On Structure-Based Inconsistency Measures and Their Computations via Closed Set Packing

(Extended Abstract)

Said Jabbour
CRIL - Univ. Artois
jabbour@cril.fr

Yue Ma
LRI - Univ. Paris-Saclay
yue.ma@lri.fr

Badran Raddaoui
CRIL - Univ. Artois
raddaoui@cril.fr

Lakhdar Sais
CRIL - Univ. Artois
sais@cril.fr

Yakoub Salhi
CRIL - Univ. Artois
salhi@cril.fr

ABSTRACT

Measuring conflicts is important for understanding the contradictory status of a knowledge base (KB). In this work, we propose a new framework called *closed set packing*, an interesting extension of the well-known set packing problem, by which we define a family of fine-grained inconsistency measures exploiting the structure of minimal inconsistent sets of a KB. We show that closed set packing also gives a general encoding for computing this new family of measures.

Categories and Subject Descriptors

I.2.4 [**Knowledge representation formalisms and methods**]

General Terms

Measurement, Theory

Keywords

Inconsistency Measures, Closed Set Packing Problem

1. INTRODUCTION

Conflicting information is often unavoidable in real-world knowledge-based systems. Indeed, conflicts among various agents are a common phenomena. If one agent has to choose another one to cooperate, the agent should prefer one that has the *least* disagreement with herself, which motivates the research on inconsistency measures [1, 4, 3]. This paper proposes a family of structure-based measures for a fine-grained discriminative analysis of inconsistency and provides encoding algorithms for their computation.

Throughout this paper, we consider a propositional language $\mathcal{L}$ built over a finite set of propositional symbols $\mathcal{P}$ using classical logical connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$. The symbol $\perp$ denotes contradiction. A KB K consists of a finite set of propositional formulas. K is inconsistent if $K \vdash \perp$, where $\vdash$ is the classical consequence relation.

Appears in: *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2015), Bordini, Elkind, Weiss, Yolum (eds.), May, 4–8, 2015, Istanbul, Turkey.*

Definition 1. *For a KB K and $M \subseteq K$, M is a Minimal Inconsistent Subset (MIS) of K iff $M \vdash \perp$ and $\forall M' \subsetneq M$, $M' \nvdash \perp$. We denote by $MISes(K)$ the set of MISes of K. We define $free(K) = \{\alpha \mid \nexists M \in MISes(K),\ \alpha \in M\}$ and $unfree(K) = K \setminus free(K)$.*

An inconsistency measure assigns a nonnegative number to every KB as its degree of conflict. In [1], a simple inconsistency measure is defined as $I_{MI}(K) = |MISes(K)|$. A second inconsistency metric proposed in [2] is defined as follows.

Definition 2 ([2]). *A MIS-partition of a KB K is a pair $\langle\{K_1, \ldots, K_n\}, R\rangle$ s.t.:*

- *$\forall i$, $K_i \subseteq K$ and $K_i \vdash \perp$, and $\forall i \neq j$, $K_i \cap K_j = \emptyset$,*
- *$MISes(K_1 \cup \ldots \cup K_n) = \biguplus_{i=1}^{n} MISes(K_i)$.*

Then, $I_{CC}(K) = m$ if there is a MIS-partition $\langle D, R\rangle$ where $|D| = m$, and there is no MIS-partition $\langle D', R'\rangle$ s.t. $|D'| > m$.

2. CSP-BASED CONFLICT MEASURES

In this section, we define our framework of measuring conflicts of a KB. First, we define a novel generalization of the well-known set packing problem, called *closed set packing* (CSP).

Definition 3. *Let U be a universe and S a family of subsets of U. A* set packing *is a subset $P \subseteq S$ such that, $\forall S_i, S_j \in P$ with $S_i \neq S_j$, $S_i \cap S_j = \emptyset$.*

The *maximum set packing problem (MSP)* is the related well-known combinatorial optimization problem, defined as finding a set packing of S with the maximum size for a collection of subsets S over a universe U.

Definition 4. *Let U be a universe and S a family of subsets of U. We define the function $f_S : 2^S \mapsto 2^S$ as $f_S(P) = \{S_i \in S \mid S_i \subseteq \cup_{S' \in P} S'\}$. Then, a set packing $P \subseteq S$ is called a* closed set packing (CSP) *if P is a fixed point of the function f_S, i.e. $f_S(P) = P$.*

Example 1. *Consider the family F built over the universe $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$:*

$$F = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_2, u_4, u_6\}, \{u_4, u_5\}, \{u_6, u_7\}, \{u_6, u_8\}\}.$$

Then, $S = \{\{u_1, u_2\}, \{u_6, u_7\}\}$ is a CSP of maximum cardinality, which is related to the MCSP problem *defined below.*

Problem: MCSP-decision

Input: a universe U, a collection of subsets S of U, and an integer k

Question: is there a closed set packing P s.t. $|P| \geq k$?

THEOREM 1. *MCSP-decision is NP-complete.*

The next result shows that $I_{\mathcal{CC}}$ measure can be nicely characterized by the closed set packing problem.

PROPOSITION 1. *$I_{\mathcal{CC}}(K)$ is the cardinality of the solution of MCSP(U, S), where $U = K$ and $S = MISes(K)$.*

Recall that $I_{\mathcal{CC}}$ measure is a lower bound for syntactic measures. Unfortunately, the lower bound considers only a subset of MISes forming a CSP of MISes (cf. Proposition 1). That is, $I_{\mathcal{CC}}$ does not take into consideration the contribution of each MIS to the whole inconsistency. A key step for designing a more accurate inconsistency metric is to consider all MISes with (possibly) different degrees.

Before defining our inconsistency metrics, we need to introduce a *partitioning* of MISes of a KB into clusters of CSP.

DEFINITION 5. *$\mathcal{P} = \{P_1, \ldots, P_n\}$ is a* csp-partition *of the MISes of a KB K if $MISes(K) = \biguplus_{1 \leq i \leq n} P_i$ s.t. P_i is a CSP. $\mathcal{P}$ is called an* ordered *csp-partition of $MISes(K)$ if $|P_1| \geq \ldots \geq |P_n|$.*

Let $\mathcal{P}_{MISes}(K)$ denote the set of ordered csp-partitions of $MISes(K)$.

DEFINITION 6. *The* csp-partition inconsistency measure *of a KB K, written $I_{\mathcal{CSP}}(K)$, is defined as:*

$$I_{\mathcal{CSP}}(K) = \max\ \{\mathcal{W}(\mathcal{P}) \mid \mathcal{P} \in \mathcal{P}_{MISes}(K)\}$$

where $\mathcal{W}(\mathcal{P}) = \sum_{P_i \in \mathcal{P}} |P_i| \times w_i$ s.t. $\{w_n\}_{n=1}^{+\infty}$ is a decreasing positive sequence s.t. $w_1 = 1$.

EXAMPLE 2. *Let $K = \{a, \neg a, a \wedge b, (a \vee c) \wedge d, \neg d, \neg c \wedge e, \neg e, \neg e \wedge f\}$. We have $I_{\mathcal{CSP}}(K) = 2 + 2w_1 + w_2 + w_3$.*

Clearly, we can get the following result from the definition of the $I_{\mathcal{CSP}}$ measure.

PROPOSITION 2. *For a KB K and $\mu = |MISes(K)| - I_{\mathcal{CC}}(K)$, it holds $(\sum_{i=2}^{\mu} w_i) + I_{\mathcal{CC}}(K) \leq I_{\mathcal{CSP}}(K) \leq I_{MI}(K)$.*

Note that the measure $I_{\mathcal{CSP}}$ reaches the maximum value when the MISes itself form a CSP. The minimum value is obtained, for example for a KB whose MISes share at least one formula. In this case, $I_{\mathcal{CSP}}(K) = \sum_{i=1}^{n} w_i$.

THEOREM 2. *$I_{\mathcal{CSP}}$ satisfies the following properties:*

- $I_{\mathcal{CSP}}(K) = 0$ iff K is consistent,
- If $K \subseteq K'$, $I_{\mathcal{CSP}}(K) \leq I_{\mathcal{CSP}}(K')$,
- $I_{\mathcal{CSP}}(K \cup \{\alpha\}) = I_{\mathcal{CSP}}(K)$ if $\alpha \in$ *free*$(K \cup \{\alpha\})$,
- $I_{\mathcal{CSP}}(M) = 1$ if $M \in MISes(K)$,
- $I_{\mathcal{CSP}}(K_1 \cup \ldots \cup K_n) = \Sigma_{i=1}^{n} I_{\mathcal{CSP}}(K_i)$, if $MISes(K_1 \cup \ldots \cup K_n) = \uplus_{i=1}^{n} MISes(K_i)$, and for all $1 \leq i \neq j \leq n$ *unfree*$(K_i) \cap$ *unfree*$(K_j) = \emptyset$.

Now, let us stress that the definition of $I_{\mathcal{CSP}}$ is a general definition that allows for a range of measures to be proposed.

PROPOSITION 3. *Let $\mathcal{K}$ be a KB and $\{w_n\}_{n=1}^{+\infty}$ a sequence s.t. $w_1 = 1$ and $\forall\ n > 1, w_n = \lambda$, where $0 \leq \lambda \leq 1$. Then,*

$$I_{\mathcal{CSP}}(K) = (1 - \lambda) \times I_{\mathcal{CC}}(K) + \lambda \times I_{MI}(K)$$

According to Proposition 3, the following result holds.

PROPOSITION 4. *Given a KB K. Then, we have:*
$I_{\mathcal{CSP}} = I_{\mathcal{CC}}$, if $\lambda = 0$,
$I_{\mathcal{CSP}} = I_{MI}$, if $\lambda = 1$,
$I_{\mathcal{CSP}} = (I_{\mathcal{CC}} + I_{MI})/2$, if $\lambda = 1/2$.

3. ON THE COMPUTATION OF $I_{\mathcal{CC}}$ AND $I_{\mathcal{CSP}}$

In this section, we provide an encoding for $I_{\mathcal{CC}}$ using *Integer Linear Programming* (ILP) allowing to use existing solvers for its computation. Notice that the encoding of $I_{\mathcal{CSP}}$ by ILP can be obtained in a similar way.

Variables: We associate a binary variable $X_e \in \{0, 1\}$ to each element e in U and a binary variable $Y_{S_i} \in \{0, 1\}$ to each subset $S_i \in S$.

Constraints: The first linear inequalities allow us to only consider the pairwise disjoint subsets in S:

$$\sum_{S_i \in S | e \in S_i} Y_{S_i} \leq 1 \qquad \forall e \in U \tag{1}$$

The following inequalities allow us to express that $Y_{S_i} = 1$ iff, for all $e \in S_i$, $X_e = 1$, i.e., $Y_{S_i} \Leftrightarrow (\sum_{e \in S_i} X_e = |S_i|)$:

$$(\sum_{e \in S_i} X_e) - |S_i| \times Y_{S_i} \geq 0 \qquad \forall S_i \in S \tag{2}$$

$$(\sum_{e \in S_i} X_e) - Y_{S_i} \leq |S_i| - 1 \qquad \forall S_i \in S \tag{3}$$

Any solution to the inequalities (1), (2) and (3) represents a closed set packing of S.

Let us now define the integer linear program:

Problem: *ILP-$I_{\mathcal{CC}}(U, S)$*

$$\max \sum_{S_i \in S} Y_{S_i}$$

$$\text{subject to } (1), (2), (3)$$

$$X_e \in \{0, 1\}, \forall e \in U, Y_{S_i} \in \{0, 1\}, \forall S_i \in S$$

PROPOSITION 5. *The integer linear program corresponding to ILP-$I_{\mathcal{CC}}(U, S)$ is a correct encoding of $I_{\mathcal{CC}}$.*

REFERENCES

[1] A. Hunter and S. Konieczny. On the measure of conflicts: Shapley inconsistency values. *Artif. Intell.*, 174(14):1007–1026, 2010.

[2] S. Jabbour, Y. Ma, and B. Raddaoui. Inconsistency measurement thanks to mus decomposition. In *AAMAS*, pages 877–884, 2014.

[3] S. Jabbour, Y. Ma, B. Raddaoui, and L. Saïs. Prime implicates based inconsistency characterization. In *ECAI*, pages 1037–1038, 2014.

[4] M. Thimm. Inconsistency measures for probabilistic logics. *Artif. Intell.*, 197:1–24, 2013.