, x_1\} )$ for every $i = 1, \ldots, 4 \lfloor \Gamma \rfloor - 1$. Then, we create some further votes according to $\Gamma$. **Case 1.** $\Gamma \equiv 0 \mod 4$. We create no further vote. **Case 2.** $\Gamma \equiv 1 \mod 4$. We create additional $k - 2$ votes defined as $(x_1, x_1+1, L\{c_1, c_3\}, L\{x_1, x_1\}, p)$. **Case 3.** $\Gamma \equiv 2 \mod 4$. We create additional $k - 2$ votes defined as $(x_1-1, x_1, p, c_1, c_2, L\{x_1-1, x_1\}, p)$. **Case 4.** $\Gamma \equiv 3 \mod 4$. We create additional $k - 2$ votes defined as $(L\{x_1-2, x_1\}, L\{x_1-1, x_1\}, p, c_2, c_3)$.

**Unregistered Votes** $\Pi_r$: For each $u \in V(G)$, let $(x_i, x_j, x_k)$ be the order of $D(u)$ with $x_i < x_j < x_k$. We create a vote $v_u = (x_i, x_j, x_k, p, L(x_i, x_j), \tilde{\Gamma}(x_j, x_k), \tilde{\Gamma}(x_k, p), L(p, c_i))$. Due to Lemma 4, either $x_i$ or $x_k$ lies consecutively with $x_j$ in $\tilde{\Gamma}$, thus, all these unregistered votes have 3 peaks $x_i, x_j, p$ and every $x_k$. Due to Lemma 4, every $x_i$ or $x_k$ lies consecutively with $x_j$ in $\tilde{\Gamma}$, thus, all these unregistered votes have 3 peaks $x_i, x_j, p$ and every $x_k$.

In the following, we prove that $E$ is a true-instance of $\mathcal{NP}$-hardness and that $E'$ is a true-instance of $\mathcal{NP}$-hardness. It is easy to see that $SCV(x) = k - 2$ for all $x = c \in C \setminus \{p, c_1, c_2, c_3\}$, $SCV(p) = 0$ and $SCV(c) \leq k - 2$ for all $c = c_1, c_2, c_3$.

**(<=)** Suppose that $E'$ is a true-instance and $S$ is an independent set of size $k$. Then we add all votes corresponding to $S$, that is, all votes in $\{v_u | u \in S\}$, to the election. Since $S$ is an independent set, no two added votes approve a common candidate except $p$; thus, each candidate except $p$ has a final score at most $k - 1$. Since each added vote approves $p$, it follows that $p$ has a final score of $k$ points, implying that $p$ becomes the unique winner in the election including the new votes.

**(=>)** Suppose that $E'$ is a true-instance and $S'$ is a solution. Clearly, $p$ has a final score of $k$ points. Since $p$ is the unique winner, for every $c \in C \setminus \{p\}$, there is at most one vote in $S'$ approving $c$. Thus, no two votes in $S'$ approve a common candidate except $p$. Due to the construction, the vertices corresponding to $S'$ must be an independent set.

The proof applies to r-AV-3 for any constant $r \geq 5$ by a similar modification as discussed in Subsect. 3.3.

5. Parameterized Complexity

In this section, we study several control problems for r-approval from the viewpoint of parameterized complexity. The first two FPT results are for the general case.

**Theorem 6.** r-DV is FPT with the number of deleted votes as the parameter, where $r$ is a constant.

**Proof.** To derive the FPT algorithm, we divide the candidates $C$ into two parts: $C_1 = \{c \in C | SCV(c) \geq SCV(p)\}$ and $C_2 = C \setminus C_1$. Meanwhile, we divide $\Pi_v$ into two parts: $\Pi_v_1 = \{\pi_v | \pi_v \in \Pi_v \land \exists c \in C_1 \land v \in \pi_v\}$ and $\Pi_v_2 = \Pi_v \setminus \Pi_v_1$. That is, $\Pi_v_2$ contains all the votes which approve at least one candidate which has at least the same score as $p$, and $\Pi_v_2$ contains other votes. We observe that every true-instance of r-DV has a solution $S$ with $S \subseteq \Pi_v_1$. Due to the observation, we can restrict our attention to $\Pi_v_1$. Since at most $R$ votes can be deleted and each deleted vote approves at most $r$ candidates, $|S| \leq r \cdot R$ holds for every true-instance. We assume now that $|S| \leq r \cdot R$ in the given instance. We say two votes have the same type if they approve the same candidates. Clearly, there are at most $O(r \cdot R^d)$ different types of votes in $\Pi_v_1$. Since every solution includes at most $R$ votes from each type of votes, we have at most $R^{O(d(R))}$ cases to check where $f(R) = R^r$, implying an FPT algorithm for r-DV.

**Theorem 7.** r-AV is FPT with the number of added votes as the parameter, where $r$ is a constant.

Faliszewski et al. [14] proved that control by deleting candidates in 1-approval is NP-hard when restricted to Swoop-SP elections. Since Swoop-SP is a special case of 2-peaked elections, 1-DV-3 with $k \geq 2$ is NP-hard. We strengthen this result by proving that 1-DC-3 is $\mathcal{W}[1]$-hard with the number of deleted candidates as the parameter.

**Theorem 8.** 1-DC-3 is $\mathcal{W}[1]$-hard with the number of deleted candidates as the parameter.

**Proof.** We prove the theorem by an FPT reduction from INDEPENDENT SET which is $\mathcal{W}[1]$-hard [10]. For a linear order $A = (a_1, a_2, \ldots, a_n)$ over $A = (a_1, a_2, \ldots, a_n)$ and a subset $B \subseteq A$, denote by $A \setminus B$ the linear order of $A \setminus B$ obtained from $A$ by deleting all elements in $B$. For an instance $E = (G = (V, E), k)$ of Independent Set we construct an instance $E'$ of 1-DC-3 as follows.

**Candidates:** $V \cup \{p, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$.

**3-Harmonious Order:** Let $F = (c_1, c_2, \ldots, c_n)$ be an (arbitrary) order of $V$. Then, the 3-harmonious order $L$ is given by $(b_1, b_2, \ldots, b_n, p, a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_n)$.

**Votes:** Let $m = |E|$. There are seven types of votes.

(1) 2m − 1 votes defined as $(L\{a, c\}, L[p, b_a])$; (2) 2m votes
defined as \((L[p, c_a], L[b, b_a])\); (3) \(2m + \kappa - 1\) votes defined as \((L[b, b_a], L[p, c_a])\); (4) for each edge \((c_i, c_j) \in E(G)\) with \(i < j\), create one vote defined as \((c_i, c_j, L[a, a_a], L[p, b], F \setminus \{c_i, c_j\})\); (5) for each vertex \(c_i\), create one vote defined as \((c_i, L[p, a_a], L[b, b_a], F \setminus \{c_i\})\) and one vote defined as \((c_i, L[a, a_a], L[p, b], F \setminus \{c_i\})\); (6) \(\kappa + 1\) votes defined as \((L[a_1, c_a], L[a, b]); (7) one vote defined as \((L[b_1, b_a], L[b, c_a])\); It is easy to verify that all constructed votes are 3-peaked with respect to \(L\).

**Number of Added Candidates:** \(R = \kappa\).

\((\Leftarrow)\) It is easy to verify that \(\mathcal{E}'\) is a true-instance implies \(\mathcal{E}\) is a true-instance: for every independent set \(S\) of size \(\kappa\), deleting the candidates \(S\) from the election clearly makes the distinguished candidate \(p\) become the unique winner.

\((\Rightarrow)\) Suppose that \(\mathcal{E}'\) is a true-instance and \(S'\) is a solution with \(|S'| \leq \kappa\). We first observe that \(b \not\in S'\). This observation is true, since otherwise, all candidates in \([b_1, b_2, ..., b_n]\) must be deleted, contradicting that \(|S'| \leq \kappa\). The same argument applies to the candidate \(a\). However, in order to beat \(b\), exactly \(\kappa\) candidates from \(V\) must be deleted so that \(p\) can get extra \(k\) points from the constructed votes of case 5. Since \(|S'| \leq \kappa\), \(S'\) must be a subset of \(V\). Moreover, no two candidates \(c_1, c_2 \in S'\) are adjacent to each other in the graph \(G\), since otherwise, the candidate \(a\) would get at least one extra point from the constructed votes of case 4, and \(p\) cannot be the unique winner. Thus, \(S'\) forms an independent set of size \(\kappa\) of \(G\). □

6. CONCLUSION

In this paper, we study the \(k\)-peaked elections which generalize the single-peaked elections by allowing at most \(k\)-peaks in each vote. We derive a dichotomy of the complexity of control problems for \(r\)-approval voting in \(k\)-peaked elections with respect to \(k\). Moreover, we present some results concerning the parameterized complexity of these problems in general as well as \(k\)-peaked elections. All of our results work for both unique-winner and nonunique-winner models. In addition, several of our results apply to approval voting and SP-AV as well. Another possible research direction could be studying more strategic behaviors for other voting systems in \(k\)-peaked elections.

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7. REFERENCES


