Belief Merging versus Judgment Aggregation

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ABSTRACT

The problem of aggregating pieces of propositional information coming from several agents has given rise to an intense research activity. Two distinct theories have emerged. On the one hand, belief merging has been considered in AI as an extension of belief revision. On the other hand, judgment aggregation has been developed in political philosophy and social choice theory. Judgment aggregation focusses on some specific issues (represented as formulas and gathered into an agenda) on which each agent has a judgment, and aims at defining a collective judgment set (or a set of collective judgment sets). Belief merging considers each source of information (the belief base of each agent) as a whole, and aims at defining the beliefs of the group without considering an agenda. In this work the relationships between the two theories are investigated both in the general case and in the fully informed case when the agenda is complete (i.e. it contains all the possible interpretations). Though it cannot be ensured in the general case that the collective judgment computed using a rational belief merging operator is compatible with the collective judgment computed using a rational judgment aggregation operator, we show that some close correspondences between the rationality properties considered in the two theories exist when the agenda is complete.

Categories and Subject Descriptors

I.2 [**Artificial Intelligence**]: Multiagent systems

Keywords

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1. INTRODUCTION

Belief merging (BM) [7, 8] is a logical setting which rules the way jointly contradictory belief bases coming from a group of n agents should be aggregated, in order to obtain a collective belief base. Belief merging operators have been defined and studied as an extension of AGM belief revision theory [4, 2, 5], and some rationality postulates for merging (the so-called IC postulates) have been pointed out.

Judgment aggregation (JA) has been developed in political philosophy and social choice theory [11, 10]. The aim of judgment aggregation is to make collective judgments on several (possibly logically related) issues, from the judgments given on each issue by the members of a group of n agents. Again, several operators (referred to as rules or correspondences) have been put forward and the rationality issue has been investigated as well.

Belief merging and judgment aggregation have similar, yet distinct goals. Especially, a belief merging process and a judgment aggregation process do not consider the same inputs and outputs. In a belief merging process, the input is a profile $E = (K_1, \ldots, K_n)$ of propositional belief bases (finite sets of propositional formulas), and a formula μ representing some integrity constraints the result of the process must comply with; it outputs an (aggregated/collective) base which satisfies the integrity constraints. In a judgment aggregation process, the input is an agenda (i.e., a set of propositional formulas $X = \{\varphi_1, \ldots, \varphi_m\}$, considered as binary questions), and a profile $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of individual judgment sets on these formulas. This profile consists of the judgment sets furnished by the agents, where each individual judgment set makes precise for each question whether it is supported, its negation is supported or none of them. Finally, the judgment aggregation process outputs a (set of) aggregated/collective judgment set(s) on these formulas.

A reasonable closed-world assumption is that each individual judgment on a formula of the agenda is fully determined by the beliefs of the corresponding agent. Stated otherwise, we assume that each agent is equipped with a so-called *projection function* p such that the judgment of the agent on φ is given by $p(K_i, \varphi)$, where K_i is the belief base of agent i.[1] Such a projection function p can be easily extended to agendas $X = \{\varphi_1, \ldots, \varphi_m\}$ by stating that $p_X(K_i) = (p(K_i, \varphi_1), \ldots, p(K_i, \varphi_m))$, and then to profiles $E = (K_1, \ldots, K_n)$ by stating that $p_X(E) = (p_X(K_1), \ldots, p_X(K_n))$. Furthermore, the standard JA setting does not take integrity constraints μ into account for characterizing the possible worlds (i.e., each world is possible). In order to make a fair comparison of BM and JA (i.e., based only on the input E and X), we thus assume that μ is valid. Under those assumptions, belief merging and judgment aggregation can be embodied in the same global aggregation setting, enabling to compare them. In the resulting framework, one can view a judgment aggregation process as a partially informed aggregation process: it is not the case (in general) that every piece of beliefs of each agent is exploited in the aggregation (the focus is laid on the specific issues of the agenda). This contrasts with a belief merging process, which is fully informed in the sense that each belief base is entirely considered. The two processes and the way they are connected are depicted on Figure 1.

Given a profile of belief bases $E = (K_1, \ldots, K_n)$ corresponding to a group of n agents, an agenda $X = \{\varphi_1, \ldots, \varphi_m\}$, and a projection function p, there are two ways to define the collec-

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[1] For the sake of simplicity, we also assume that all the agents share the same projection function.

$$
\begin{tikzcd}
E = (K_1, \ldots, K_n) \arrow[r, "\Delta"] \arrow[d, "p_{\{\varphi_1,\ldots,\varphi_m\}}"] & \Delta(E) \arrow[d, "p_{\{\varphi_1,\ldots,\varphi_m\}}"'] \\
\Gamma = (\gamma_1, \ldots, \gamma_n) \arrow[r, "Ag"] & \varphi_{Ag}/\varphi_{\Delta}
\end{tikzcd}
$$

Figure 1: Belief Merging vs. Judgment Aggregation

tive judgment (interpreted as a proposition formula) of the n agents on the agenda. On the one hand, by following the judgment aggregation path: one first takes advantage of p to compute a profile Γ gathering the individual judgments of the n agents on the m questions of the agenda; then using a judgment aggregation operator Ag, one computes the collective judgment on the agenda. On the other hand, by following the merging path: the bases of the input profile are first merged using Δ, which leads to a merged base $\Delta(E)$, and then one computes directly the collective judgment on the agenda using p. The two paths correspond to two collective judgments: $\varphi_{\Delta} = p_X(\Delta(E))$, and $\varphi_{Ag} = Ag_{p_X(E)}$.

The contribution of the paper is as follows. First, we point out some natural requirements on the projection function and show that there exists a unique projection function satisfying them. Given this projection function, we first consider the general case when no constraints are imposed on the agenda. In this case, we show that the two approaches for computing collective judgments given above are hardly compatible; especially, ensuring that Δ is an IC merging operator and that Ag satisfies an expected unanimity condition is not enough to guarantee that φ_{Δ} and φ_{Ag} are jointly consistent. Then we focus on the case of complete agendas, i.e., the questions are the set of all interpretations over the underlying propositional language. In this case, the projection function is invertible: one can recover the belief base of each agent (up to logical equivalence) starting from her judgments on all interpretations. As a consequence, given a profile of belief bases, every belief merging operator Δ can be associated with a unique judgment aggregation method Ag^{Δ}, and vice-versa. We show that $\varphi_{\Delta} \equiv \varphi_{Ag^{\Delta}}$. We also show how imposing IC conditions on the belief merging operator comes down to imposing quite standard rationality conditions on the associated judgment aggregation method.

The layout of the rest of the paper is as follows. In the next section the key definitions of the propositional belief merging framework are recalled. In Section 3 some basics of judgment aggregation are provided. In Section 4 we study the projection functions enabling to obtain judgments from belief bases given an agenda. In Section 5 we show that imposing standard rationality conditions on Δ and Ag is not enough to ensure that the corresponding collective judgments are jointly consistent. Section 6 is dedicated to the case of complete agendas. Section 7 discusses the compatibility of the two aggregation approaches. Finally Section 8 concludes the paper.

2. ON PROPOSITIONAL MERGING

We consider a propositional language $\mathcal{L}$ defined from a finite set P of propositional symbols and the usual connectives.

An interpretation (or state of the world) ω is a total function from P to $\{0, 1\}$. Ω is the set of all interpretations. An interpretation is usually denoted by a bit vector whenever a strict total order on P is specified. It is also viewed as the formula $\bigwedge_{p \in P \mid \omega(p)=1} p \wedge \bigwedge_{p \in P \mid \omega(p)=0} \neg p$.

ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $\models$ denotes logical entailment and $\equiv$ denotes logical equivalence. $[\phi]$ denotes the set of models of formula ϕ, i.e., $[\phi] = \{\omega \in \Omega \mid \omega \models \phi\}$.

A *belief base* K is a finite set of propositional formulas, interpreted conjunctively (i.e., viewed as the formula which is the conjunction of its elements). We suppose that each belief base is non-trivial, i.e., it is consistent but not valid.

A *profile* E represents the beliefs of a group of n agents involved in the merging process; formally E is given by a vector $(K_1, \ldots, K_n)$ of belief bases, where K_i is the belief base of agent i. $\bigwedge E$ denotes the conjunction of all elements of E, and $\sqcup$ denotes the vector union. Two profiles $E = (K_1, \ldots, K_n)$ and $E = (K'_1, \ldots, K'_n)$ are equivalent, noted $E \equiv E'$, iff there exists a permutation π over $\{1, \ldots, n\}$ s.t. for each $i \in 1, \ldots, n$, we have $K_i \equiv K'_{\pi(i)}$.

An *integrity constraint* μ is a consistent formula restricting the possible results of the merging process.

A *merging operator* $\triangle$ is a mapping which associates with a profile E and an integrity constraint μ a (*merged*) base $\triangle_{\mu}(E)$. $\Delta(E)$ is a short for $\triangle_{\top}(E)$.

The logical properties given in [7] for characterizing IC belief merging operators are:

DEFINITION 1. *A merging operator $\triangle$ is an* IC merging operator *iff it satisfies the following properties:*

(IC0) $\triangle_{\mu}(E) \models \mu$
(IC1) *If μ is consistent, then $\triangle_{\mu}(E)$ is consistent*
(IC2) *If $\bigwedge E$ is consistent with μ, then $\triangle_{\mu}(E) \equiv \bigwedge E \wedge \mu$*
(IC3) *If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$*
(IC4) *If $K_1 \models \mu$ and $K_2 \models \mu$, then $\triangle_{\mu}((K_1, K_2)) \wedge K_1$ is consistent if and only if $\triangle_{\mu}((K_1, K_2)) \wedge K_2$ is consistent*
(IC5) $\triangle_{\mu}(E_1) \wedge \triangle_{\mu}(E_2) \models \triangle_{\mu}(E_1 \sqcup E_2)$
(IC6) *If $\triangle_{\mu}(E_1) \wedge \triangle_{\mu}(E_2)$ is consistent, then $\triangle_{\mu}(E_1 \sqcup E_2) \models \triangle_{\mu}(E_1) \wedge \triangle_{\mu}(E_2)$*
(IC7) $\triangle_{\mu_1}(E) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(E)$
(IC8) *If $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(E) \models \triangle_{\mu_1}(E)$*

See [7] for some explanations of these properties.

Let us now give some examples of IC merging operators from the family of distance-based merging operators [6]:

DEFINITION 2. *A* (pseudo-)distance *between interpretations is a function $d : \Omega \times \Omega \to \mathbb{R}^+$ such that for any $\omega_1, \omega_2 \in \Omega$:*

- $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$
- $d(\omega_1, \omega_2) = 0$ *iff* $\omega_1 = \omega_2$

Let *diff* (ω, ω') be the set of propositional variables on which ω and ω' differ:

DEFINITION 3. *A distance d is* normal *iff* $\forall \omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$, $d(\omega_1, \omega_2) \leq d(\omega_3, \omega_4)$ *whenever* $\mathit{diff}(\omega_1, \omega_2) \subseteq \mathit{diff}(\omega_3, \omega_4)$.

This normality property expresses a very natural idea: if ω_1 and ω_2 differ on a given subset D of variables and ω_3 and ω_4 differ on a superset of D, then ω_3 should not be considered closer to ω_4 than ω_1 is to ω_2. All usual distances are normal, in particular the Hamming distance and the Drastic distance [7] are normal distances.

DEFINITION 4. *An* aggregation function *is a mapping*[2] *f from $\mathbb{R}^m$ to $\mathbb{R}$, which satisfies:*

[2]Strictly speaking, it is a family of mappings, one for each m.

- *if $x_i \geq x'_i$, then $f(x_1, ..., x_i, ..., x_m) \geq f(x_1, ..., x'_i, ..., x_m)$* **(non-decreasingness)**
- $f(x_1, \ldots, x_m) = 0$ *if* $\forall i,\ x_i = 0$ **(minimality)**
- $f(x) = x$ **(identity)**
- *If σ is a permutation over $\{1, \ldots, m\}$, then* $f(x_1, \ldots, x_m) = f(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ **(symmetry)**

Some additional properties can also be considered for f, especially:

- *if $x_i > x'_i$, then $f(x_1, ..., x_i, ..., x_m) > f(x_1, ..., x'_i, ..., x_m)$* **(strict non-decreasingness)**
- *If $f(x_1, \ldots, x_n, z) \leq f(y_1, \ldots, y_n, z)$, then $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$* **(decomposition)**
- *If $\forall i, z > y_i$, then $f(z, x_1, \ldots, x_n) > f(y_1, \ldots, y_{n+1})$* **(strict preference)**

DEFINITION 5. *Let d and f be respectively a distance between interpretations and an aggregation function. The* distance-based merging operator $\triangle^{d,f}$ *is defined by*

$$[\triangle^{d,f}_{\mu}(E)] = \min([\mu], \leq_E)$$

where the total pre-order $\leq_E$ on Ω is defined in the following way (with $E = (K_1, \ldots, K_n)$):

- $\omega \leq_E \omega'$ *iff* $d(\omega, E) \leq d(\omega', E)$
- $d(\omega, E) = f(d(\omega, K_1), \ldots, d(\omega, K_n))$
- $d(\omega, K) = \min_{\omega' \models K} d(\omega, \omega')$

For usual aggregation functions, and whatever the chosen distance, the corresponding distance-based operators exhibit good logical properties:

PROPOSITION 1 ([7]). *For any distance d, if f is equal to Σ, leximax*[3]*, leximin, or Σ^n (sum of the n^{th} powers), then $\triangle^{d,f}$ is an IC merging operator.*

3. ON JUDGMENT AGGREGATION

We briefly present some definitions and notations used in the following.[4]

An *agenda* $X = \{\varphi_1, \ldots, \varphi_m\}$ is a finite, non-empty and totally ordered set of non-trivial (i.e., consistent but not valid) propositional formulas.

A *judgment* on a formula φ_k of X is an element of $D = \{1, 0, \star\}$, where 1 means that φ_k is supported, 0 that $\neg\varphi_k$ is supported, $\star$ that neither φ_k nor $\neg\varphi_k$ are supported. A *judgment set* on X is a mapping γ from X to D, also viewed as a m-vector over D, when the cardinality of X is m. For each φ_k of X, γ is supposed to satisfy $\gamma(\neg\varphi_k) = \neg\gamma(\varphi_k)$, where $\neg\gamma$ is given by $\neg\gamma(\varphi_k) = \star$ iff $\gamma(\varphi_k) = \star$, $\neg\gamma(\varphi_k) = 1$ iff $\gamma(\varphi_k) = 0$, and $\neg\gamma(\varphi_k) = 0$ iff $\gamma(\varphi_k) = 1$.

Judgment sets are often asked to be consistent and resolute: A judgment set is *resolute* iff $\forall\varphi_k \in X$, $\gamma(\varphi_k) = 0$ or $\gamma(\varphi_k) = 1$. A judgment set γ on X is *consistent* iff the associated formula (judgment) $\widehat{\gamma} = \bigwedge_{\{\varphi_k \in X \mid \gamma(\varphi_k)=1\}} \varphi_k \wedge \bigwedge_{\{\varphi_k \in X \mid \gamma(\varphi_k)=0\}} \neg\varphi_k$ is consistent.

Aggregating judgments consists in associating a set of collective judgment sets with a profile of individual judgment sets: a *profile* $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of judgment sets on X is a non-empty vector of judgments sets on X. Γ is *consistent* (resp. *resolute*) when each judgment set in it is consistent (resp. resolute).

For each agenda X, a *judgment aggregation method Ag* associates with a consistent profile Γ on X a non-empty set Ag_Γ of collective judgment sets γ_Γ on X, also viewed as a formula (the collective judgment) $\widehat{Ag_\Gamma} = \bigvee_{\gamma_\Gamma \in Ag_\Gamma} \widehat{\gamma_\Gamma}$. For $\varphi_k \in X$, we note $Ag_\Gamma(\varphi_k) = 1$ (resp. $Ag_\Gamma(\varphi_k) = 0$) if and only if $\forall\gamma_\Gamma \in Ag_\Gamma$, $\gamma_\Gamma(\varphi_k) = 1$ (resp. $\forall\gamma_\Gamma \in Ag_\Gamma, \gamma_\Gamma(\varphi_k) = 0$), and $Ag_\Gamma(\varphi_k) = *$ in the remaining case. When Ag_Γ is a singleton for each Γ, the judgment aggregation operator is called a (deterministic) judgment aggregation rule, and it is called a judgment aggregation correspondence otherwise [9]. In this paper, we mainly focus on the more general case of judgment aggregation correspondences.

Usual rationality properties pointed out so far for judgment aggregation (JA) operators are:

Universal domain. The domain of Ag is the set of all consistent profiles.

Collective rationality. For any profile Γ in the domain of Ag, Ag_Γ is a set of consistent collective judgment sets.

Collective resoluteness. For any profile Γ in the domain of Ag, Ag_Γ is a set of resolute collective judgment sets.

Anonymity. For any two profiles $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_n)$ in the domain of Ag which are permutations one another, we have $Ag_\Gamma = Ag_{\Gamma'}$.

Neutrality. For any φ_p, φ_q in the agenda X and profile Γ in the domain of Ag, if $\forall i\ \gamma_i(\varphi_p) = \gamma_i(\varphi_q)$, then $Ag_\Gamma(\varphi_p) = Ag_\Gamma(\varphi_q)$.

A more demanding property is independence:

Independence. For any φ_p, φ_q in the agenda X and profiles Γ and Γ' in the domain of Ag, if $\forall i\ \gamma_i(\varphi_p) = \gamma'_i(\varphi_q)$, then $Ag_\Gamma(\varphi_p) = Ag_{\Gamma'}(\varphi_q)$.

The above properties are quite standard [10]. In previous works we criticize both neutrality and independence [3], but here we stick with the standard JA definitions.

Other properties are also very attractive for JA operators, such as unanimity [3] and majority preservation [9].

Unanimity. For any $\varphi_k \in X$, for any profile Γ in the domain of Ag, if $\exists x \in \{0, 1\}$ s.t. $\forall\gamma_i \in \Gamma, \gamma_i(\varphi_k) = x$, then for every $\gamma_\Gamma \in Ag_\Gamma$, we have $\gamma_\Gamma(\varphi_k) = x$. Note that unanimity is not required when $x = \star$, since in this case it makes sense to let the acceptance of φ_k depends on the acceptance of other (logically related) formulas.

Majority preservation. If the judgment set obtained using the majority rule is consistent and resolute,[5] then Ag_Γ is a singleton which consists of this set.

Majority preservation[6] [9, 13] is a very natural property, stating that if the simple majority vote on each issue leads to a consistent judgment set, then the judgment aggregation correspondence must output precisely this set. Indeed, it is sensible to stick to the result furnished by a simple majority vote when no doctrinal paradox occurs.

Let us now review some of the judgment aggregation operators that have been put forward in the literature. Usual judgment aggregation operators are majority, supermajority, premise-based,

[3] Also referred to as $Gmax$.

[4] Most of these notations depart from (but are equivalent to) the ones usually used in judgment aggregation papers.

[5] Several definitions are possible for the majority rule when abstention is allowed. Here, one considers that the majority rule gives 1 (resp. 0) when the number of agents reporting 1 (resp. 0) is strictly greater than the number of agents reporting 0 (resp. 1), and it gives $\star$ otherwise.

[6] Called strong majority preservation in [13].

conclusion-based, sequential priority [10]. In [12] distance-based (merging-based) operators are pointed out. In [9] several families of judgment aggregation operators based on minimization, inspired from operators considered in voting theory and in AI, are defined. Majority preservation is presented as a natural requirement for such operators. In [3] another family of operators, called ranked majority operators, based on the number of votes received by each formula, has been introduced.

4. PROJECTING A BELIEF BASE ON AN AGENDA

As explained previously, belief merging and judgment aggregation consider different inputs. In belief merging, an input profile consists of a profile of belief bases, representing the beliefs of a group of agents. In judgment aggregation, agents answer "yes" (1), "no" (0) or "undetermined" ($\star$) to a set of questions (the agenda), and the input profile is a vector of such answers (alias judgment sets). Of course agents might use their beliefs to answer the questions, but it is out of the scope of judgment aggregation methods to specify how.

Imagine that the beliefs K_i of an agent i are known, given a question φ_k, what could be the opinion of the agent on the question? Suppose that an agent only believes that $a \wedge b$ is true, and questions her about a: she will probably answer "yes" to the question because she necessarily believes that a is true. If the question is $\neg b$, she will probably answer "no" because b being false is incompatible with her beliefs. Suppose now that the agent just believes that a is true, and that the question is $a \wedge b$. In this case the agent probably has no opinion on the question (the question is contingent given her beliefs), thus she will probably answer "undetermined".

What we need to define to make it formal is a notion of projection function, which characterizes the answers (i.e., the judgment set) an agent can give to the questions of the agenda, depending on her current belief base. We call such projection functions *decision policies*, and our purpose is first to characterize axiomatically the rational ones:

DEFINITION 6. *A decision policy $p : \mathcal{L} \times \mathcal{L} \rightarrow \{0, 1, \star\}$ is a mapping associating an element of $\{0, 1, \star\}$ with any pair of non-trivial formulas (K, φ) and satisfying:*

1. *if $K_1 \equiv K_2$, then $\forall \varphi$, $p(K_1, \varphi) = p(K_2, \varphi)$*

2. *if $\varphi_1 \equiv \varphi_2$, then $\forall K$, $p(K, \varphi_1) = p(K, \varphi_2)$*

3. $p(\varphi, \varphi) = 1$

Conditions 1 and 2 can be viewed as a formal counterpart, respectively, of a neutrality condition and of an anonymity condition for decision policies, but we will refrain from using such a terminology here because of a possible confusion with the corresponding rationality conditions on judgment aggregation methods (in particular, the "neutrality" and "anonymity" conditions here do not entail respectively the neutrality property or the anonymity property of a judgment aggregation correspondence as defined previously).

Given an agenda $X = \{\varphi_1, \ldots, \varphi_m\}$ and a belief base K (respectively a profile $E = (K_1, \ldots, K_n)$ of belief bases), every decision policy p induces a judgment set $p_X(K) = (p(K, \varphi_1), \ldots, p(K, \varphi_m))$ (resp. a profile of judgment sets $p_X(E) = (p_X(K_1), \ldots, p_X(K_n))$).

Examples of decision policies are the following ones:

- $p_B(K, \varphi) = \begin{cases} 1 & \text{if } K \models \varphi \\ 0 & \text{if } K \models \neg\varphi \\ \star & \text{otherwise} \end{cases}$
- $p_C(K, \varphi) = \begin{cases} 1 & \text{if } K \wedge \varphi \not\models \bot \\ 0 & \text{otherwise} \end{cases}$

The *belief decision policy* p_B makes sense when beliefs are considered. According to it an agent answers "yes" (resp. "no") to a given question precisely when it (resp. its negation) is a logical consequence of her belief base; in the remaining case she answers "undetermined".

Observe that with the *consistency decision policy* p_C it is possible to have together $p_C(K_i, \varphi_k) = 1$ and $p_C(K_i, \neg\varphi_k) = 1$ (for instance a belief base equivalent to a is consistent with b and with $\neg b$). In order to avoid this problem, some additional conditions must be ensured:

DEFINITION 7. *Let $p : \mathcal{L} \times \mathcal{L} \rightarrow \{0, 1, \star\}$ be a decision policy. It is a* rational *decision policy if it satisfies the two following conditions:*

4. *if $p(K, \varphi) = 1$, then $p(K, \neg\varphi) = 0$*

5. *If $K_1 \wedge K_2$ is consistent and if $p(K_1, \varphi) = 1$ then $p(K_1 \wedge K_2, \varphi) = 1$*

It turns out that these two additional conditions fully characterize the belief decision policy:

PROPOSITION 2. *p is a rational decision policy iff $p = p_B$.*

PROOF. First, it is easy to check that p_B satisfies the conditions 1 to 5. Second, let p be any rational decision policy. We have to show that $p = p_B$. In this proof, we take advantage of the formulas $\varphi_\omega \equiv \bigvee_{\omega' \in \Omega, \omega' \neq \omega} \omega'$: φ_ω is the formula of $\mathcal{L}$ whose models are all interpretations except ω. Any formula φ can be written as a conjunction of formulas φ_ω, with $\omega \models \neg\varphi$: $\varphi \equiv \bigwedge_{\omega \models \neg\varphi} \varphi_\omega$.

Suppose that $K \models \varphi$. Then any model ω s.t. $\omega \models \neg\varphi$ satisfies $\omega \models \neg K$. We can write: $K \equiv \bigwedge_{\omega \models \neg K} \varphi_\omega \equiv \bigwedge_{\omega \models \neg\varphi} \varphi_\omega \wedge \bigwedge_{\omega \models \neg K \wedge \varphi} \varphi_\omega$.

Then $K \equiv \varphi \wedge \bigwedge_{\omega \models \neg K \wedge \varphi} \varphi_\omega$.

From rules 3 and 5, $p(\varphi \wedge \bigwedge_{\omega \models \neg K \wedge \varphi} \varphi_\omega, \varphi) = 1$. From rule 1, $p(K, \varphi) = 1$. As a consequence, if $K \models \varphi$, then $p(K, \varphi) = 1$.

Suppose that $K \models \neg\varphi$. From the previous point, we know that $p(K, \neg\varphi) = 1$. From rule 4, we deduce that $p(K, \varphi) = 0$.

Finally, suppose that $K \not\models \varphi$ and $K \not\models \neg\varphi$.

Assume that $p(K, \varphi) = 1$. As K is consistent with $\neg\varphi$, from rule 5, we get $p(K \wedge \neg\varphi, \varphi) = 1$. But as $K \wedge \neg\varphi$ is consistent, from rule 3 and 5, $p(K \wedge \neg\varphi, \neg\varphi) = 1$ and from rule 4 we get $p(K \wedge \neg\varphi, \varphi) = 0$: contradiction.

Assume that $p(K, \varphi) = 0$. Then $p(K, \neg\varphi) = 1$ from rule 3, and a demonstration similar to the one above leads to a contradiction. So if $K \not\models \varphi$ and $K \not\models \neg\varphi$, $p(K, \varphi) \neq 1$ and $p(K, \varphi) \neq 0$, so $p(K, \varphi) = \star$. We can thus conclude that $p = p_B$. $\square$

We also have the following expected property when p_B is used:

PROPOSITION 3. *p_B guarantees individual consistency: whatever the belief base K and the agenda X, if γ is the judgment set on X induced by p_B given K, then the associated judgment $\widehat{\gamma}$ is consistent.*

The last two propositions justify to focus on the p_B policy, and this is what we do in the following.

5. BM VS JA: THE GENERAL CASE

Our objective is first to determine whether some logical connections between the formulas φ_Δ and φ_{Ag} exist whenever Δ and Ag are "rational". Especially, we focus on the unanimity condition and the majority preservation condition on Ag which are natural ones.

One first needs to give a couple of notations:

DEFINITION 8. *Let $E = (K_1, \ldots, K_n)$ be a profile of belief bases and let p be a decision policy. Let Δ be a belief merging operator and Ag be a judgment aggregation correspondence. Let $X = \{\varphi_1, \ldots, \varphi_m\}$ be an agenda. There are two ways to define a collective judgment on X (see Figure 1):*

- *if the project-then-aggregate path $Ag \circ p$ is followed, then the output is $\varphi_{Ag} = \widehat{Ag_{p_X(E)}}$,*
- *if the merge-then-project path $p \circ \Delta$ is followed, then the output is $\varphi_\Delta = \widehat{p_X(\Delta(E))}$.*

Each of $Ag \circ p$ and $p \circ \Delta$ can be viewed as an aggregation operator associating with a profile E of belief bases and an agenda X a (set of) collective judgment set(s), interpreted as a propositional formula (a collective judgment). The point is that the two resulting collective judgments are not necessarily compatible, even when Δ and Ag are rational operators. More precisely:

PROPOSITION 4. *There exist an IC merging operator Δ, a judgment aggregation operator Ag satisfying unanimity such that for a profile E of belief bases and a singleton agenda X, $\varphi_{Ag} \wedge \varphi_\Delta$ is inconsistent.*

PROOF. Consider the profile $E = (K_1, K_2, K_3)$ where $K_1 \equiv \neg a \wedge \neg b$, and $K_2 = K_3 \equiv a \wedge b$. Consider also the singleton agenda $X = \{\varphi\}$ with $\varphi = a \Leftrightarrow b$. The merged base $\Delta^{d_H, \Sigma^2}(E)$ obtained using the IC distance-based operator induced by the Hamming distance and Σ^2 as aggregation function is equivalent to $a \Leftrightarrow \neg b$. Thus the projection of $\Delta^{d_H, \Sigma^2}(E)$ on X gives a judgment set leading to $\varphi_\Delta = a \Leftrightarrow \neg b$. However, every belief base K_i is such that $K_i \models \varphi$, thus we have $\varphi_{Ag} \equiv a \Leftrightarrow b$ for every judgment aggregation operator satisfying unanimity.[7] $\square$

PROPOSITION 5. *Let $\triangle^{d,f}$ be a distance-based merging operator with d any normal distance and f any strictly non-decreasing function, and let Ag satisfies majority preservation, one can find a profile E of belief bases and a (singleton) agenda X such that $\varphi_{Ag} \wedge \varphi_\Delta$ is inconsistent.*

PROOF. A simple example is enough to prove that the results may be jointly inconsistent. Consider a profile $E = (K_1, K_2, K_3, K_4, K_5)$ where $K_1 \equiv a \wedge b$, $K_2 \equiv a \wedge \neg b$, $K_3 \equiv \neg a \wedge \neg b$, $K_4 \equiv \neg a \wedge b$ $K_5 \equiv \neg a \wedge b$. The agenda X consists of one question $\varphi = \neg a \wedge b$. The corresponding judgment sets are given in Table 1.

	γ_1	γ_2	γ_3	γ_4	γ_5	$Majority$
φ	0	0	0	1	1	0

Table 1: Judgment sets

The distances one can obtain with any distance-based merging operator are reported in Table 2.

	K_1	K_2	K_3	K_4	K_5
00	$d(00,11)$	$d(00,10)$	0	$d(00,01)$	$d(00,01)$
01	$d(01,11)$	$d(01,10)$	$d(01,00)$	0	0
10	$d(10,11)$	0	$d(10,00)$	$d(10,01)$	$d(10,01)$
11	0	$d(11,10)$	$d(11,00)$	$d(11,01)$	$d(11,01)$

Table 2: Distances

Suppose that d is a normal distance, we note $d_{\{a\}}$, $d_{\{b\}}$ and $d_{\{a,b\}}$ the distance between two interpretations which differ respectively only on the sets $\{a\}$, $\{b\}$ and $\{a, b\}$. The results given in Table 3 can be obtained.

	K_1	K_2	K_3	K_4	K_5
00	$d_{\{a,b\}}$	$d_{\{a\}}$	0	$d_{\{b\}}$	$d_{\{b\}}$
01	$d_{\{a\}}$	$d_{\{a,b\}}$	$d_{\{b\}}$	0	0
10	$d_{\{b\}}$	0	$d_{\{a\}}$	$d_{\{a,b\}}$	$d_{\{a,b\}}$
11	0	$d_{\{b\}}$	$d_{\{a,b\}}$	$d_{\{a\}}$	$d_{\{a\}}$

Table 3: Results

We can observe in Table 3 that the interpretation 01 has a distance to the profile strictly lower than the distance of the profile to any other interpretation. So, for any aggregation function f which is strictly non-decreasing, $\Delta^{d,f}(E) \equiv \neg a \wedge b$. Then $\Delta^{d,f}(E)$ accepts φ whereas any judgment aggregation function respecting majority preservation rejects φ. $\square$

This result is quite important, because most reasonable distances between interpretations (Hamming distance, Drastic distance) are normal ones and most reasonable aggregation functions (Σ, leximax, leximin, Σ^n, ...) satisfy strict non-decreasingness. Thus in the general case the results obtained by using rational IC merging methods can be inconsistent with the results obtained by using rational JA methods.

6. BM VS JA: COMPLETE AGENDAS

Let us now investigate the connections between belief merging and judgment aggregation in the case when the two approaches are equally informed, i.e., when the agenda X gathers all interpretations of Ω.

In the following, in order to simplify the notations, since X is fixed, we write $p(K_i)$ instead of $p_X(K_i)$ and $p(E)$ instead of $p_X(E)$.[8] For any belief base K_i of E and any $\omega \in X$, we have $p(K_i, \omega) = 0$ iff $K_i \models \neg\omega$, i.e., $p(K_i, \omega) = 0$ iff $\omega \not\models K_i$. So $p(K_i, \omega) \neq 0$ iff $\omega \models K_i$. Observe that when questions are interpretations, a belief base K_i that is not complete (i.e., with more than one model) cannot lead to answer 1, but only to $\star$ or to 0. Whatever the case, $p(K_i)$ contains necessarily at most one 1 and at least one 0 (since K_i is supposed to be non-trivial).

We assume in what follows that the judgment aggregation correspondence Ag under consideration satisfies both the collective resoluteness condition and the collective rationality condition. This is a harmless assumption when $X = \Omega$ provided that Ag outputs at least one consistent judgment set (which is not a very demanding condition). Indeed, let $Ag_\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be the set of collective judgment sets given by Ag on the profile Γ of individual judgment sets on X. For each $\gamma_i \in Ag_\Gamma$ such that γ_i is consistent, let γ_i^R be the set of resolute and consistent collective judgment sets obtained by replacing in γ_i precisely one $\star$ by 1 when γ_i does not contain any 1, and the other $\star$ by 0 (so that γ_i^R contains e elements whenever γ_i contains e $\star$ but no 1). Let $Ag_\Gamma^R = \bigcup_{\gamma_i \in Ag_\Gamma} \gamma_i^R$. We have $\widehat{Ag_\Gamma} \equiv \widehat{Ag_\Gamma^R}$. So the collective judgments are the same ones for Ag_Γ and Ag_Γ^R and this explains why one can safely suppose that collective resoluteness holds. We call *abstention-free* correspondence associated with Ag the judgment aggregation correspondence which associates Ag_Γ^R with the input profile Γ.

Interestingly, when $X = \Omega$, we can recover the belief base of any agent from her judgment set γ (and not just deduce her judgment set from her belief base, unlike what happens in the general case). Thus the inverse mapping p^{-1} of p can be defined as follows (up to logical equivalence): $[p^{-1}(\gamma)] = \{\omega \in \Omega \mid \gamma(\omega) \neq 0\}$. $[p^{-1}(\gamma)]$ is the set of models of the belief base of the agent reporting the judgment set γ.

[7]This proof uses a majority operator, but the same example holds for the arbitration operator $\Delta^{d_H, Gmax}$.

[8]Remember that p denotes here the belief decision policy.

On this ground, one can define a judgment aggregation correspondence $Ag = Ag^{\Delta}$ from a merging operator Δ and a merging operator $\Delta = \Delta^{Ag}$ from a judgment aggregation correspondence Ag. Given an interpretation ω (also viewed as a formula), let the induced judgment set γ_ω be equal to $p(\omega)$. By construction, $[\widehat{\gamma_\omega}] = \{\omega\}$, thus γ_ω is consistent.

DEFINITION 9.
- *Given a merging operator Δ and a profile $E = (K_1, \ldots, K_n)$, we define $Ag^{\Delta}_{p(E)} = \{\gamma_\omega \mid \omega \models \Delta(E)\}$= $\{\gamma_\omega \mid p(\Delta(E), \omega) \neq 0\}$.*
- *Given a judgment aggregation correspondence Ag and a profile $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of judgment sets, we note $p^{-1}(\Gamma) = (p^{-1}(\gamma_1), \ldots, p^{-1}(\gamma_n))$ and $\Delta^{Ag}(p^{-1}(\Gamma)) = \{\omega \in \Omega \mid \gamma_\omega \in Ag_\Gamma\}$.*

It is easy to prove that in the case $X = \Omega$ the two aggregation paths corresponding respectively to Δ and to Ag^{Δ} lead to equivalent results:

PROPOSITION 6. *Let X be a complete agenda. Let $\varphi_{Ag} = \widehat{Ag^{\Delta}_{p(E)}}$ and $\varphi_\Delta = p(\widehat{\Delta(E)})$. We have $\varphi_{Ag} \equiv \varphi_\Delta$.*

Furthermore, when X is the complete agenda Ω, every belief merging operator corresponds to one judgment aggregation correspondence, and vice-versa. More precisely, we have that:

PROPOSITION 7. Δ=$\Delta^{(Ag^{\Delta})}$ *and* Ag=$Ag^{(\Delta^{Ag})}$

Thus, Definition 9 induces a one-to-one mapping between the merging operator and the corresponding judgment aggregation correspondence. This bijection will be used in the following to show how IC postulates and judgment aggregation properties are related.

Let us now parse the IC postulates and determine their counterparts in judgment aggregation (when they exist):

(IC0) By construction of Δ^{Ag} (IC0) is satisfied, so (IC0) does not correspond to any non-trivial condition on Ag.

(IC1) If μ is consistent, then $\triangle_\mu(E)$ is consistent

PROPOSITION 8. *Δ^{Ag} satisfies (IC1) iff Ag satisfies universal domain.*

PROOF. Suppose that Δ^{Ag} does not satisfy (IC1). There exists a profile $E = (K_1, \ldots, K_n)$, such that there is no model in $\Delta^{Ag}(E)$. Thus $\forall\omega \in \Omega, \gamma_\omega \notin Ag_{p(E)}$. So $Ag_{p(E)}$ is empty. Hence Ag does not satisfy universal domain.

Conversely, suppose that Ag does not satisfy universal domain. Then there is a profile of judgment sets $\Gamma = (\gamma_1, \ldots, \gamma_n)$ on $X = \Omega$ s.t. Ag_Γ is empty. Therefore there is a profile $E = (K_1, \ldots, K_n)$ of belief bases such that $K_1 = p^{-1}(\gamma_1)$, ..., $K_n = p^{-1}(\gamma_n)$ and $\Delta^{Ag}(p^{-1}(\Gamma))$ is inconsistent. So (IC1) is not satisfied. □

(IC2) Let us define an additional property for JA methods:

DEFINITION 10. *Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a profile of judgment sets on an agenda X.*

- $\varphi \in X$ *is* unanimous *for* Γ *iff* $\forall i \in \{1, \ldots, n\}$, $\gamma_i(\varphi) \neq 0$.
- Γ *is* consensual *iff there exists $\varphi \in X$ which is unanimous for* Γ.
- *A judgment aggregation correspondence Ag satisfies* consensuality *iff for every consensual profile Γ of judgment sets on an agenda X, for every $\varphi \in X$, $Ag_\Gamma(\varphi) \neq 0$ iff φ is unanimous for* Γ.

PROPOSITION 9. *Δ^{Ag} satisfies (IC2) iff Ag satisfies consensuality.*

PROOF. Suppose that Δ^{Ag} satisfies (IC2). Consider a consensual profile $\Gamma = (\gamma_1, \ldots, \gamma_n)$ on Ω and suppose that ω is one of the unanimous interpretations. As ω is unanimous, $\forall i \in \{1, \ldots, n\}$, $\gamma_i(\omega) \neq 0$. Then $\forall i \in \{1, \ldots, n\}$, $\omega \models p^{-1}(\gamma_i)$, so $\bigwedge p^{-1}(\gamma_i)$ is consistent, and since Δ^{Ag} satisfies (IC2), Δ^{Ag} $((p^{-1}(\gamma_1), \ldots, p^{-1}(\gamma_n))) \equiv \bigwedge p^{-1}(\gamma_i)$.
Hence, the models of $\Delta^{Ag}((p^{-1}(\gamma_1), \ldots, p^{-1}(\gamma_n)))$ are unanimous interpretations. Then for each unanimous interpretation ω, $\gamma_\omega \in Ag_\Gamma$ and for each non-unanimous interpretation ω, $\gamma_\omega \notin Ag_\Gamma$: Ag is consensual.

Conversely, suppose that Ag is consensual. Consider a profile $E = (K_1, \ldots, K_n)$ s.t. $\bigwedge E$ is consistent. Let $\omega \models \bigwedge E$. As $\forall i \in \{1, \ldots, n\}$, $\omega \models K_i$, we have that $\forall i \in \{1, \ldots, n\}$, $\gamma(K_i)(\omega) \neq 0$. Since Ag is consensual, we have $Ag_\Gamma(\omega) \neq 0$ iff ω is unanimous for Γ iff $\omega \models \bigwedge E$. Hence, $[\Delta^{Ag}(E)] = \{\omega \mid \omega$ is unanimous for $\Gamma\} = [\bigwedge E]$: (IC2) is satisfied. □

(IC3) If $E_1 \equiv E_2$, then $\Delta(E_1) \equiv \Delta(E_2)$

PROPOSITION 10. *Δ^{Ag} satisfies (IC3) iff Ag satisfies anonymity.*

PROOF. Suppose that Ag satisfies anonymity. Suppose $E_1 \equiv E_2$. Since $E_1 \equiv E_2$, the profiles of judgment sets $\Gamma_1 = p(E_1)$ and $\Gamma_2 = p(E_2)$ are permutations of each other. Since Ag satisfies anonymity, $Ag_{\Gamma_1} = Ag_{\Gamma_2}$, hence $\Delta^{Ag}(E_1) \equiv \Delta^{Ag}(E_2)$ and (IC3) is satisfied.

Conversely, suppose that Δ^{Ag} satisfies (IC3). Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_n)$ be two profiles which are permutations of each other. Then $E_1 = p^{-1}(\Gamma)$ and $E_2 = p^{-1}(\Gamma')$ are equivalent. Since Δ^{Ag} satisfies (IC3), we have $\Delta^{Ag}(E_1) \equiv \Delta^{Ag}(E_2)$ and $Ag_\Gamma = Ag_{\Gamma'}$. □

(IC4) The neutrality condition on Ag is not sufficient to ensure that Δ^{Ag} satisfies (IC4).

(IC5) Let us now define two additional properties for JA operators, based on the consistency condition for voting methods [14, 1]. These two properties correspond respectively to (IC5) and (IC6).

Weak consistency. Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_p)$ be two profiles of judgment sets on an agenda X and in the domain of Ag. For any $\varphi \in X$, if $Ag_\Gamma(\varphi) = 1$ and $Ag_{\Gamma'}(\varphi) = 1$, then $Ag_{\Gamma \sqcup \Gamma'}(\varphi) = 1$.

This property states that if a formula is not accepted by a profile Γ, and by a profile Γ', then it must be accepted by the union of the profiles.

Consistency. Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_p)$ be two profiles of judgment sets on an agenda X and in the domain of Ag. If there is $\varphi \in X$ s.t. $Ag_\Gamma(\varphi) = 1$ and $Ag_{\Gamma'}(\varphi) = 1$, then for every $\psi \in X$, if $Ag_{\Gamma \sqcup \Gamma'}(\psi) = 1$ then $Ag_\Gamma(\psi) = 1$ and $Ag_{\Gamma'}(\psi) = 1$.

This property states that if there is at least a formula that is accepted by two subprofiles Γ and Γ', then each formula that is accepted by the whole profile $\Gamma \sqcup \Gamma'$ should be accepted by each of the two subprofiles Γ and Γ'.

Quite surprisingly these conditions have not been considered as standard ones for judgment aggregation methods (we are only aware of [9, 13] which point out the consistency condition, under the name "separability").

PROPOSITION 11. *Δ^{Ag} satisfies (IC5) iff Ag satisfies weak consistency*

PROOF. Suppose that Δ satisfies (IC5). Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_p)$ be any profiles of judgment sets, and let $\omega \in \Omega$ be any interpretation s.t. $\gamma_\omega \in Ag^{\Delta}_{\Gamma}$ and $\gamma_\omega \in Ag^{\Delta}_{\Gamma'}$.

Let $E_1 = (p^{-1}(\gamma_1), \ldots, p^{-1}(\gamma_n))$ and $E_2 = (p^{-1}(\gamma'_1), \ldots, p^{-1}(\gamma'_p))$. As $\gamma_\omega \in Ag^{\Delta}_{\Gamma}$ and $\gamma_\omega \in Ag^{\Delta}_{\Gamma'}$, we have $\omega \models \Delta(E_1)$ and $\omega \models \Delta(E_2)$. So $\omega \models \Delta(E_1) \wedge \Delta(E_2)$. With (IC5), $\omega \models \Delta(E_1 \sqcup E_2)$, and $\gamma_\omega \in Ag^{\Delta}_{\Gamma \sqcup \Gamma'}$: Ag^{Δ} satisfies weak consistency.

Conversely, suppose Ag that satisfies weak consistency. Let $E_1 = (K_1, \ldots, K_n)$ and $E_2 = (K'_1, \ldots, K'_p)$ be two profiles of belief bases. If $\Delta^{Ag}(E_1) \wedge \Delta^{Ag}(E_2)$ is not consistent, then (IC5) is satisfied. Otherwise, let ω be any model of $\Delta^{Ag}(E_1) \wedge \Delta^{Ag}(E_2)$. Then for $\Gamma = (p(K_1), \ldots, p(K_n))$ and $\Gamma' = (p(K'_1), \ldots, p(K'_p))$, we get $\gamma_\omega \in Ag_{\Gamma}$ (as a consequence, $Ag_{\Gamma}(\omega) = 1$) and $\gamma_\omega \in Ag_{\Gamma'}$ (so $Ag_{\Gamma'}(\omega) = 1$). Since Ag satisfies weak consistency, we have $Ag_{\Gamma \sqcup \Gamma'}(\omega) = 1$ and $\gamma_\omega \in Ag_{\Gamma \sqcup \Gamma'}$. Hence $\omega \models \Delta^{Ag}(E_1 \sqcup E_2)$ and (IC5) is satisfied. $\square$

(IC6) If $\triangle(E_1) \wedge \triangle(E_2)$ is consistent, then $\triangle(E_1 \sqcup E_2) \models \triangle(E_1) \wedge \triangle(E_2)$

PROPOSITION 12. *Δ^{Ag} satisfies (IC6) iff Ag satisfies consistency*

PROOF. Suppose that Δ satisfies (IC6). Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_p)$ be two profiles of judgment sets. Suppose that there is $\omega \in \Omega$ s.t. $Ag^{\Delta}_{\Gamma}(\omega) = 1$ (i.e., $\gamma_\omega \in Ag^{\Delta}_{\Gamma}$) and $Ag^{\Delta}_{\Gamma'}(\omega) = 1$ (i.e., $\gamma_\omega \in Ag^{\Delta}_{\Gamma'}$). Let $E_1 = (p^{-1}(\gamma_1), \ldots, p^{-1}(\gamma_n))$ and $E_2 = (p^{-1}(\gamma'_1), \ldots, p^{-1}(\gamma'_p))$. Since $\gamma_\omega \in Ag^{\Delta}_{\Gamma}$ and $\gamma_\omega \in Ag^{\Delta}_{\Gamma'}$, we have that $\Delta(E_1) \wedge \Delta(E_2)$ is consistent. Due to (IC6), $\Delta(E_1 \sqcup E_2) \models \Delta(E_1) \wedge \Delta(E_2)$. As a consequence, any interpretation ω' s.t. $Ag^{\Delta}_{\Gamma \sqcup \Gamma'}(\omega') = 1$ (i.e., $\gamma_{\omega'} \in Ag^{\Delta}_{\Gamma \sqcup \Gamma'}$) is a model of $\Delta(E_1 \sqcup E_2)$, so it is a model of $\Delta(E_1) \wedge \Delta(E_2)$. Then $\gamma_{\omega'} \in Ag^{\Delta}_{\Gamma}$. Hence $Ag^{\Delta}_{\Gamma}(\omega) = 1$ and $\gamma_{\omega'} \in Ag^{\Delta}_{\Gamma'}$ so $Ag^{\Delta}_{\Gamma'}(\omega) = 1$: Ag^{Δ} satisfies consistency.

Conversely, suppose that Ag satisfies consistency. Let $E_1 = (K_1, \ldots, K_n)$ and $E_2 = (K'_1, \ldots, K'_p)$ be two profiles of belief bases s.t. $\Delta^{Ag}(E_1) \wedge \Delta^{Ag}(E_2)$ is consistent. If $\Gamma = (p(K_1), \ldots, p(K_n))$ and $\Gamma' = (p(K'_1), \ldots, p(K'_p))$, then there is $\omega \in \Omega$ s.t. $\gamma_\omega \in Ag_{\Gamma}$ and $\gamma_\omega \in Ag_{\Gamma'}$. Let ω' be any model of $\Delta^{Ag}(E_1 \sqcup E_2)$. We have $\gamma_{\omega'} \in Ag_{\Gamma \sqcup \Gamma'}$. As Ag satisfies consistency, we have $\gamma_{\omega'} \in Ag_{\Gamma}$ and $\gamma_{\omega'} \in Ag_{\Gamma'}$. So $\omega' \models \Delta^{Ag}(E_1)$ and $\omega' \models \Delta^{Ag}(E_2)$: $\omega' \models \Delta^{Ag}(E_1) \wedge \Delta^{Ag}(E_2)$. Hence Δ^{Ag} $(E_1 \sqcup E_2)$ $\models \Delta^{Ag}(E_1) \wedge \Delta^{Ag}(E_2)$ and (IC6) is satisfied. $\square$

We do not consider (IC7) and (IC8) since they are obviously satisfied by any merging operator when the integrity constraints μ_1 and μ_2 are valid.

The following proposition sums up the results:

PROPOSITION 13.

- *If Ag satisfies collective rationality, consensuality, anonymity, neutrality, weak consistency, consistency, then Δ^{Ag} satisfies (IC0) to (IC3) and (IC5), (IC6).*
- *If Δ is an IC merging operator, then Ag^{Δ} satisfies collective rationality, consensuality, anonymity, weak consistency, and consistency.*

7. ON THE COMPATIBILITY OF BM AND JA

In Proposition 13 we pointed out a list of properties required for a JA operator to correspond to an IC merging operator in the complete agenda case. A key question is whether these properties can be satisfied by some judgment aggregation operator.

We give a positive answer to this issue, considering some JA correspondences $\delta^{RM_{\oplus}}$ defined in [3]. Roughly, each $\delta^{RM_{\oplus}}$ correspondence consists in selecting in the set of all consistent and resolute judgment sets the "best score" ones, where the score of each judgment set is defined as the $\oplus$-aggregation of an m-vector of values (one value per question in the agenda X, reflecting the number of agents supporting the question in the input profile Γ). Note that, by construction, the sets of collective judgment sets computed using $\delta^{RM_{\oplus}}$ contain only consistent and resolute judgment sets (thus, $\delta^{RM_{\oplus}}$ coincides with the abstention-free correspondence associated with it). Finally, when $X = \Omega$, the set of all consistent and resolute judgment sets coincide with $\{\gamma_\omega \mid \omega \in \Omega\}$.

PROPOSITION 14. *When the agenda is complete, for any $\oplus$ satisfying strict non-decreasingness, the ranked majority judgment aggregation correspondence $\delta^{RM_{\oplus}}$ satisfies universal domain, collective rationality, collective resoluteness, anonymity, neutrality, unanimity, consensuality, and majority preservation. It does not satisfy independence. For $\oplus = \Sigma$, weak consistency and consistency are also satisfied.*

PROOF. Collective rationality and collective resoluteness are satisfied by construction by any $\delta^{RM_{\oplus}}$ correspondence. For universal domain, anonymity, neutrality, and majority preservation, the results are given in [3].

For unanimity, consider a profile Γ of individual judgment sets. Suppose first that there exists $\omega \in X$ s.t. $\forall \gamma_i \in \Gamma$, we have $\gamma_i(\omega) = 1$. This implies that ω is the unique model of each belief base K_i, and as a consequence, for any $\omega' \in X$ s.t. $\omega' \neq \omega$, we have $\gamma_i(\omega') = 0$. Thus, all the input judgment sets γ_i of Γ coincide, and are equal to the judgment set γ_ω where only ω is supported. The score of any other $\gamma_{\omega'}$ is thus strictly lower than the score of γ_ω, which is the unique judgment set which is selected by $\delta^{RM_{\oplus}}$. As expected, we have $\gamma_\omega(\omega) = 1$. Now, suppose that there is no $\omega \in X$ s.t. $\forall \gamma_i \in \Gamma$, we have $\gamma_i(\omega) = 1$ but there exists at least one $\omega \in X$ s.t. $\forall \gamma_i \in \Gamma$, we have $\gamma_i(\omega) = 0$. Let $\{\omega_{u_1}, \ldots, \omega_{u_k}\}$ be the set of all $\omega_{u_j} \in X$ s.t. $\forall \gamma_i \in \Gamma$, we have $\gamma_i(\omega_{u_j}) = 0$. To get the result, we have to show that the score of any γ_ω, where $\omega \notin \{\omega_{u_1}, \ldots, \omega_{u_k}\}$ is strictly greater than the score of any $\gamma_{\omega_{u_j}}$ ($j \in \{1, \ldots, k\}$). Observe that such a γ_ω necessarily exists, because Γ contains consistent judgment sets γ_i: it cannot be the case that $\gamma_i(\omega) = 0$ for every $\omega \in \Omega$. Now, by construction, γ_ω and $\gamma_{\omega_{u_j}}$ differ only on ω and ω_{u_j}. Since $\oplus$ is symmetric in each argument and strictly non-decreasing, it is enough to compare the supports of ω and ω_{u_j} in Γ in order to compare the scores of γ_ω and $\gamma_{\omega_{u_j}}$. The number of judgment sets γ_i in Γ agreeing with γ_ω on ω_{u_j} is n. The number of judgment sets γ_i in Γ agreeing with γ_ω on ω is a, in the range 0 to $n-1$. The number of judgment sets γ_i in Γ agreeing with $\gamma_{\omega_{u_j}}$ on ω_{u_j} is 0. The number of judgment sets γ_i in Γ agreeing with $\gamma_{\omega_{u_j}}$ on ω is b, in the range 0 to $n-1$. Thus the two vectors of scores associated with γ_ω and $\gamma_{\omega_{u_j}}$ contain the same values except that the one corresponding to γ_ω contains n, a where the one corresponding to $\gamma_{\omega_{u_j}}$ contains 0, b. Now, a is at least equal to 0 and b is at most equal to $n-1$. Since $\oplus$ is symmetric in each argument and strictly non-decreasing we get that the score of γ_ω is strictly greater than the score of $\gamma_{\omega_{u_j}}$, so that $\gamma_{\omega_{u_j}}$ cannot be selected. Since for every γ_ω where $\omega \notin \{\omega_{u_1}, \ldots, \omega_{u_k}\}$ and for every $j \in \{1, \ldots, k\}$ we have that $\gamma_\omega(\omega_{u_j}) = 0$, the result follows.

For consensuality, consider a consensual profile $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of judgment sets on $X = \Omega$ and a unanimous interpretation $\omega_u \in X$ for Γ. For each $\gamma_i \in \Gamma$, if $\gamma_i(\omega_u) = 1$ then for every $\omega \in X$ s.t. $\omega \neq \omega_u$, we have $\gamma_i(\omega) = 0$, and if $\gamma_i(\omega_u) = \star$ then for every $\omega \in X$ s.t. $\omega \neq \omega_u$, we have $\gamma_i(\omega) \neq 1$; indeed, if we had $\gamma_i(\omega) = 1$ for some $\omega \in X$ s.t. $\omega \neq \omega_u$, then we would have $[K_i] = \{\omega\}$, and as a consequence we would have $\gamma_i(\omega_u) = 0$. If the set of unanimous interpretations $\{\omega_{u_1}, \ldots, \omega_{u_k}\}$ is not a singleton, then for each $\gamma_i \in \Gamma$ and for each unanimous interpretation ω_{u_i} ($i \in \{1, \ldots, k\}$) we have $\gamma_i(\omega_{u_i}) = \star$. Consider now two judgment sets γ_ω and $\gamma_{\omega'}$ for $\omega, \omega' \in \Omega$, with $\omega \neq \omega'$. By construction, γ_ω and $\gamma_{\omega'}$ differ only on ω and ω'. Since $\oplus$ is symmetric in each argument and strictly non-decreasing, it is enough to compare the supports of ω and ω' in Γ in order to compare the scores of γ_ω and $\gamma_{\omega'}$. Suppose first that ω is a unanimous interpretation and that ω' is not. The number of judgment sets γ_i in Γ agreeing with γ_ω on ω is in the range 0 to n. The number of judgment sets γ_i in Γ agreeing with γ_ω on ω' is in the range 1 to n. The number of judgment sets γ_i in Γ agreeing with $\gamma_{\omega'}$ on ω is 0. The number of judgment sets γ_i in Γ agreeing with $\gamma_{\omega'}$ on ω' is 0. Since $\oplus$ is strictly non-decreasing we get that the score of γ_ω is strictly greater than the score of $\gamma_{\omega'}$, so that $\gamma_{\omega'}$ cannot be selected. Suppose now that ω and ω' are two unanimous interpretations. We have $\gamma_\omega(\omega) = \gamma_\omega(\omega') = \gamma_{\omega'}(\omega) = \gamma_{\omega'}(\omega') = \star$. As a consequence, the score of γ_ω is equal to the score of $\gamma_{\omega'}$, and both judgment sets are selected. To sum up, the resulting set $\delta_\Gamma^{RM\oplus}$ of collective judgment sets is equal to $\{\gamma_\omega \mid \omega$ is a unanimous interpretation for $\Gamma\}$, showing that $\delta_\Gamma^{RM\oplus}$ satisfies consensuality.

For independence, consider the following profiles $\Gamma = (\gamma_1, \gamma_2, \gamma_3)$ and $\Gamma' = (\gamma'_1, \gamma'_2, \gamma'_3)$ on the same complete agenda $X = \{\neg a \wedge \neg b, \neg a \wedge b, a \wedge \neg b, a \wedge b\}$.

	γ_1	γ_2	γ_3
$\neg a \wedge \neg b$	0	0	$*$
$\neg a \wedge b$	$*$	0	$*$
$a \wedge \neg b$	$*$	$*$	0
$a \wedge b$	0	$*$	0

	γ'_1	γ'_2	γ'_3
$\neg a \wedge \neg b$	0	0	$*$
$\neg a \wedge b$	$*$	0	$*$
$a \wedge \neg b$	$*$	$*$	0
$a \wedge b$	$*$	$*$	$*$

For each $i \in \{1, 2, 3\}$, we have $\gamma_i(a \wedge \neg b) = \gamma'_i(a \wedge \neg b)$. However, we have $\delta_\Gamma^{RM\oplus}(a \wedge \neg b) \neq \delta_{\Gamma'}^{RM\oplus}(a \wedge \neg b)$. □

Let us now step back to the general case, when the agenda X is not complete. First, let us observe that in this case, no JA correspondence can satisfy both consensuality and majority preservation.

PROPOSITION 15. *The consensuality property and the majority preservation property cannot be satisfied together in the general case.*

PROOF. Consider two propositional variables a and b in P, and a profile $\Gamma = (\gamma_1, \gamma_2, \gamma_3)$ consisting of three individual judgment sets. a is unanimous for Γ, but b receives also a majority of votes. Let Ag be a JA correspondence. The consensuality property requires that $Ag_\Gamma(b) = 0$, whereas the majority preservation property requires that $Ag_\Gamma(b) = 1$.

	γ_1	γ_2	γ_3
a	1	1	1
b	1	1	0

□

Unsurprisingly, unanimity and consensuality are connected:

PROPOSITION 16. *Consensuality implies unanimity.*

Unfortunately, the quite good behaviour of $\delta^{RM\oplus}$ in the complete agenda case does not lift to the general case:

PROPOSITION 17. *In the general case, $\delta^{RM\oplus}$ satisfies universal domain, collective rationality, collective resoluteness, anonymity, neutrality. For any $\oplus$ satisfying strict non-decreasingness, $\delta^{RM\oplus}$ satisfies majority preservation, but does not satisfy weak consistency, consistency, or consensuality. If $\oplus$ satisfies strict preference and decomposition, then $\delta^{RM\oplus}$ satisfies unanimity. Finally, $\delta^{RM\oplus}$ does not satisfy independence.*

8. CONCLUSION AND DISCUSSION

We investigated the relationships between propositional merging operators and judgment aggregation ones. This required the definition of a projection function. We pointed out some natural requirements on it and showed that there exists a unique projection function satisfying them. Starting with a profile of belief bases and an agenda, we showed that the beliefs generated from the merged base projected onto the agenda are in general hardly compatible with the beliefs obtained by aggregating the judgments obtained by projecting each base first. The majority preservation property, which is natural and advocated as an important property for judgment aggregation, is at the core of such an incompatibility when incomplete agendas are considered. Focussing on the fully informative case (when the agenda consists of all possible interpretations) we showed that a close correspondence between some IC merging postulates and some judgment aggregation properties is reached.

We did not focus on the merging postulates (IC7) and (IC8) in the paper in order to obtain a fair comparison of belief merging and judgment aggregation (indeed, judgment aggregation does not take account for such integrity constraints). However we believe that one fundamental distinction between belief merging and judgment aggregation lies in these two postulates. Especially, these two postulates mainly formalize that the notion of closeness to the given profile of belief bases considered in belief merging is independent from the chosen integrity constraint. Accordingly, merging can be viewed as a two step process: one first measures the closeness of *each* interpretation to the profile, and then the integrity constraints are exploited to retain the closest interpretations *amongst* the models of the constraints. Contrastingly, for judgment aggregation methods, the agenda (whose role is somehow related to integrity constraints in belief merging, in the sense that it reduces the scope of investigation of the aggregation) defines what "close to the profile" of individual judgment sets means. This is inherent to the fact that judgment aggregation is based on a partially informed setting (in the general case), where the only information provided by the agents are the individual judgment sets on the questions of the agenda. To make it more formal, let us give a translation of (IC7) and (IC8) in terms of judgment aggregation (in the case of complete agendas):

Sen's property $\boldsymbol{\alpha}$. Let Γ be a profile of judgment sets on an agenda X and let $X' \subseteq X$. Let $\varphi \in X$ s.t. $Ag_\Gamma(\varphi) = 1$. If $\varphi \in X'$, then $Ag_{\Gamma_{X'}}(\varphi) = 1$.

This property states that if a formula is accepted given a agenda X then it should remain accepted in any subagenda $X' \subseteq X$.

Sen's property $\boldsymbol{\beta}$. Let Γ be a profile of judgment sets on an agenda X and let $X' \subseteq X$. Let $\varphi_1, \varphi_2 \in X'$ s.t. $Ag_{\Gamma_{X'}}(\varphi_1) = 1$ and $Ag_{\Gamma_{X'}}(\varphi_2) = 1$. Then $Ag_\Gamma(\varphi_1) = 1$ iff $Ag_\Gamma(\varphi_2) = 1$.

This property states that if two formulas are accepted when a subagenda X' is considered, and that one of these two formulas is also accepted when X is considered, then the other formula should be also accepted in this case.

Basically, in the complete agenda case, Sen's property $\boldsymbol{\alpha}$ corresponds to (IC7), and Sen's property $\boldsymbol{\beta}$ corresponds to (IC8) (in the presence of (IC7)). However, none of these properties can be considered as reasonable for judgment aggregation, because they do not take into account the interactions between formulas of the agenda. For instance Sen's property $\boldsymbol{\beta}$ does not take account for the fact that φ_1, φ_2 may interact differently with the formulas of the agenda X which are not in X', which justifies the fact that φ_1, φ_2 are not necessarily expected to be treated in the same way. Thus, Sen's property $\boldsymbol{\alpha}$ and Sen's property $\boldsymbol{\beta}$ properties lead to similar problems as systematicity [3] and should not be required.

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