Altruistic Hedonic Games

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ABSTRACT

Hedonic games are coalition formation games in which players have preferences over the coalitions they can join. All models of representing hedonic games studied so far are based upon selfish players only. Among the known ways of representing hedonic games compactly, we focus on friend-oriented hedonic games and propose a novel model for them that takes into account not only a player's own preferences but also her friends' preferences under three degrees of altruism. We study both the axiomatic properties of these games and the computational complexity of problems related to various stability concepts.

Keywords

Hedonic games; coalition formation; stability; altruism

1. INTRODUCTION

Hedonic games, proposed by Drèze and Greenberg [17] and later formally modelled by Bogomolnaia and Jackson [10] and Banerjee et al. [8], are coalition formation games in which players have preferences over coalitions (subsets of players) they can be part of. In the context of decentralized coalition formation, several stability concepts and representations have been studied from an axiomatic and a computational complexity point of view; see Woeginger's survey [33] on this topic and the book chapters by Aziz and Savani [6] and Elkind and Rothe [18] for an overview.

Dimitrov et al. [16] proposed a model that allows for compact representation of hedonic games, namely, the friend-and-enemy encoding of the players' preferences, where each player divides the set of players into friends and enemies. Based on such a network of friends, they suggest two models of preference extensions: appreciation of friends and aversion to enemies. In *friend-oriented hedonic games*, a coalition A is preferred to another coalition B if A contains either more friends than B or the same number of friends as B but fewer enemies than B. This setting corresponds to a network of players represented as a graph. Since we study symmetric friendship relations for stability reasons, this graph is undirected. For example, suppose there are four players, 1, 2, 3, and 4, and let 1 be friends with 2 but neither with 3 nor with 4, while 2 and 3 are friends with each other but not with 4. The corresponding network is displayed in Figure 1.

1 —— 2 —— 3   4

Figure 1: Example of a network of friends

Now, in the friend-oriented extension model player 2 prefers teaming up with 1 and 3 to forming a coalition with 1 and 4. Player 1, on the other hand, is indifferent between coalitions $\{1,2,3\}$ and $\{1,2,4\}$. Intuitively, however, 1 would have an advantage from being in a coalition with 2 and 3, since 2 and 3—being friends—can be expected to cooperate better than 2 and 4. Also, 1 can be expected to care about her friend 2's interests and thus might prefer a coalition in which 2 is satisfied ($\{1,2,3\}$) to one in which 2 is less satisfied ($\{1,2,4\}$). In order to model these kinds of preferences we will introduce several degrees of altruism, starting from friend-oriented hedonic games. Taking friends' preferences into account does not contradict the idea of hedonic games: In hedonic games player i's utility function depends only on the coalitions that contain i. Since player i is also interested in her friends' satisfaction (with varying degrees), we incorporate this notion into player i's utility function. Note that player i's utility is still a function of the coalitions containing her, which in addition takes her friends' preferences that are in the *same* coalition as player i into account.

Our Contribution and Related Work: Focusing on the friend-oriented encoding of preferences and taking the idea of players caring about their friends' preferences into account, we propose hedonic games with altruistic influences. In particular, we define three degrees of altruism, from being selfish first, over aggregating opinions of a player and her friends equally, to altruistically letting one's friends decide first. The latter is the most altruistic case, as we assume that from a player's perspective only friends can be consulted, while agents further away (such as a friend's friend that is one's enemy) cannot be communicated with or cannot be trusted. In a social network, for example, the whole set of players other than friends might not even be known. The proposed games are compactly representable but not fully expressive. However, they can express other hedonic games than those representable by popular compact representations in the literature. We study both the axiomatic properties of these games and the computational complexity of problems related to common stability concepts.

From a *noncooperative* game-theoretic point of view, the interests of not only selfish, but altruistic agents have been modelled and studied by, for example, Hoefer and Skopalik [24], Chen et al. [14], Apt and Schäfer [2], and Rahn and Schäfer [30]. Salehi-Abari and Boutilier [31] study social choice with empathetic preferences. Their local empathetic model is related to our model. Altruism has also been studied in (experimental) economics [27]. Brânzei and Larson [12] study social distance games: In contrast to degrees of altruism as proposed here, a player's opinion on her friends (players of distance one) has the highest weight while her opinion on play-

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ers farther away counts less. This is similar (but not equivalent) to our selfish-first model to be defined in Section 3.2.

Furthermore, the study of other agents' influence on opinions has gained increasing interest in collective decision making [22, 23]. In the context of voting scenarios, preference extensions and their properties have been studied by Endriss [20]. An overview of axiomatic properties of preference orders can be found, e.g., in the book chapters by Barberà et al. [9] and Lang and Rothe [26]. The work by Darmann et al. [15] combines aspects of voting theory and the theory of coalition formation games: They define a model for selection scenarios for a number of group activities that can also be represented by hedonic games and they study the complexity of stability concepts in this model. Aziz et al. [5] provide a survey of known results for additively separable hedonic games; in particular, two impressive complexity results—$\mathrm{NP}^{\mathrm{NP}}$-completeness for the existence of (strictly) core-stable coalition structures—are due to Woeginger [34] and Peters [28]. The concept of Pareto optimality has been studied by Aziz et al. [3] for a number of encodings of hedonic games, while recent work of Brandl et al. [11] is concerned with the complexity of various stability concepts in fractional hedonic games [4]. Lang et al. [25] introduce a new type of hedonic game where agents rank their friends and their enemies (and may, in addition, feel "neutral" about some other players), and these preferences over players are extended to preferences over coalitions.

2. PRELIMINARIES

A *hedonic game* is a pair $(N,\succeq)$, where $N=\{1,\dots,n\}$ is a set of players and $\succeq\ = (\succeq_1,\dots,\succeq_n)$ is a list of the players' preferences. For $i \in N$, let $\mathcal{N}^i = \{C \subseteq N \mid i \in C\}$ denote the set of coalitions containing i. Player i's preference relation $\succeq_i\ \in \mathcal{N}^i \times \mathcal{N}^i$ induces a complete, weak preference order over $\mathcal{N}^i$. For $A,B \in \mathcal{N}^i$, we say that player i *weakly prefers* A *to* B if $A \succeq_i B$, that i *prefers* A *to* B ($A \succ_i B$) if $A \succeq_i B$ but not $B \succeq_i A$, and that i is *indifferent between* A *and* B ($A \sim_i B$) if $A \succeq_i B$ and $B \succeq_i A$. We call $C \in \mathcal{N}^i$ *acceptable for player* i if $C \succeq_i \{i\}$. A *coalition structure* is a partition $\Gamma = \{C_1,\dots,C_k\}$ of the players into k coalitions $C_1,\dots,C_k \subseteq N$ (i.e., $\bigcup_{r=1}^k C_r = N$ and $C_r \cap C_s = \emptyset$ for all r and s, $1 \le r \ne s \le k$). The unique coalition in Γ containing player $i \in N$ is denoted by $\Gamma(i)$.

In order to avoid exponentially large preference orders in the number of players, a common way to represent players' preferences is to consider a network of friends [16]. Each player $i \in N$ has a set of friends $F_i \subseteq N \setminus \{i\}$ and a set of enemies $E_i = N \setminus (F_i \cup \{i\})$. Visually, let the players in $N=\{1,\dots,n\}$ be represented by the vertices in a graph $G=(N,H)$, and let a directed edge $(i,j) \in H$ denote that j is i's friend, that is, the open neighborhood of i represents the set of i's friends $F_i = \{j \mid (i,j) \in H\}$. Since in the context of stability it is reasonable to consider symmetric friendship relations [33], we will focus on undirected graphs representing networks of friends. In the *friend-oriented* preference extension [16] more friends are preferred to fewer friends, and in case of an equal number of friends, fewer enemies are preferred. Formally, define

$$A \succeq_i^F B \iff |A\cap F_i| > |B \cap F_i| \text{ or } (|A\cap F_i| = |B\cap F_i| \text{ and } |A \cap E_i| \le |B \cap E_i|). \quad (1)$$

Note that friend-oriented preferences can be represented additively, by assigning a value of $n=|N|$ to each friend and a value of -1 to each enemy [16]: For any player $i \in N$ and for any coalition $A \in \mathcal{N}^i$, define the *value of a coalition* by

$$v_i(A) = n|A \cap F_i| - |A \cap E_i|.$$

Then, for $A,B \in \mathcal{N}^i$, we have $A \succeq_i^F B \iff v_i(A) \ge v_i(B)$.

Relatedly, other representations and their preference extensions are enemy-oriented preferences [16], additively separable [32] and fractional hedonic games [4], and singleton encodings [13] (which each are compactly representable but not fully expressive or complete), individually rational encodings [7], and hedonic coalition nets [19] (which are fully expressive but not compact in the sense of polynomial-size representation). See Section 4 for a discussion of how our models differ from representations known from the literature.

2.1 Properties of Preference Extensions

Below we give a selection of properties of preference extensions inspired by various related topics such as voting theory and resource allocation. Let $N=\{1,\dots,n\}$ be a set of players and F_i and E_i the sets of player i's friends and enemies, respectively. Let $G=(N,H)$ be the corresponding network of friends. Consider player i's preference relation $\succeq_i$ on $\mathcal{N}^i$. We say $\succeq_i$ is *reflexive* if $A \succeq_i A$ for each coalition $A \in \mathcal{N}^i$; $\succeq_i$ is *transitive* if for any three coalitions $A,B,C \in \mathcal{N}^i$, $A \succeq_i B$ and $B \succeq_i C$ implies $A \succeq_i C$; $\succeq_i$ is *polynomial-time computable* if for a given player i and two given coalitions $A,B \in \mathcal{N}^i$, it can be decided in polynomial time whether or not $A \succeq_i B$; and $\succeq_i$ is *anonymous* if renaming the players in N does not change $\succeq_i$. Clearly, the first three properties are necessary to have efficiently computable and rational preferences, and anonymity means that only the structure of the friendship network is important. We further define the following properties.

Weak Friend-Orientedness: If coalition A is acceptable for i, then $A \cup \{f\}$ is also acceptable for i, where $f \in F_i \setminus A$.

Favoring Friends: If $x \in F_i$ and $y \in E_i$ then $\{x,i\} \succ_i \{y,i\}$.

Indifference between Friends: If $x,y \in F_i$ then $\{x,i\} \sim_i \{y,i\}$.

Indifference between Enemies: If $x,y \in E_i$ then $\{x,i\} \sim_i \{y,i\}$.

Note that these four properties hold for friend-oriented preferences, see the work of Alcantud and Arlegi [1].

Sovereignty of Players: For a fixed player i and each $C \in \mathcal{N}^i$, there exists a network of friends such that C ends up as i's most preferred coalition.

Monotonicity: Let $j \ne i$ be a player with $j \in E_i$ and $A,B \in \mathcal{N}^i$, and $\succeq_i'$ be the preference relation resulting from $\succeq_i$ when j turns from being i's enemy to being i's friend (all else being equal). We call $\succeq_i$ *type-I-monotonic* if it holds that (1) if $A \succ_i B$, $j \in A \cap B$, and $A \succeq_j^F B$, then $A \succ_i' B$, and (2) if $A \sim_i B$, $j \in A\cap B$, and $A \succeq_j^F B$, then $A \succeq_i' B$. We call $\succeq_i$ *type-II-monotonic* if it holds that (1) if $A \succ_i B$ and $j \in A \setminus B$, then $A \succ_i' B$, and (2) if $A \sim_i B$ and $j \in A \setminus B$, then $A \succeq_i' B$.
Type-I-monotonicity ensures i's preference of A over B not to become worse if an enemy j who is contained in both coalitions, turns into i's friend, while j is weakly preferring A to B. Type-II-monotonicity, on the other hand, requires that j is only in A (hence has no opinion on B), but still i's preference of A over B should not become worse.

Symmetry: Let j and k be two distinct players with $j \ne i \ne k$. We say that $\succeq_i$ is *symmetric* if it holds that if swapping the positions of j and k in G is an automorphism then $\big(\forall\, C \in \mathcal{N}^i \setminus (\mathcal{N}^j \cup \mathcal{N}^k)\big)\ [C \cup \{j\} \sim_i C \cup \{k\}]$.

Local Friend Dependence: The preference order $\succeq_i$ can depend on the sets of friends $F_1,\dots,F_n$. Let $A,B \in \mathcal{N}^i$. We say that comparison (A,B) is

- *friend-dependent in* $\succeq_i$ if (1) $A \succeq_i B$ is true (false) and (2) can be made false (true) by changing the set of friends of some players (except for i);
- *locally friend-dependent in* $\succeq_i$ if (1) $A \succeq_i B$ is true (false), (2) can be made false (true) by changing the set of friends

of some players that are in A or B and are i's friends, and (3) changing the set of friends of all other players in $N \setminus (\{i\} \cup (F_i \cap (A \cup B))$ does not affect the status of the comparison.

We say $\succeq_i$ is *locally friend-dependent* if (1) there are $A, B \in \mathcal{N}^i$ such that (A,B) is friend-dependent in $\succeq_i$ and (2) every (A', B') that is friend-dependent in $\succeq_i$ is locally friend-dependent in $\succeq_i$.

This property says that an agent's preference over some coalition can change if the set of a friend's friends changes. This friend also has to be a member of a coalition that is under consideration. Thus local friend dependence is a crucial property that tries to capture the essence of the proposed approach to altruism in hedonic games.

Friend-Oriented Unanimity: Let $A, B \in \mathcal{N}^i$ with $A \cap F_i = B \cap F_i$. We say that $\succeq_i$ is *friend-orientedly unanimous* if $A \succ_j^F B$ for each $j \in (F_i \cup \{i\}) \cap A$ implies that $A \succ_i B$.

Note that the definition of friend-oriented unanimity covers all cases where the same subset of friends is consulted who all have a unanimous opinion in terms of friend-oriented preferences, in particular the case considering all friends' opinions: $F_i \subseteq A \cap B$.

2.2 Stability Concepts

The following stability concepts are commonly studied in hedonic games.

Definition 1. *Let $(N, \succeq)$ be a hedonic game and Γ be a coalition structure. A coalition $C \subseteq N$* blocks Γ *if for each $i \in C$ it holds that $C \succ_i \Gamma(i)$. If there is at least one $i \in C$ with $C \succ_i \Gamma(i)$ while $C \succeq_j \Gamma(j)$ holds for the other players $j \neq i$ in C, we call C* weakly blocking. *A coalition structure Γ is said to be*

1. individually rational *if for all $i \in N$, $\Gamma(i)$ is acceptable;*
2. Nash-stable *if for all $i \in N$ and for each $C \in \Gamma \cup \{\emptyset\}$ with $\Gamma(i) \neq C$, it holds that $\Gamma(i) \succeq_i C \cup \{i\}$;*
3. individually stable *if for all $i \in N$ and for each $C \in \Gamma \cup \{\emptyset\}$, it either holds that $\Gamma(i) \succeq_i C \cup \{i\}$ or there is a player $j \in C$ with $C \succ_j C \cup \{i\}$;*
4. contractually individually stable *if for all $i \in N$ and for each $C \in \Gamma \cup \{\emptyset\}$, it either holds that $\Gamma(i) \succeq_i C \cup \{i\}$, or there is a player $j \in C$ with $C \succ_j C \cup \{i\}$, or there is a player $k \in \Gamma(i)$ with $i \neq k$ and $\Gamma(i) \succ_k \Gamma(i) \setminus \{i\}$;*
5. strictly popular *if it beats every other coalition structure $\Gamma' \neq \Gamma$ in pairwise comparison, that is, if $|\{i \in N \mid \Gamma(i) \succ_i \Gamma'(i)\}| > |\{i \in N \mid \Gamma'(i) \succ_i \Gamma(i)\}|$;*
6. (strictly) core-stable *if there is no (weakly) blocking coalition;*
7. perfect *if for all $i \in N$ and for all $C \in \mathcal{N}^i$, it holds that $\Gamma(i) \succeq_i C$.*

3. ALTRUISTIC HEDONIC GAMES

In this section, we introduce our new model that refines friend-oriented hedonic games by taking altruistic influences into account. In this model, each player still wants to be with as many friends and as few enemies as possible, but in addition she wants her friends to be as satisfied as possible.

3.1 Naïve Approach

A first attempt to formalize this idea (that will turn out to fail) is the following. Consider the scenario where $i \in N$ has a friend-oriented preference extension (according to Equivalence (1)) except that, whenever the number of friends in A and B is the same and so is the number of enemies in A and B (i.e., $A \sim_i^F B$), i now prefers A to B if more of i's friends that are contained in A and B prefer A to B than B to A (according to Equivalence (1)). Formally:

$$
\begin{aligned}
A \succeq_i^{NA} B \iff & |A \cap F_i| > |B \cap F_i| \text{ or} \\
& (|A \cap F_i| = |B \cap F_i| \text{ and } |A \cap E_i| < |B \cap E_i|) \text{ or} \\
& (|A \cap F_i| = |B \cap F_i| \text{ and } |A \cap E_i| = |B \cap E_i| \text{ and} \\
& |\{j \in A \cap B \cap F_i \mid A \succ_j^F B\}| \geq \\
& |\{j \in A \cap B \cap F_i \mid B \succ_j^F A\}|).
\end{aligned} \tag{2}
$$

Intuitively, according to (2), a player is selfish first, but as soon as she is indifferent between two coalitions in the sense of (1), she cares about her friends' preferences. A major disadvantage of this definition, however, is that *irrational* preference orders can arise, i.e., preference orders that are not transitive in general: Consider, e.g., the hedonic game $(N, \succeq^{NA})$ with $N = \{1,2,3,4,5,6,7\}$ and the network of friends shown in Figure 2a. For coalitions $A = \{1,2,3,5\}$, $B = \{1,2,4,7\}$, and $C = \{1,3,4,6\}$, it holds that $A \succ_1^{NA} B$ and $B \succ_1^{NA} C$, yet $C \succ_1^{NA} A$, violating transitivity.

(a) Used, e.g., in the proof of Proposition 7

(b) Illustrating distinct degrees of altruism in Example 1

Figure 2: Two networks of friends representing hedonic games

In order to ensure transitivity, we have to add an extra condition to Equivalence (2). One idea would be to demand indifference between all coalitions that are involved in a $\succ_i^{NA}$-cycle by (2). This, however, can lead to a comparison of all coalitions containing a player, so determining a relation between two coalitions might comprise an exponential number of steps in the number of players. Then it would have been easier to give an arbitrary preference order as an input in the first place. Another idea would be to include the preferences of all friends, not only of those contained in the considered coalitions, but this would contradict the concept of hedonic game. In the following, we take a different approach.

3.2 Modelling Altruistic Influences

Given the failure of extending friend-oriented preferences by breaking ties with "majority voting," we consider the following model instead: Player $i \in N$ prefers coalition A over B if the average value of i's friends in A is larger than the average value of i's friends in B. In more detail, using the friend-oriented encoding, we obtain a friend j's opinion on a coalition containing both player i and j, which can have an influence on i's preference relation in the following ways. Since we consider friends to be equally important and focus on the average valuation, assigning a weight to player i's own contribution in comparison to her friends' influence on her preference, we will distinguish between three *degrees of altruism*: A player may (a) be selfish first and ask her friends only in case of indifference, (b) treat her friends and herself equally, or (c) be truly altruistic by asking her friends first and deciding herself only in case of indifference. Next to the definition we will show that the preferences capture the intuitive ideas behind them. For $i \in N$ and $A \in \mathcal{N}^i$, let

$$
\mathrm{avg}_i^F(A) = \sum_{a \in A \cap F_i} \frac{v_a(A)}{|A \cap F_i|}.
$$

In each of the three cases below, a player's *utility* u_i of a coalition is used as a measure of comparison combining the values v_i and v_j for $j \in F_i$. Note that $u_i = v_i$ under friend-oriented extensions.

(a) Selfish First: A player initially decides upon her preference over two coalitions friend-orientedly (i.e., according to (1)) and, if and only if she is indifferent between them, she asks her friends for a vote. For $M \geq n^5$, we define:

$$A \succeq_i^{SF} B \iff M(n|A\cap F_i| - |A\cap E_i|) + \sum_{a\in A\cap F_i} \frac{n|A\cap F_a| - |A\cap E_a|}{|A\cap F_i|} \geq M(n|B\cap F_i| - |B\cap E_i|) + \sum_{b\in B\cap F_i} \frac{n|B\cap F_b| - |B\cap E_b|}{|B\cap F_i|}. \quad (3)$$

THEOREM 1. *For $M \geq n^5$, $v_i(A) > v_i(B)$ implies $A \succ_i^{SF} B$.*

PROOF. The claim clearly holds for $\mathrm{avg}_i^F(A) \geq \mathrm{avg}_i^F(B)$. For $\mathrm{avg}_i^F(A) < \mathrm{avg}_i^F(B)$, it holds if and only if $M > \frac{\mathrm{avg}_i^F(B)-\mathrm{avg}_i^F(A)}{v_i(A)-v_i(B)}$. The numerator is upper-bounded by $\frac{n|B\cap F_i|\cdot|B|}{|B\cap F_i|} + \frac{|A\cap F_i|\cdot|A|}{|A\cap F_i|} \leq n^2+n$. Since $v_i(A)$ and $v_i(B)$ are integral, $v_i(A) - v_i(B) \geq 1$. Thus $M > n^2+n$ suffices. □

(b) Equal Treatment: A player and her friends "vote" friend-orientedly at the same time, equally taking part in the decision:

$$A \succeq_i^{EQ} B \iff \sum_{a\in A\cap(F_i\cup\{i\})} \frac{n|A\cap F_a| - |A\cap E_a|}{|A\cap(F_i\cup\{i\})|} \geq \sum_{b\in B\cap(F_i\cup\{i\})} \frac{n|B\cap F_b| - |B\cap E_b|}{|B\cap(F_i\cup\{i\})|}. \quad (4)$$

(c) Altruistic Treatment: A player first asks her friends for their opinion on a coalition they are contained in and adopts their average opinion; if and only if the consensus is indifference, the player decides for herself. For $M \geq n^5$, we define:

$$A \succeq_i^{AL} B \iff n|A\cap F_i| - |A\cap E_i| + M\sum_{a\in A\cap F_i} \frac{n|A\cap F_a| - |A\cap E_a|}{|A\cap F_i|} \geq n|B\cap F_i| - |B\cap E_i| + M\sum_{b\in B\cap F_i} \frac{n|B\cap F_b| - |B\cap E_b|}{|B\cap F_i|}. \quad (5)$$

THEOREM 2. *For $M \geq n^5$, $\mathrm{avg}_i^F(A) > \mathrm{avg}_i^F(B)$ implies $A \succ_i^{AL} B$.*

PROOF. The claim clearly holds for $v_i(A) \geq v_i(B)$. For $v_i(A) < v_i(B)$, the claim holds if and only if $M > \frac{v_i(B)-v_i(A)}{\mathrm{avg}_i^F(A)-\mathrm{avg}_i^F(B)}$. The numerator is upper-bounded by n^2+n. Note that $v_i(A) < v_i(B)$ implies $|A\cap F_i| \leq |B\cap F_i|$. We distinguish three cases.

1. If $\mathrm{avg}_i^F(A)|A\cap F_i| > \mathrm{avg}_i^F(B)|B\cap F_i|$, then by the integrality of v, we have $\mathrm{avg}_i^F(A)|A\cap F_i| - \mathrm{avg}_i^F(B)|B\cap F_i| \geq 1$. Thus $\mathrm{avg}_i^F(A) - \mathrm{avg}_i^F(B) \geq \frac{\mathrm{avg}_i^F(A)|A\cap F_i| - \mathrm{avg}_i^F(B)|B\cap F_i|}{|A\cap F_i|} \geq \frac{1}{n}$, as $\mathrm{avg}_i^F(X)$ is always positive if X contains a friend of i and $|A\cap F_i| \leq |B\cap F_i|$.
2. If $\mathrm{avg}_i^F(A)|A\cap F_i| = \mathrm{avg}_i^F(B)|B\cap F_i|$, then we have $\sum_{a\in A\cap F_i} v_a(A)\left(\frac{1}{|A\cap F_i|} - \frac{1}{|B\cap F_i|}\right) \geq |A\cap F_i|\left(\frac{1}{|A\cap F_i|} - \frac{1}{|B\cap F_i|}\right)$. This is equal to $1 - \frac{|A\cap F_i|}{|B\cap F_i|}$, which is lower-bounded by $\frac{1}{n}$: Suppose that the lower bound does not hold. Then $n(|B\cap F_i| - |A\cap F_i|) < |B\cap F_i|$. Hence, we have the contradiction $n < |B\cap F_i|$.
3. If $\mathrm{avg}_i^F(A)|A\cap F_i| < \mathrm{avg}_i^F(B)|B\cap F_i|$, then $|B\cap F_i| > |A\cap F_i|$. Suppose that $\mathrm{avg}_i^F(A) - \mathrm{avg}_i^F(B) < \frac{1}{n^2}$. Since by the premise we have $|B\cap F_i| \sum_{a\in A\cap F_i} v_a(A) - |A\cap F_i| \sum_{b\in B\cap F_i} v_b(B) \geq 1$, this is equivalent to $n^2 < \frac{|A\cap F_i|\cdot|B\cap F_i|}{|B\cap F_i|\sum_{a\in A\cap F_i} v_a(A) - |A\cap F_i|\sum_{b\in B\cap F_i} v_b(B)}$. The right-hand side is upper-bounded by n^2-n, implying the contradiction $n^2 < n^2-n$.

Overall, $M > n^4+n^3$ suffices in all three cases. □

For consistency we choose $M \geq n^5$. In all three cases in the proof of Theorem 2, normalization by the number of i's friends in a coalition prevents a "tyranny of the many" (otherwise, large coalitions would be preferred merely by the fact that the total number of friends is larger). The following example represents the different approaches to altruism in hedonic games.

EXAMPLE 1. *Consider the game with five players $N = \{1,2,3,4,5\}$ and the network in Figure 2b. Table 1 gives an overview of the relevant values and average values needed to determine player 1's utilities for different acceptable coalitions depending on the degree of altruism. A dash indicates that a value does not exist.*

C	$\{1,2,3\}$	$\{1,2,3,4\}$	$\{1,2,3,5\}$	N	$\{1,2\}$	$\{1,3\}$	$\{1,2,4\}$	$\{1,2,5\}$	$\{1,3,4\}$
$v_1(C)$	10	9	9	8	5	5	4	4	4
$v_2(C)$	4	3	9	8	5	–	4	10	–
$v_3(C)$	4	9	3	8	–	5	–	–	10
$\mathrm{avg}_1^F(C)$	4	6	6	8	5	5	4	10	10
EQ: $u_1(C)$	6	7	7	8	5	5	4	7	7

Table 1: Values and average values of the players in Example 1

All four weak preference orders are different. Under the friend-oriented preference extension (1)*, player 1's weak preference order is given in the first line according to the values of v_1. For the* selfish-first *extension* (3)*, the order remains the same; however, indifferences are dissolved, as is the case here with $\{1,2,5\} \succ_1^{SF} \{1,2,4\}$ by Theorem 1. Under the* equal-treatment *extension* (4)*, the grand coalition is the most preferred one; intuitively, because all friends have a large number of friends at the same time. Finally, under the most altruistic extension* (5)*, player 1's friends consider $\{1,2,5\}$ and $\{1,3,4\}$ the best coalition. As they agree on that, player 1 altruistically adopts this opinion by Theorem 2.*

A player's utility of a coalition can also be deduced from the corresponding network of friends itself.

PROPOSITION 3. *Let G be a network of friends, i a player, and $C \in \mathcal{N}^i$ a coalition. Let λ be the number of edges $\{i,j\}$ where $j\in C$, i.e., $\lambda = |F_i \cap C|$. Let μ be the number of edges between i's friends in C, i.e., $\mu = |\{\{j,k\} \mid j,k \in F_i\cap C\}|$, and let ν denote the number of edges between i's friends in C and those friends of j in C who are i's enemies, i.e., $\nu = |\{\{j,k\} \mid j\in F_i\cap C,\ k\in F_j\cap C,\ k\notin F_i\}|$. Then i's utility of C under selfish-first preferences is*

$$M\cdot\lambda(n+1) + M + n + 2 - (M+1)|C| + \frac{(n+1)(2\mu+\nu)}{\lambda};$$

under equal-treatment preferences it is

$$\frac{(2\lambda+2\mu+\nu)(n+1)}{\lambda+1} - |C| + 1;$$

and under altruistic-treatment preferences it is

$$M(n+2) + \lambda(n+1) + 1 - (M+1)|C| + \frac{M(n+1)(2\mu+\nu)}{\lambda}.$$

The proof of Proposition 3 is omitted due to limitation of space.

4. PROPERTIES OF HEDONIC GAMES WITH ALTRUISTIC INFLUENCES

In this section, we show which of the desirable properties from Section 2.1 are satisfied by our model. First, however, we start with a discussion of expressiveness, focusing on model (4):

First, as the original definition of friend-oriented preferences is recovered for coalitions that only consist of enemies, our models are not fully expressive. This follows from indifference between friends and enemies, respectively.

Second, we show that the expressiveness of model (4) is incomparable to (additively) separable hedonic games, fractional hedonic games, hedonic games with $\mathcal{B}$- or $\mathcal{W}$-preferences, and B- and W-hedonic games (see, for example, the book chapter by Aziz and Savani [6] for the definitions of these representations of hedonic games). In all of the above models two players' preference orders are independent but in our model they might depend on each other. Players are independent in choosing friends; however, the induced preferences depend crucially on friends' relations to other players. In other words, a player's preference is constrained by her friends' preferences. Hence, there is a tradeoff between the expressiveness of preferences and the expressiveness of profiles.

Third, model (4) can express preferences that are not separable: Consider a game with players $N = \{i,a,b,j\}$ and $F_i = \{a,b,j\}$, $F_a = \{i,b\}, F_b = \{i,a\}, F_j = \{i\}$. Denote the left-hand side of the inequality in (4) by $u_i^*(A) = \sum_{a' \in A\cap(F_i\cup\{i\})} \frac{n|A\cap F_{a'}|-|A\cap E_{a'}|}{|A\cap(F_i\cup\{i\})|}$. Then $u_i^*(\{i,a,b\}) = 2n$ and $u_i^*(\{i,a,b,j\}) = 2n-1$. Thus $\{i,a,b\} \succ_i^{EQ} \{i,a,b,j\}$ but $\{i,j\} \succ_i^{EQ} \{i\}$, because j is a friend. However, additively separable preferences can express strict preferences over coalitions with a single friend, which is not possible in model (4) because of indifference between friends. Similarly, fractional preferences can express strict preferences over pairs. In addition, they can express nonseparable relations by losing the ability to express indifference between pairs. $\mathcal{B}$- and $\mathcal{W}$-preferences can express indifference between pairs but this constraints the preferences over larger coalitions. In this case, however, depending on the network of friends, model (4) can express every possible relation between some specific coalitions. For B- and W-hedonic games, there is a simple example with two coalitions of size two only one of which is acceptable, where the implied relation over the coalition with both players does not hold under model (4).

Overall, neither is model (4) more expressive than any of the other considered models nor the other way around. Similar examples also exist for the other two models. It is not hard to see that all three degrees of altruism are locally friend-dependent, because (except for player i's own preference) only her friends in the current coalition affect its value. Note that this is a crucial property that distinguishes our model from previous work. We omit the proof of Proposition 4.

PROPOSITION 4. *Under all three degrees of altruism* (3)–(5)*, the following properties are satisfied: reflexivity, transitivity, polynomial-time computability, as well as anonymity.*

THEOREM 5. *Under all three degrees of altruism* (3)–(5)*, weak friend-orientedness, favoring friends, indifference between friends, indifference between enemies, sovereignty of players, symmetry, and friend-oriented unanimity are satisfied.*

PROOF. We show these properties only for equal treatment (4).

Weak friend-orientedness: Suppose that A is acceptable for $i \in A$, that is, we have $A \succeq_i^{EQ} \{i\}$, which is equivalent to the inequality

$$\sum_{a\in A\cap(F_i\cup\{i\})} \frac{v_a(A)}{|A\cap(F_i\cup\{i\})|} \geq \frac{v_i(\{i\})}{|\{i\}\cap(F_i\cup\{i\})|} = 0.$$

$A\cup\{f\} \succeq_i^{EQ} \{i\}$ if and only if $\sum_{a\in(A\cup\{f\})\cap(F_i\cup\{i\})} v_a(A\cup\{f\}) \geq 0$. The left-hand side of this inequality equals

$$\sum_{a\in A\cap(F_i\cup\{i\})} v_a(A) + \sum_{a\in A\cap(F_i\cup\{i\})} v_a(\{f\}) + v_f(A\cup\{f\}). \quad (6)$$

The second sum and $v_f(A\cup\{f\})$ are greater than zero because i is f's friend (and vice versa) and there are at most $n-1$ enemies for f. Thus, in total, we have that (6) is nonnegative and, therefore, $A\cup\{f\}$ is acceptable for i as well.

Favoring friends: This property holds due to the fact that

$$\sum_{a\in\{x,i\}\cap(F_i\cup\{i\})} \frac{n|\{x,i\}\cap F_a| - |\{x,i\}\cap E_a|}{|\{x,i\}\cap(F_i\cup\{i\})|} = \frac{v_x(\{x,i\})}{|\{x,i\}\cap(F_i\cup\{i\})|} + \frac{v_i(\{x,i\})}{|\{x,i\}\cap(F_i\cup\{i\})|} = n \;\geq\; -1 \;=\; \sum_{b\in\{y,i\}\cap(F_i\cup\{i\})} \frac{v_b(\{y,i\})}{|\{y,i\}\cap(F_i\cup\{i\})|}.$$

Indifference between friends: Let $x,y\in F_i$. As their names can be swapped, the sum for both friends is n and, therefore, we have both $\{x,i\} \succeq_i^{EQ} \{y,i\}$ and $\{y,i\} \succeq_i^{EQ} \{x,i\}$.

Indifference between enemies: Analogously to the indifference between friends, both sums equal -1.

Sovereignty of players: We construct the network G such that for all pairs of players $x,y \in C$, $x \neq y$, there is an edge $\{x,y\}$ in G, while there are no other edges in G.

Symmetry: Suppose that swapping the positions of j and k in G is an automorphism. If j is a friend of a player i, so is k. The same holds for enemies. Therefore, the sum over all players in $(C\cup\{j\})\cap(F_i\cup\{i\})$ equals the sum over all players in $(C\cup\{k\})\cap(F_i\cup\{i\})$ if j (and k) are not in F_i. If j (and, therefore, k) are i's friends, the additional summands equal each other, as j is neither in F_j nor in E_j and as k is neither in F_k nor in E_k.

Friend-oriented unanimity: As $A\cap F_i = B\cap F_i$ and both $A,B \in \mathcal{N}^i$, we have $|A\cap(F_i\cup\{i\})| = |B\cap(F_i\cup\{i\})|$. Thus it suffices to show that $\sum_{a\in A\cap(F_i\cup\{i\})} v_a(A) > \sum_{a\in A\cap(F_i\cup\{i\})} v_a(B)$. According to the friend-oriented model, we subdivide the sum on the left-hand side above as follows:

$$\sum_{a\in A\cap(F_i\cup\{i\})} v_a(A) = \sum_{a_1\in A\cap(F_i\cup\{i\})} v_{a_1}(A) + \sum_{a_2\in A\cap(F_i\cup\{i\})} v_{a_2}(A),$$

where a_1 are those $a\in A\cap(F_i\cup\{i\})$ for which $|A\cap F_a| > |B\cap F_a|$, and a_2 are those $a\in A\cap(F_i\cup\{i\})$ for which $|A\cap F_a| = |B\cap F_a|$ and $|A\cap E_a| < |B\cap E_a|$.

Considering a_1, we note that $v_{a_1}(A) > v_{a_1}(B)$, as even for $|A\cap F_{a_1}| = |B\cap F_{a_1}|+1$ and $|A\cap E_{a_1}| = n-1$ and $|B\cap E_{a_1}| = 0$, we have $n(|B\cap F_{a_1}|+1)-(n-1) > n|B\cap F_{a_1}|$. For a_2, we see that $n|A\cap F_{a_2}| - |A\cap E_{a_2}| = n|B\cap F_{a_2}| - |A\cap E_{a_2}| > n|B\cap F_{a_2}| - |B\cap E_{a_2}|$. Thus $A \succ_i^{EQ} B$, showing friend-oriented unanimity.

This proves (4); the proof for (3) and (5) is similar. □

Regarding the property of symmetry, note that whenever two interchangeable players have a distance of at most two, then the statement $\left(\forall C\in\mathcal{N}^i\setminus(\mathcal{N}^j\cup\mathcal{N}^k)\right)[C\cup\{j\}\sim_i C\cup\{k\}]$ implies that swapping i and j in G is an automorphism.

THEOREM 6. *Selfish-first preferences* (3) *are type-I-monotonic and type-II-monotonic.*

PROOF. Let $i\in N$, $A,B\in\mathcal{N}^i$, and $j\in E_i$. We denote with $\widehat{\succeq_i^{SF}}$ the preference of i resulting from $\succeq_i^{SF}$ when j is moved from E_i to F_i and i is moved from E_j to F_j (all else being equal), and we denote the new friend sets of i and j with $F_i' = F_i\cup\{j\}$ and $F_j' = F_j\cup\{i\}$. Recall that $A \succeq_j^{SF} B$ means that (i) if $A \sim_j^{SF} B$ then $A \sim_j^{F} B$ has to hold and $\text{avg}_j^F(A) = \text{avg}_j^F(B)$ (where $\text{avg}_j^F(A) = \sum_{a \in A \cap F_j} \frac{v_a(A)}{|A \cap F_j|}$), and (ii) if $A \succ_j^{SF} B$ then either (a) $A \sim_j^F B$ and $\text{avg}_j^F(A) > \text{avg}_j^F(B)$, or (b) $A \succ_j^F B$. Suppose $A \sim_i^{SF} B$. Then we have that $A \sim_i^F B$ and

$$\text{avg}_i^F(A) = \text{avg}_i^F(B). \tag{7}$$

Since $j \in A \cap B$, $A \widehat{\sim_i^{SF}} B$, so we have to compare $\sum_{a \in A \cap F_i'} \frac{v_a(A)}{|A \cap F_i|+1}$ and $\sum_{b \in B \cap F_i'} \frac{v_b(B)}{|B \cap F_i|+1}$, the former being equal to

$$\underbrace{\frac{|A \cap F_i|}{|A \cap F_i|+1}}_{\text{(part 1)}} \left(\underbrace{\frac{n|A \cap F_j| + n - |A \cap E_j| + 1}{|A \cap F_i|}}_{\text{(part 2)}} + \underbrace{\text{avg}_i^F(A)}_{\text{(part 3)}} \right) \tag{8}$$

and the latter being equal to

$$\underbrace{\frac{|B \cap F_i|}{|B \cap F_i|+1}}_{\text{(part 4)}} \left(\underbrace{\frac{n|B \cap F_j| + n - |B \cap E_j| + 1}{|B \cap F_i|}}_{\text{(part 5)}} + \underbrace{\text{avg}_i^F(B)}_{\text{(part 6)}} \right). \tag{9}$$

We know from $A \sim_i^F B$ that (part 1) = (part 4), we know from (7) that (part 3) = (part 6), and we know that the denominators in (part 2) and (part 5) are equal. So calculating (part 2) − (part 5) leads to

$$n|A \cap F_j| - |A \cap E_j| - (n|B \cap F_j| - |B \cap E_j|) \begin{cases} = 0 & \text{if } A \sim_j^F B, \\ \geq 0 & \text{if } A \succeq_j^F B. \end{cases}$$

In total, we have (8) $\geq$ (9) if $A \succeq_j^F B$, thus $A \widehat{\succeq_i^{SF}} B$.

Assume now that $A \succ_i^{SF} B$. If $A \succ_i^F B$, nothing changes by making j a friend of i. If $A \sim_i^F B$, then it has to hold that

$$\sum_{a \in A \cap F_i} \frac{v_a(A)}{|A \cap F_i|} > \sum_{b \in B \cap F_i} \frac{v_b(B)}{|B \cap F_i|}. \tag{10}$$

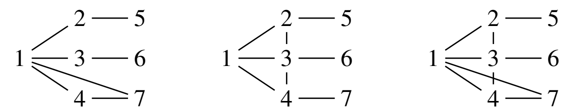
By making j a friend of a, the relation $A \sim_i^F B$ is not changed. So, analogously to the first part, we have to compare the sums in (8) and (9). By $A \sim_i^F B$, we know that (part 1) = (part 4) and from (10) we know that (part 3) > (part 6). By the same argument as above, we know that (part 2) $\geq$ (part 5) if $A \succeq_j^F B$, and, in total, we have that $A \widehat{\succ_i^{SF}} B$.

It remains to prove type-II monotonicity. In this case, we have $j \in A \setminus B$. Suppose $A \sim_i^{SF} B$. This implies that $A \cap F_i = B \cap F_i$. Since j is not contained in B, we have that $|A \cap F_i'| > |B \cap F_i'|$, which gives $A \widehat{\succ_i^{SF}} B$. Assume now that $A \succ_i^{SF} B$. There are two cases. First, if $A \sim_i^F B$, then $A \widehat{\succ_i^{SF}} B$ follows by the same argumentation as above. Second, if $A \succ_i^F B$, then adding a further friend to A who is not contained in B does not change this relation. Thus $A \widehat{\succ_i^{SF}} B$. □

PROPOSITION 7. *Equal-treatment preferences* (4) *and altruistic-treatment preferences* (5) *are not type-II-monotonic.*

PROOF. Let $\mathcal{G}_1$ be a game with the network of friends shown in Figure 2a. We see that $7 \notin F_1$ and $A \sim_1^{EQ} B$ for $A = \{1,2,3,7\}$ and $B = \{1,3,4,5\}$. Making 7 a friend of 1 leads to $\mathcal{G}_1'$ with the network of friends shown in Figure 3a, and in $\mathcal{G}_1'$ we have that $B \succ_1^{EQ} A$, violating type-II monotonicity. With an analogous argumentation for the games $\mathcal{G}_2$ and $\mathcal{G}_2'$ (illustrated in Figure 3b and 3c), altruistic-treatment preferences are not type-II-monotonic. □

Note that the above result is a desirable outcome since this behavior exactly captures the intuition behind the definition of the equal treatment and the altruistic treatment.



(a) Network of $\mathcal{G}_1'$ (b) Network of $\mathcal{G}_2$ (c) Network of $\mathcal{G}_2'$

Figure 3: Network of friends in the proof of Proposition 7

Based on a similar line of thought we note the following: In addition to the axiomatic properties in Section 2.1, one could consider notions of independence (see, e.g., [1] for a characterization of friend-oriented preferences using an independence axiom). Classic independence axioms say that a relation between two coalitions, A and B, continues to hold even if a new (and the same) player is introduced to both coalitions. However, independence axioms of this type are not desirable in our model because the new player can be valued very differently in both coalitions. This would be the case, for example, if the new player were an enemy to most of i's friends in A but were loved by most of i's friends in B. Similarly, $\mathcal{B}$- and $\mathcal{W}$-preferences [13] are natural extensions from singleton encodings that are not independent.

5. STABILITY

In this section, we study several common stability concepts in our model. Questions of interest are how hard it is to verify whether a given coalition structure satisfies a certain concept in a given hedonic game and whether stable coalition structures for certain concepts always exist. If, for some concept, such a coalition structure does not always exist, we are also interested in the computational complexity of deciding whether or not some such coalition structure exists in a given hedonic game. Recently, Peters and Elkind [29] established metatheorems that help proving NP-hardness results for stability concepts in hedonic games. However, their results do not seem to be immediately applicable.

OBSERVATION 8. *Under all three degrees of altruism* (3)–(5)*, a coalition structure* Γ *is individually rational if and only if for each* $i \in N$, $\Gamma(i) \cap F_i \neq \emptyset$ *or* $\Gamma(i) = \{i\}$.

PROPOSITION 9. *For all three degrees of altruism* (3)–(5)*, it can be tested in polynomial time whether a given coalition structure in a given game is Nash-stable, individually stable, or contractually individually stable.*

PROOF. Let Γ be a coalition structure. We need to check if for each player $i \in N$ and for each existing coalition C in Γ or for $C = \emptyset$, i prefers $\Gamma(i)$ to being added to C. For n players, there are at most $n+1$ such coalitions, and the preference relation can be verified in polynomial time by Proposition 4. Similar arguments apply to individual and contractually individual stability. □

LEMMA 10. *For all three degrees of altruism the following hold:*
1. For each player i, each $j \in F_i$ *assigns a positive value to any coalition* $C \in \mathcal{N}^i \cap \mathcal{N}^j$.
2. If a player has at least one friend, her favorite coalition contains at least one friend.

PROOF. 1. For symmetric friendship relationships, a friend always has at least one friend in a coalition she is asked to evaluate. Therefore, if a valuation of a friend is considered to influence a preference, it is always positive.

2. Suppose that a coalition contains player i and none of her friends. Then the overall value is at most zero.

This completes the proof. □

THEOREM 11. *For all three degrees of altruism* (3)–(5)*, there always exist Nash-stable, individually stable, and contractually individually stable coalition structures.*

PROOF. Let $E = \{i \in N \mid F_i = \emptyset\}$ and rename its members by $E = \{e_1, \dots, e_k\}$. The coalition structure $\{\{e_1\}, \dots, \{e_k\}, N \setminus E\}$ is Nash-stable: For each $i \in E$, $v_i(N \setminus E) < 0$, since there are no friends to be evaluated positively nor to be asked for their valuation, i would rather stay alone. For each $i \notin E$, $v_i(N \setminus E) > 0$, since there is at least one friend who leads to a positive value and i herself contributes a positive value by Lemma 10.1, i would rather like to stay in $N \setminus E$ than to move alone to the empty coalition.

Nash stability implies individual stability, which in turn implies contractually individual stability. □

On the other hand, for all three degrees of altruism, there exists a game such that no coalition structure is strictly popular.

EXAMPLE 2. *Consider the game from Example 1 and the coalition structures* $\Gamma_1 = \{\{1,2,5\},\{3,4\}\}$, $\Gamma_2 = \{\{1,3,4\},\{2,5\}\}$, $\Gamma_3 = \{\{1,2,3,4\},\{5\}\}$, $\Gamma_4 = \{\{1,2,3,5\},\{4\}\}$, *and* $\Gamma_5 = \{N\}$.

1. Under selfish-first preferences (3), Γ_1 *and* Γ_2 *are more popular than all other coalition structures, but are in a tie.*

2. Under equal-treatment preferences (4)*, even three coalition structures are in a tie:* Γ_3, Γ_4*, and* Γ_5.

3. Under altruistic-treatment preferences (5), Γ_2 *is more popular than* Γ_3*, which in turn is more popular than* Γ_5. Γ_5 *and* Γ_2 *are in a tie. Further,* Γ_1 *is more popular than* Γ_4*; the two coalition structures behave analogously to* Γ_2 *and* Γ_3*, respectively, due to symmetries. There is no other coalition structure that is not beaten by any of the above-mentioned coalition structures. Hence, no coalition structure is strictly popular.*

We now turn to the complexity of the verification and the existence problem for strict popularity in selfish-first hedonic games.

THEOREM 12. *Under selfish-first preferences* (3)*, the problem of whether a given coalition structure in a given game is strictly popular is* coNP*-complete and the problem of whether there exists a strictly popular coalition structure in a given game is* coNP*-hard.*

PROOF. The first problem belongs to coNP, since the complementary problem can be decided by nondeterministically choosing another coalition structure and verifying whether a larger number of players prefer the latter to the former than the other way around. Verification can be done in polynomial time by Proposition 4. We show coNP-hardness by means of a polynomial-time many-one reduction from EXACT COVER BY 3-SETS (XC_3) to the complement of the first problem. An XC_3 instance consists of a set $B = \{1, \dots, 3k\}$ and a family $\mathscr{S} = \{S_1, \dots, S_m\}$ of subsets $S_i \subseteq B$ with $|S_i| = 3$, for each i, $1 \le i \le m$, and the question is whether there exists an exact cover for B in $\mathscr{S}$, i.e., a subfamily $\mathscr{S}' \subseteq \mathscr{S}$ such that each element of B occurs in exactly one set in $\mathscr{S}'$. We may assume that each $b \in B$ occurs at most three times in the sets of $\mathscr{S}$ (see [21]). The following construction is inspired by methods used by Sung and Dimitrov [32], which will be adapted in a nontrivial way, though. Given an XC_3 instance $(B, \mathscr{S})$, we consider the set of players $N = \{\beta_b \mid b \in B\} \cup \{\zeta_{S,\ell} \mid S \in \mathscr{S}, 1 \le \ell \le 3k\} \cup \{\eta_{S,j} \mid S \in \mathscr{S}, 1 \le j \le 3k+3\}$. The network of friends, as displayed in Figure 4, is the following: All players in $\{\beta_b \mid b \in B\}$ are friends with each other; β_b and $\zeta_{S,\ell}$ are each others' friends if $b \in S$, for each $S \in \mathscr{S}$, $1 \le \ell \le 3k$; for each $S \in \mathscr{S}$, all players in $Q_S = \{\zeta_{S,\ell}, \eta_{S,j} \mid 1 \le \ell \le 3k, 1 \le j \le 3k+3\}$ are each others' friends; and there are no other friendship relations.

Define the coalition structure $\Gamma = \{\{\beta_1, \dots, \beta_{3k}\}, Q_{S_1}, \dots, Q_{S_m}\}$. We show that Γ is strictly popular if and only if there exists no exact cover for B in $\mathscr{S}$.

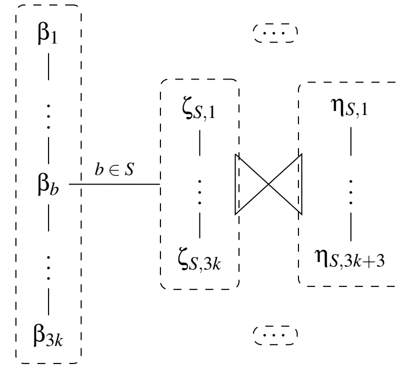


Figure 4: Network of friends in the proof of Theorem 12

Only if: Assume there exists an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ such that $\bigcup_{S \in \mathscr{S}'} S = B$ and $|\mathscr{S}'| = k$. Then, for the coalition structure $\Delta = \{\{\beta_b \mid b \in S\} \cup Q_S \mid S \in \mathscr{S}'\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}'\}$, it holds that $|\{i \mid \Delta(i) \succ_i^{SF} \Gamma(i)\}| = 3k + k \cdot 3k = k(3k+3) = |\{j \mid \Gamma(j) \succ_j^{SF} \Delta(j)\}|$. Hence, Γ cannot be strictly popular.

If: If, on the other hand, Γ is not strictly popular, there exists some coalition structure Δ that is preferred to Γ by at least as many players as the number of those players preferring Γ to Δ. This is only possible if such a Δ implies the existence of an exact cover for B. Note that while eliminating all other possibilities, the friends' influences on a player $\zeta_{S,\ell}$ would be employed in favor of a clique Q_S in comparison to other coalitions that $\zeta_{S,\ell}$ may be indifferent between from a selfish-first point of view.

For the second problem, consider the same reduction as above, except that Γ is not given. If, on the one hand, there is no exact cover for B in $\mathscr{S}$, a strictly popular coalition structure exists, namely, Γ as considered above. If, on the other hand, there is an exact cover for B in $\mathscr{S}$, note that Γ beats every other coalition structure in pairwise comparison, but is in a tie in comparison to Δ as defined above. Therefore, Γ as well as any other coalition structure cannot be strictly popular. □

Theorem 13 is inspired by a result of Dimitrov et al. [16].

THEOREM 13. *In games with selfish-first preferences* (3)*, there always exists a (strictly) core-stable coalition structure.*

PROOF. For a given game with the player set N and selfish-first preferences, we assume that the corresponding network of friends consists of k connected components. We show that the coalition structure $\Gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_k\}$ consisting of these k components is (strictly) core-stable. We know that the players from different coalitions in Γ are not friends: Each $i \in N$ has all of her friends in $\Gamma(i)$. Clearly, a (weakly) blocking coalition C cannot contain players from different Γ_ℓ, for $1 \le \ell \le k$, as each of these players would prefer their assigned coalition because it either contains more friends than C or, if all friends are also in C, less enemies.

This allows us to argue for the special case of a network consisting of only one connected component and we show that the coalition structure $\Gamma = \{N\}$ is (strictly) core-stable. For a contradiction, we assume that there exists a blocking coalition $C \subset N$. So, for every $j \in C$ we have that $C \succ_j^{SF} N$, which implies that either (a) $C \succ_j^F N$ or (b) $C \sim_j^F N$ and $\mathrm{avg}_i^F(C) > \mathrm{avg}_i^F(N)$.

All of j's friends are contained in N, so the first case can only occur if C contains all of j's friends and a proper subset of j's enemies. The same has to hold for each $i \in F_j \subseteq C$, and for each of i's friends, as well. This can be continued following the paths through the network of friends, and since we assume that the network is connected, we obtain that $C = \bigcup_{i \in N} F_i = N$. In the second case, $C \sim_j^F N$ implies that $v_j(C) = v_j(N)$, thus $C = N$.

Overall, there is no blocking coalition. The same argumentation also holds for a weakly blocking coalition: At least one $j \in C$ has to prefer C to N while we need $C \succeq_i^{SF} N$ to hold for the remaining $i \in C$. The latter also implies that for each $i \in C$, the set of friends has to be in C, leading to $C = N$ as above. □

Under selfish-first preferences, it is easy to figure out whether there exists a perfect coalition structure: This is the case if and only if each connected component is a clique.

LEMMA 14. *Let C be player i's most preferred coalition in a game with equal-treatment preferences* (4). *If a friend j is in $F_i \cap C$, then $F_j \setminus F_i \subseteq C$.*

PROOF. If i has at least one friend, the value of C is positive by Lemma 10.1. Assume that there is a $k \in F_j \setminus F_i$ with $k \notin C$. Then

$$\begin{aligned}
&\frac{\sum_{a \in (F_i \cup \{i\}) \cap (C \cup \{k\})} v_a(C \cup \{k\})}{|(F_i \cup \{i\}) \cap (C \cup \{k\})|} - \frac{\sum_{a \in C \cap (F_i \cup \{i\})} v_a(C)}{|(F_i \cup \{i\}) \cap C|} \\
&= \frac{\sum_{a \in C \cap (F_i \cup \{i\})} (v_a(C \cup \{k\}) - v_a(C))}{|F_i \cap C| + 1} \\
&= \frac{\sum_{a \in C \cap (F_i \cup \{i\}) \cap F_k} (n) - \sum_{a \in C \cap (F_i \cup \{i\}) \setminus F_k} (1)}{|F_i \cap C| + 1} \geq \frac{n - (n-2)}{|F_i \cap C| + 1} > 0.
\end{aligned}$$

Thus $C \cup \{k\} \succ_i^{EQ} C$, since i asks the same number of friends and the value of $C \cup \{k\}$ increases by n for at least one player and decreases by 1 for at most $n-2$ players. □

PROPOSITION 15. *Whenever a perfect coalition structure exists under equal-treatment preferences* (4)*, it is unique and consists of all connected components.*

PROOF. Let C be a coalition in a perfect coalition structure. Due to Lemma 10.2, C is connected. Suppose C is a proper subset of a connected component. Then there exists an edge $\{k, \ell\}$ with $k \in C$ and $\ell \notin C$. By Lemma 10.2, there exists another friend j of k's in C.

Case 1: Assume that there exists a player j with $\ell \notin F_j$. Then, by Lemma 14, this is a contradiction to C being j's favorite coalition, because $C \cup \{\ell\} \succ_j^{EQ} C$.

Case 2: For each $j \in F_k \cap C$, it holds that $\ell \in F_j$ (and $j \in F_\ell$ by symmetry). (a) Assume that there exists another $x \in C$ with $\ell \notin F_x$. By Lemma 10.2, there exists a $j \in F_k$ with $x \in F_j$ (and $j \in F_x$). Again, with $\ell \in F_j$ this is a contradition to C being x's most preferred coalition by Lemma 14. (b) Finally, for each $x \in C$, $\ell \in F_x$. This implies that $v_\ell(C \cup \{\ell\}) = n \cdot |C| - 0$. Thus, comparing coalitions $C \cup \{\ell\}$ and C from k's point of view, and letting λ denote $|F_k \cap C|$, we obtain:

$$\begin{aligned}
&\frac{v_k(C \cup \{\ell\}) + \sum_{j \in F_k \cap C} v_j(C \cup \{\ell\}) + v_\ell(C \cup \{\ell\})}{1 + \lambda + 1} - \frac{v_k(C) + \sum_{j \in F_k \cap C} v_j(C)}{1 + \lambda} \\
&\geq \frac{n + |E_k \cap C| + \lambda(1+\lambda)n - \lambda|C|n + (1+\lambda)(n \cdot |C|)}{(2+\lambda)(1+\lambda)} \\
&= \frac{n + |E_k \cap C| + \lambda(1+\lambda)n + n \cdot |C|}{(2+\lambda)(1+\lambda)} > 0.
\end{aligned}$$

Therefore, $C \cup \{\ell\} \succ_k^{EQ} C$, which means that C has to be the whole connected component. This completes the proof. □

COROLLARY 16. *If there exists a perfect coalition structure under equal treatment* (4)*, all connected components have a diameter of at most two.*

There do exist networks with a diameter of at most two that do not allow a perfect coalition structure, e.g., stars (i.e., one central vertex connected to a number of leaves).

PROPOSITION 17. *Under equal treatment* (4)*, trees with at least three vertices do not allow a perfect coalition structure.*

PROOF. Note that trees with diameter two are stars. Let i be the central player and j a leaf. It holds that $N \setminus \{j\} \succ_i^{EQ} N$ such that $\{N\}$ is not perfect, which in turn implies that there cannot be a perfect coalition structure by Proposition 15. □

6. CONCLUSIONS AND FUTURE WORK

We have introduced and studied hedonic games with altruistic influences where the agents' utility functions depend on their friends' preferences. Axiomatically, we have defined desirable properties and have shown that these are satisfied by our model, depending on the degree of altruism. When tailored to other well-studied preference models, such as friend-oriented, enemy-oriented, additively separable, and $\mathcal{B}$- and $\mathcal{W}$-preferences, we note that all of these five extension principles fulfill the introduced properties of anonymity, symmetry, and type-II-monotonicity, while only the former three satisfy independence.

In terms of stability, hedonic games with altruistic influences always admit, e.g., Nash-stable coalition structures. However, both the verification and the existence problems of strictly popular coalition structures are computationally intractable.

We consider it important future work to completely characterize when certain properties hold or stable coalition structures exist (e.g., to characterize when the grand coalition is perfect). Also, it might be useful to extend the model and normalize by the size of the coalition to consider only relative contributions of friend-of-a-friend relationships. This can be compared to a friend-oriented restriction of a fractional hedonic game [4]. For example, one could define

$$\begin{aligned}
A \succeq_i^{EQ_f} B \iff &\sum_{a \in A \cap (F_i \cup \{i\})} \frac{n|A \cap F_a| - |A \cap E_a|}{|A| \cdot |A \cap (F_i \cup \{i\})|} \\
&\geq \sum_{b \in B \cap (F_i \cup \{i\})} \frac{n|B \cap F_b| - |B \cap E_b|}{|B| \cdot |B \cap (F_i \cup \{i\})|}.
\end{aligned} \tag{11}$$

This definition clearly extends to the altruistic case. For the selfish-first case, the normalization is without effect. Hence, the fractional variant is equivalent to the selfish-first case we have considered.

In addition, we propose considering restrictions of the input such as constraining networks to special graph classes (such as interval graphs, where the width of an interval represents an agent's "tolerance"), studying problems of strategic influence (e.g., misreporting preferences to friends, pretending to be a friend while one in fact is an enemy, asserting control over the game as a whole). The model can be extended in multiple ways. To model more realistic situations, it would be suitable to allow for different degrees of altruism for distinct players and other representations of preferences and aggregators. So far, a player takes only her friends' preferences into consideration, that is, a player tries to satisfy her friends with respect to their preferences. Since players derive utility based on their own preferences and their friends' preferences, an interesting model would be to consider players that try to maximize their friends utilities (see, e.g., [31]). In a similar vein, the model can be extended to edge-weighted graphs, where the influence of a friend (or of a friend's friend) diminishes with the distance as in, e.g., social distance games [12].

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