# Game-Theoretic Semantics for Alternating-Time Temporal Logic

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## ABSTRACT

We introduce versions of game-theoretic semantics (GTS) for Alternating-Time Temporal Logic (ATL). In GTS, truth is defined in terms of existence of a winning strategy in a semantic evaluation game, and thus the game-theoretic perspective appears in the framework of ATL on two semantic levels: on the object level, in the standard semantics of the strategic operators, and on the meta-level, where game-theoretic logical semantics can be applied to ATL. We unify these two perspectives into semantic evaluation games specially designed for ATL. The novel game-theoretic perspective enables us to identify new variants of the semantics of ATL, based on limiting the time resources available to the verifier and falsifier in the semantic evaluation game; we introduce and analyse an *unbounded* and *bounded* GTS and prove these to be equivalent to the standard (Tarski-style) compositional semantics. We also introduce a non-equivalent *finitely bounded* semantics and argue that it is natural from both logical and game-theoretic perspectives.

## Keywords

Logic and game theory; Logics for agents and multi-agent systems; Argumentation-based dialogue and protocols

## 1. INTRODUCTION

*Alternating-Time Temporal Logic* ATL was introduced in [3] as a multi-agent extension of the branching-time temporal logic CTL. The semantics of ATL is defined over *multi-agent transition systems*, also known as *concurrent game models*, in which agents take simultaneous actions at the current state and the resulting collective action determines the subsequent state transition. The logic ATL and its extensions such as ATL$^*$ have gradually become the most popular logical formalisms for reasoning about strategic abilities of agents in synchronous multi-agent systems.

*Game-theoretic semantics* (GTS) of logical languages has a complex history going back to Hintikka [8], Lorenzen [11] and others. For an overview of the topic, see [10]. In GTS, truth of a logical formula $\varphi$ is determined in a formal *debate* between two players, *Eloise* and *Abelard*. Eloise is trying to verify $\varphi$, while Abelard is opposing her. Each logical operator is associated with a related rule in the game. The framework of GTS has turned out to be particularly useful for the purpose of defining variants of semantic approaches to different logics. For example, IF-logic of Hintikka and Sandu [9] is an extension of first-order logic which was originally developed using GTS. Also, the game-theoretic approach to semantics has lead to new methods for solving decision problems of logics, e.g., via using parity games for the $\mu$-calculus.

In this article we introduce game-theoretic semantics for ATL. In that framework, the rules corresponding to strategic operators involve scenarios where Eloise and Abelard are both controlling (or leading) coalitions of agents with opposing objectives. The perspective offered by GTS enables us to develop novel approaches to ATL based on different time resources available to the players. In *unbounded* GTS, a coalition trying to verify an until-formula is allowed to continue without a time limit, the price of an infinite play being a loss in the game. In *bounded* GTS, the coalition must commit to *finishing in finite time* by submitting an *ordinal number* in the beginning of the game; the ordinal controls available time resources in the game and *guarantees a finite play*. In fact, even safety games (for release-formulae) will be determined in finite time, and thus the bounded and unbounded approaches to GTS are conceptually different.

Despite the differences between the two semantics, we show that they are in fact equivalent to the standard compositional (i.e., Tarski-style) semantics of ATL and therefore to each other. Furthermore, we introduce a restriction of the bounded GTS, called *finitely bounded* GTS, where the ordinals controlling time flow must always be finite. This is a particularly simple system of semantics where the players will always announce the ultimate (always finite) duration of the game before the game begins. We show that the finitely bounded GTS is equivalent to the standard ATL semantics on *image finite models*, and therefore provides an alternative approach to ATL sufficient for most practical purposes.

Since the finitely bounded semantics is new, we also develop an equivalent (over all models) Tarski-style semantics for it. We note that the difference between the finitely bounded and unbounded semantics is conceptually linked to the difference between *for-loops* and *while-loops*.

The main contributions of this paper are twofold: the development of game-theoretic semantics for ATL and the introduction of new resource-sensitive versions of logics for multi-agent strategic reasoning. The latter relates conceptu-

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ally to the study of other resource-bounded versions of ATL, see [2], [12], [1].

The current article is a short version of the self-contained full paper [7] that includes more technical details and full proofs. The structure of the paper is as follows. After the preliminaries in Section 2, we develop the bounded and unbounded GTS in Section 3. We analyse the frameworks in Section 4, where we show, inter alia, that the two game-theoretic frameworks are equivalent. In Section 5, we compare the game-theoretic and standard Tarski-style semantics and establish the equivalences between them stated above.

It is worth pointing out that some of our technical results could be derived using more general alternative methods from coalgebraic modal logic. We will discuss this matter in more detail in the concluding section 5.

# 2. PRELIMINARIES

In this section we define concurrent game models as well as the syntax and standard compositional semantics of ATL.

DEFINITION 2.1. *A **concurrent game model** (CGM) $\mathcal{M}$ is a tuple $(\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ which consists of the following non-empty sets: **agents** $\mathbb{A}\mathrm{gt} = \{1, \dots, k\}$, **states** St, **proposition symbols** $\Pi$, **actions** Act, **action function** $d : \mathbb{A}\mathrm{gt} \times \mathrm{St} \to \mathcal{P}(\mathrm{Act}) \setminus \{\emptyset\}$ assigning a non-empty set of actions available to each agent at each state, and a **transition function** $o$ assigning a unique **outcome state** $o(q, \vec{\alpha})$ to each state $q \in \mathrm{St}$ and **action profile** (a tuple of actions $\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$ such that $\alpha_i \in d(i, q)$ for each $i \in \mathbb{A}\mathrm{gt}$), and a **valuation function** $v : \Pi \to \mathcal{P}(\mathrm{St})$.*

Sets of agents $A \subseteq \mathbb{A}\mathrm{gt}$ are also called **coalitions**. The complement $\overline{A} = \mathbb{A}\mathrm{gt} \setminus A$ of a coalition $A$ is called the **opposing coalition** (of $A$). We also define the set of action tuples that are *available* to coalition $A$ at a state $q \in \mathrm{St}$: $\mathsf{action}(A, q) := \{(\alpha_i)_{i \in A} \mid \alpha_i \in d(i, q) \text{ for each } i \in A\}$.

DEFINITION 2.2. *Let $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a concurrent game model. A **strategy**[1] for an agent $a \in \mathbb{A}\mathrm{gt}$ is a function $s_a : \mathrm{St} \to \mathrm{Act}$ such that $s_a(q) \in d(a, q)$ for each $q \in \mathrm{St}$. A **collective strategy** $S_A$ for $A \subseteq \mathbb{A}\mathrm{gt}$ is a tuple of individual strategies, one for each agent in $A$. A **path** in $\mathcal{M}$ is a sequence of states $\Lambda$ such that $\Lambda[n+1] = o(\Lambda[n], \vec{\alpha})$ for some admissible action profile $\vec{\alpha}$, where $\Lambda[n]$ is the $n$-th state in $\Lambda$ ($n \in \mathbb{N}$). The function $\mathsf{paths}(q, S_A)$ returns the set of all paths that can be formed when the agents in $A$ play according to $S_A$, beginning from the state $q$.*

The **formulae of** ATL are defined as follows[2]:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \varphi \mid \langle\!\langle A \rangle\!\rangle (\varphi \,\mathsf{U}\, \varphi) \mid \langle\!\langle A \rangle\!\rangle (\varphi \,\mathsf{R}\, \varphi)$$

Other Boolean connectives are defined as usual, and the combined operators $\langle\!\langle A \rangle\!\rangle \mathsf{F}\, \varphi$ and $\langle\!\langle A \rangle\!\rangle \mathsf{G}\, \varphi$ are defined respectively by $\langle\!\langle A \rangle\!\rangle \top \mathsf{U}\, \varphi$ and $\langle\!\langle A \rangle\!\rangle \bot \mathsf{R}\, \varphi$.

DEFINITION 2.3. *Let $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a CGM, $q \in \mathrm{St}$ a state and $\varphi$ an ATL-formula. Truth of $\varphi$ in $\mathcal{M}$ and $q$, denoted by $\mathcal{M}, q \models \varphi$, is defined as follows:*

- $\mathcal{M}, q \models p$ *iff* $q \in v(p)$ *(for $p \in \Pi$).*
- $\mathcal{M}, q \models \neg\psi$ *iff* $\mathcal{M}, q \not\models \psi$.
- $\mathcal{M}, q \models \psi \vee \theta$ *iff* $\mathcal{M}, q \models \psi$ *or* $\mathcal{M}, q \models \theta$.
- $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi$ *iff there exists $S_A$ such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, we have $\mathcal{M}, \Lambda[1] \models \psi$.*
- $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{U}\, \theta$ *iff there exists $S_A$ such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, there is $i \geq 0$ such that $\mathcal{M}, \Lambda[i] \models \theta$ and $\mathcal{M}, \Lambda[j] \models \psi$ for every $j < i$.*
- $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{R}\, \theta$ *iff there exists $S_A$ such that for each $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$, we have $\mathcal{M}, \Lambda[i] \models \theta$ or there is $j < i$ such that $\mathcal{M}, \Lambda[j] \models \psi$.*

[1]Unless otherwise specified, a 'strategy' hereafter will mean a positional and deterministic strategy.

[2]The operator R (Release) was not part of the original syntax of ATL but has been commonly added later.

# 3. GAME-THEORETIC SEMANTICS

In this section we will introduce unbounded, bounded and finitely bounded evaluation games for ATL. By defining the truth of a formula as the existence of a winning strategy for the verifier in the corresponding evaluation game, these variants of evaluation games lead to three different versions of game-theoretic semantics for ATL.

## 3.1 Unbounded evaluation games

Given a CGM $\mathcal{M}$, a state $q_{in}$ and a formula $\varphi$, the **evaluation game** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$ is intuitively an argument between two opponents, **Eloise** (**E**) and **Abelard** (**A**), about whether the formula $\varphi$ is true at the state $q_{in}$ in the model $\mathcal{M}$. Eloise claims that $\varphi$ is true, so she adopts (initially) the role of a **verifier** in the game, and Abelard tries to prove the formula false, so he is (initially) the **falsifier**. These roles can swap in the course of the game when negations are encountered in the formula to be evaluated.

We will often use the following notation: if $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$, then $\overline{\mathbf{P}}$ denotes the **opponent** of $\mathbf{P}$, i.e., $\overline{\mathbf{P}} \in \{\mathbf{A}, \mathbf{E}\} \setminus \{\mathbf{P}\}$.

DEFINITION 3.1. *Let $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a CGM, $q_{in} \in \mathrm{St}$ and $\varphi$ an ATL-formula. The **unbounded evaluation game** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$ between the players **A** and **E** is defined as follows.*

- *A **position** of the game is a tuple $\mathsf{Pos} = (\mathbf{P}, q, \psi)$ where $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$, $q \in \mathrm{St}$ and $\psi$ is a subformula of $\varphi$. The **initial position** of the game is $\mathsf{Pos}_0 := (\mathbf{E}, q_{in}, \varphi)$.*
- *In every position $(\mathbf{P}, q, \psi)$, the player $\mathbf{P}$ is called the **verifier** and $\overline{\mathbf{P}}$ the **falsifier** for that position.*
- *Each position of the game is associated with a rule. The rules for positions where the related formula is either a proposition symbol or has a Boolean connective as its main connective, are defined as follows.*

1. *If $\mathsf{Pos}_i = (\mathbf{P}, q, p)$, where $p \in \Pi$, then $\mathsf{Pos}_i$ is called an **ending position** of the evaluation game. If $q \in v(p)$, then $\mathbf{P}$ wins the evaluation game. Else $\overline{\mathbf{P}}$ wins.*
2. *Let $\mathsf{Pos}_i = (\mathbf{P}, q, \neg\psi)$. The game then moves to the next position, $\mathsf{Pos}_{i+1} = (\overline{\mathbf{P}}, q, \psi)$.*
3. *Let $\mathsf{Pos}_i = (\mathbf{P}, q, \psi \vee \theta)$. Then the player $\mathbf{P}$ decides whether $\mathsf{Pos}_{i+1} = (\mathbf{P}, q, \psi)$ or $\mathsf{Pos}_{i+1} = (\mathbf{P}, q, \theta)$.*

*In order to deal with the strategic operators, we now define a **one step game**, denoted by $\mathsf{step}(\mathbf{P}, A, q)$, where $A \subseteq \mathbb{A}\mathrm{gt}$. This game consists of the following two actions.*

*i) First $\mathbf{P}$ chooses an action $\alpha_i \in d(i, q)$ for each $i \in A$.*

*ii) Then $\overline{\mathbf{P}}$ chooses an action $\alpha_i \in d(i, q)$ for each $i \in \overline{A}$.*

*The **resulting state** of the one step game $\mathsf{step}(\mathbf{P}, A, q)$ is the state $q' := o(q, \alpha_1, \dots, \alpha_k)$ arising from the combined action of the agents. We now define how the evaluation game proceeds in positions where the formula is of type $\langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi$:*

4. *Let $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi)$. The next position $\mathsf{Pos}_{i+1}$ is $(\mathbf{P}, q', \psi)$, where $q'$ is the resulting state of $\mathsf{step}(\mathbf{P}, A, q)$.*

*The rules for the other strategic operators are obtained by iterating the one step game. For this purpose, we now define the* ***embedded game*** $\mathbf{G} := \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$*, where both* $\mathbf{V}, \mathbf{C} \in \{\mathbf{E}, \mathbf{A}\}$*,* $A$ *is a coalition,* $q_0$ *a state, and* $\psi_{\mathbf{C}}$ *and* $\psi_{\overline{\mathbf{C}}}$ *are formulae. The player* $\mathbf{V}$ *is called the* ***verifier*** *(of the embedded game) and* $\mathbf{C}$ *the* ***controller****. These may, but need not be, the same player. We let* $\overline{\mathbf{V}}$ *and* $\overline{\mathbf{C}}$ *denote the opponents of* $\mathbf{C}$ *and* $\mathbf{V}$*, respectively.*

*The embedded game* $\mathbf{G}$ *starts from the* ***initial state*** $q_0$ *and proceeds from any state* $q$ *according to the following rules, applied in the order below, until an* ***exit position*** *is reached.*

*i)* $\mathbf{C}$ *may end the game at the exit position* $(\mathbf{V}, q, \psi_{\mathbf{C}})$*.*
*ii)* $\overline{\mathbf{C}}$ *may end the game at the exit position* $(\mathbf{V}, q, \psi_{\overline{\mathbf{C}}})$*.*
*iii) If the game has not ended due to the above rules, the one step game* $\mathsf{step}(\mathbf{V}, A, q)$ *is played to produce a resulting state* $q'$*. The embedded game is continued from* $q'$*.*

*If the embedded game* $\mathbf{G}$ *continues an infinite number of rounds, the controller* $\mathbf{C}$ *loses the entire evaluation game* $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$*. Else the evaluation game resumes from the exit position of the embedded game.*

*We now define the rules of the evaluation game for the remaining strategic operators as follows:*

5. *Consider a position* $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\langle A\rangle\rangle\, \psi \,\mathsf{U}\, \theta)$*. The next position* $\mathsf{Pos}_{i+1}$ *is the exit position of the embedded game* $\mathbf{g}(\mathbf{P}, \mathbf{P}, A, q, \theta, \psi)$*. (Note the order of the formulae.)*
6. *Consider a position* $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\langle A\rangle\rangle\, \psi \,\mathsf{R}\, \theta)$*. The next position* $\mathsf{Pos}_{i+1}$ *is the exit position of the embedded game* $\mathbf{g}(\mathbf{P}, \overline{\mathbf{P}}, A, q, \theta, \psi)$*.*

The embedded game $\mathbf{g}(\mathbf{V}, \mathbf{C}, A, q, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ can be seen as a 'simultaneous reachability game' where both players have a goal they are trying to reach before the opponent reaches her/his goal. The verifier $\mathbf{V}$ leads the coalition $A$ and the falsifier $\overline{\mathbf{V}}$ leads the opposing coalition $\overline{A}$. The goal of both $\mathbf{V}$ and $\overline{\mathbf{V}}$ is defined by a formula. When $\mathbf{V} = \mathbf{C}$, the goal of $\mathbf{V}$ is to verify $\psi_{\mathbf{C}}$ and the goal of $\overline{\mathbf{V}}$ is to *falsify* $\psi_{\overline{\mathbf{C}}}$. When $\mathbf{V} \neq \mathbf{C}$, the goal of $\mathbf{V}$ is to verify $\psi_{\overline{\mathbf{C}}}$ and that of $\overline{\mathbf{V}}$ is to falsify $\psi_{\mathbf{C}}$. Both players $\mathbf{V}$ and $\overline{\mathbf{V}}$ have the possibility to end the game when they believe that they have reached their goal. However, the controller is responsible for ending the embedded game in finite time, and (s)he will lose if the game continues infinitely long. If both players reach their targets at the same time, the controller $\mathbf{C}$ has the priority to end the embedded game first.

## 3.2 Bounded evaluation games

The difference between bounded and unbounded evaluation games is that in the bounded case, the embedded games are associated with a time limit. In a bounded evaluation game, the controller must first announce some possibly infinite ordinal $\gamma$ which will decrease in each round. This will guarantee that the embedded game, and in fact the entire evaluation game, will end after a finite number of rounds.

Bounded evaluation games $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$ have an additional parameter $\Gamma$, which is an ordinal that fixes an upper bound for the ordinals that the players can announce during the related embedded games. Different parameters $\Gamma$ give rise to different kinds of evaluation games and thus lead to different kinds of game-theoretic semantics, as we will see.

DEFINITION 3.2. *Let* $\mathcal{M}$ *be a* $\mathsf{CGM}$*,* $q_{in} \in \mathrm{St}$*,* $\varphi$ *an* $\mathsf{ATL}$*-formula and* $\Gamma$ *an ordinal. The* ***bounded evaluation game*** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$ *is defined as the unbounded evaluation game* $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$*, the only difference between the two games being the treatment of until- and release-formulae.*

*Let* $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ *be an embedded game that arises from a position* $\mathsf{Pos}$ *in* $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$*. In that same position* $\mathsf{Pos}$ *in the bounded evaluation game* $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$*, the player* $\mathbf{C}$ *first chooses some ordinal* $\gamma_0 < \Gamma$ *as the* ***initial time limit*** *for the embedded game* $\mathbf{G}$*. This choice leads to a* ***bounded embedded game*** *that is denoted by* $\mathbf{G}[\gamma_0]$*.*

*A* ***configuration*** *of* $\mathbf{G}[\gamma_0]$ *is a pair* $(\gamma, q)$*, where* $\gamma$ *is a (possibly infinite) ordinal called the* ***current time limit*** *and* $q \in \mathrm{St}$ *a state called the* ***current state****. The bounded embedded game* $\mathbf{G}[\gamma_0]$ *starts from the* ***initial configuration*** $(\gamma_0, q_0)$ *and proceeds from any configuration* $(\gamma, q)$ *according to the following rules, applied in the given order.*

*i) If* $\gamma = 0$*, the game ends at the exit position* $(\mathbf{V}, q, \psi_{\mathbf{C}})$*.*
*ii)* $\mathbf{C}$ *may end the game at the exit position* $(\mathbf{V}, q, \psi_{\mathbf{C}})$*.*
*iii)* $\overline{\mathbf{C}}$ *may end the game at the exit position* $(\mathbf{V}, q, \psi_{\overline{\mathbf{C}}})$*.*
*iv) If the game has not ended due to the previous rules, then* $\mathsf{step}(\mathbf{V}, A, q)$ *is played in order to produce a resulting state* $q'$*. Then the bounded embedded game continues from the configuration* $(\gamma', q')$*, where* $\gamma' = \gamma - 1$ *if* $\gamma$ *is a successor ordinal, and if* $\gamma$ *is a limit ordinal, then* $\gamma'$ *is an ordinal smaller than* $\gamma$ *and chosen by* $\mathbf{C}$*.*

*We denote the set of configurations in* $\mathbf{G}[\gamma_0]$ *by* $\mathsf{Conf}_{\mathbf{G}[\gamma_0]}$*. After the bounded embedded game* $\mathbf{G}[\gamma_0]$ *has reached an exit position—which it will, because ordinals are well-founded—the evaluation game resumes from the exit position.*

It is clear that bounded evaluation games end after a finite number of rounds because bounded embedded games do. Note that if time limits are infinite ordinals, they do not directly refer to the number of rounds left in the game, but instead they are related to the game duration in a more abstract way. Different kinds of ways to use ordinals in game-theoretic considerations go way back. An important and relatively early reference is [13] which contains references to even earlier related articles.

It is possible to analyse embedded games as separate entities independent of evaluation games. An embedded game of the form $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ can be played without a time limit as in unbounded evaluation games, or it can be given some time limit $\gamma_0$ as a parameter, which leads to the related bounded embedded game $\mathbf{G}[\gamma_0]$. When we use the plain notation $\mathbf{G}$ (as opposed to $\mathbf{G}[\gamma_0]$), we always assume that the embedded game $\mathbf{G}$ is not bounded—we may even emphasize this by calling $\mathbf{G}$ an *unbounded* embedded game.

Evaluation games of the form $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \omega)$ constitute a particularly interesting subclass of bounded evaluation games. We call the games in this class **finitely bounded evaluation games.** In these games, only *finite* time limits are allowed to be announced for bounded embedded games.

## 3.3 Game-theoretic semantics

A strategy for a player $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ will be defined below to be a function on game positions; in positions where the player $\mathbf{P}$ is not required to make a move, the strategy of $\mathbf{P}$ will output a special value "$\mathsf{void}$". We occasionally also give the value $\mathsf{void}$ to some other functions when the output is not relevant (e.g., when formulating a winning strategy, we may assign $\mathsf{void}$ for losing positions).

DEFINITION 3.3. *Let* $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ *be an embedded game and* $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$*.* ***A strategy for the player*** $\mathbf{P}$ ***in*** $\mathbf{G}$ *is a function* $\sigma_{\mathbf{P}}$ *whose domain is* $\mathrm{St}$ *and* *whose range is specified below. Firstly, for any $q \in \mathrm{St}$, it is possible to define $\sigma_{\mathbf{P}}(q) \in \{\psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}}\}$; then $\sigma_{\mathbf{P}}$ instructs $\mathbf{P}$ to end the game at the state $q$. Here it is required that if $\mathbf{P} = \mathbf{C}$, then $\sigma_{\mathbf{P}}(q) = \psi_{\mathbf{C}}$ and if $\mathbf{P} = \overline{\mathbf{C}}$, then $\sigma_{\mathbf{P}}(q) = \psi_{\overline{\mathbf{C}}}$. If $\sigma_{\mathbf{P}}(q) \notin \{\psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}}\}$, then the following conditions hold.*

- *If $\mathbf{P} = \mathbf{V}$, then $\sigma_{\mathbf{P}}(q)$ is a tuple of actions in $\mathsf{action}(A, q)$.*
- *If $\mathbf{P} = \overline{\mathbf{V}}$, then $\sigma_{\mathbf{P}}(q)$ is defined to be a* ***response function*** *$f : \mathsf{action}(A, q) \to \mathsf{action}(\overline{A}, q)$ that assigns a tuple of actions for $\overline{A}$ as a response to any tuple of actions for $A$.*

*Let $\gamma_0$ be an ordinal. A strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$ is defined in the same way as a strategy in $\mathbf{G}$, but the domain of this strategy is the set of all possible configurations $\mathsf{Conf}_{\mathbf{G}[\gamma_0]}$.*

Note that strategies in embedded games are positional, i.e., they depend only on the current state in the unbounded case and the current configuration in the bounded case.

Any strategy $\sigma_{\mathbf{P}}$ for an unbounded embedded game $\mathbf{G}$ can be used also in any bounded embedded game $\mathbf{G}[\gamma_0]$: we simply use the same action $\sigma_{\mathbf{P}}(q)$ for each configuration $(\gamma, q) \in \mathsf{Conf}_{\mathbf{G}[\gamma_0]}$. Also note that if a strategy $\sigma_{\mathbf{P}}$ for a bounded embedded game $\mathbf{G}[\gamma_0]$ is independent of time limits (and thus depends on states only), it can also be used in the unbounded embedded game $\mathbf{G}$.

Definition 3.4. *Let $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$. A* ***strategy for player $\mathbf{P}$ in an unbounded evaluation game*** *$\mathcal{G} = \mathcal{G}(\mathcal{M}, q_{in}, \varphi)$ is a function $\Sigma_{\mathbf{P}}$ defined on the set of positions $\mathsf{POS}$ of $\mathcal{G}$ (with the range specified below) satisfying the following conditions.*

1. *If $\mathsf{Pos} = (\mathbf{P}, q, \psi \vee \theta)$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos}) \in \{\psi, \theta\}$.*
2. *If $\mathsf{Pos} = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi)$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos})$ is a tuple of actions in $\mathsf{action}(A, q)$ for the one step game $\mathsf{step}(\mathbf{P}, A, q)$.*
3. *If $\mathsf{Pos} = (\overline{\mathbf{P}}, q, \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi)$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos})$ is a response function $f : \mathsf{action}(A, q) \to \mathsf{action}(\overline{A}, q)$ for $\mathsf{step}(\mathbf{P}, A, q)$.*
4. *Let $\mathsf{Pos} = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \psi \mathsf{T} \theta)$ or $\mathsf{Pos} = (\overline{\mathbf{P}}, q, \langle\!\langle A \rangle\!\rangle\, \psi \mathsf{T} \theta)$, where $\mathsf{T} \in \{\mathsf{U}, \mathsf{R}\}$. Then $\Sigma_{\mathbf{P}}(\mathsf{Pos})$ is a strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ in the respective embedded game $\mathbf{g}(\mathbf{V}, \mathbf{C}, A, q, \theta, \psi)$.*
5. *In all other cases, $\Sigma_{\mathbf{P}}(\mathsf{Pos}) = \mathsf{void}$.*

*We say that $\Sigma_{\mathbf{P}}$ is a* ***winning strategy*** *for $\mathbf{P}$ in $\mathcal{G}$ if $\mathbf{P}$ wins all plays of $\mathcal{G}$ where (s)he plays according to that strategy.*

Definition 3.5. *A* ***strategy for player $\mathbf{P}$ in a bounded evaluation game*** *$\mathcal{G} = \mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$ is defined as in Definition 3.4, with the exception of positions with until- and release-formulae, which are treated as follows.*

4. *Let $\mathsf{Pos} = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \psi \mathsf{T} \theta)$ or $\mathsf{Pos} = (\overline{\mathbf{P}}, q, \langle\!\langle A \rangle\!\rangle\, \psi \mathsf{T} \theta)$, where $\mathsf{T} \in \{\mathsf{U}, \mathsf{R}\}$, and let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q, \theta, \psi)$ denote the embedded game related to $\mathsf{Pos}$. If $\mathbf{P} = \mathbf{C}$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos}) = (\gamma_0, t, \sigma_{\mathbf{P}})$ where the following conditions hold.*
   - *$\gamma_0 < \Gamma$ is an ordinal. It is the choice for the initial time limit that leads to the bounded embedded game $\mathbf{G}[\gamma_0]$.*
   - *$t$ is a function, called* ***timer***, *on pairs $(\gamma, q)$, where $\gamma \le \gamma_0$ is a limit ordinal and $q \in \mathrm{St}$. The timer $t$ gives an instruction how to lower the time limit $\gamma$ after a transition to $q$ has been made; the value $t(\gamma, q)$ must be an ordinal less than $\gamma$.*
   - *$\sigma_{\mathbf{P}}$ is a strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$.*

   *Finally, if $\mathbf{P} \neq \mathbf{C}$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos})$ is a function that maps any ordinal $\gamma_0 < \Gamma$ to some strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$.*

In finitely bounded evaluation games, only finite time limits $\gamma_0 < \omega$ may be announced by $\mathbf{C}$. Since no limit ordinal is reached, the timer $t$ can be omitted from the strategy.

Different choices for time limit bounds $\Gamma$ give rise to different semantic systems, and most results in the next section will be proven for an arbitrary choice of $\Gamma$. However, in this paper we mainly focus on the cases $\Gamma = \omega$ (where $\omega$ is the smallest infinite ordinal) and $\Gamma = 2^{\kappa}$, where $\kappa$ is the cardinality of the model. We will prove later that time limit bounds greater than $2^{\kappa}$ are not needed.

Definition 3.6. *Let $\mathcal{M}$ be a $\mathsf{CGM}$, $q \in \mathrm{St}$ and $\varphi$ an $\mathsf{ATL}$-formula. Let $\kappa$ be the cardinality of the model $\mathcal{M}$. We define three different notions of truth of $\varphi$ in $\mathcal{M}$ and $q$ based on three different evaluation games, thereby defining the* ***unbounded, bounded*** *and* ***finitely bounded semantics*** *(denoted, respectively, by $\models^g_u$, $\models^g_b$, and $\models^g_f$) as follows.*

- *$\mathcal{M}, q \models^g_u \varphi$ iff $\mathbf{E}$ has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi)$.*
- *$\mathcal{M}, q \models^g_b \varphi$ iff $\mathbf{E}$ has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, 2^{\kappa})$.*
- *$\mathcal{M}, q \models^g_f \varphi$ iff $\mathbf{E}$ has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, \omega)$.*

*We also write more generally that $\mathcal{M}, q \models^g_{\Gamma} \varphi$ iff $\mathbf{E}$ has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, \Gamma)$.*

We will prove that *both* the bounded and unbounded semantics are equivalent to the standard compositional semantics of Definition 2.3. The finitely bounded sematics, on the other hand, is equivalent to a natural variant of the compositional semantics to be introduced in Section 5. The following example shows that the finitely bounded $\mathsf{GTS}$ differs from the unbounded and bounded cases. In particular, the fixed point property of the temporal operator $\mathsf{F}$ fails:

Example 3.7. *Let $\mathcal{M} = (\{a\}, \{q_0\} \cup \mathbb{N} \times \mathbb{N}, \{p\}, \mathbb{N}, d, o, v)$, where $v(p) = \{(i, i) \mid i \in \mathbb{N}\}$, $d(a, q_0) = \mathbb{N}$, $d(a, (i, j)) = \{0\}$, $o(q_0, i) = (i, 0)$ and $o((i, j), 0) = (i, j+1)$. In this model $\mathcal{M}, q_0 \not\models^g_f \langle\!\langle \emptyset \rangle\!\rangle \mathsf{F}\, p$ while $\mathcal{M}, q_0 \models^g_f \langle\!\langle \emptyset \rangle\!\rangle \mathsf{X} \langle\!\langle \emptyset \rangle\!\rangle \mathsf{F}\, p$. This is because for every time limit $n < \omega$ chosen by Eloise, Abelard may select the action $n$ in the first round for the agent $a$, so it will take $n+1$ rounds to reach a state where $p$ is true. But after the first step, the game will be at a state $(i, 0)$ for some $i \in \mathbb{N}$, whence Eloise can choose any time limit $n \ge i$ and reach a state where $p$ is true before time runs out.*

*However, $\mathcal{M}, q_0 \models^g_b \langle\!\langle \emptyset \rangle\!\rangle \mathsf{F}\, p$, since Eloise can choose $\omega$ as the time limit in the beginning of the game and then lower it to $i < \omega$ when the next state $(i, 0)$ is reached. Also, $\mathcal{M}, q_0 \models^g_u \langle\!\langle \emptyset \rangle\!\rangle \mathsf{F}\, p$ since a state where $p$ is true will always be reached in finite time. Still, we will show that the three semantics become equivalent over image finite models.*

# 4. ANALYSING EMBEDDED GAMES

In this section we will examine the properties of different versions of embedded games that occur as part of evaluation games. We associate each state with a winning time label which describes how good that state is for the players. Using the optimal labels will lead to a canonical strategy which will be a winning strategy whenever there exists one.

## 4.1 Winning time labels

Different values of the time limit bound $\Gamma$ correspond to different classes of bounded embedded games $\mathbf{G}[\gamma_0]$ where $\gamma_0 < \Gamma$. In this section—unless otherwise specified—we use a fixed value of $\Gamma$ and assume that all bounded embedded games are part of some evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$. Since $\Gamma$ could have any ordinal value, our results will hold for both the bounded and finitely bounded semantics.

If $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ is an embedded game and $q \in \mathrm{St}$, we write $\mathbf{G}[q] := \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$. We also use the abbreviation $\mathbf{G}[q,\gamma] := (\mathbf{G}[q])[\gamma]$. This notation is useful, since by the recursive nature of bounded embedded games, any configuration $(\gamma, q)$ of $\mathbf{G}[\gamma_0]$ (where $\gamma_0 < \Gamma$) is the initial configuration of $\mathbf{G}[q,\gamma]$. Note that since the players use positional strategies, they do not see any difference between initial configurations and other configurations.

We next define winning strategies for embedded games. "Winning an embedded game" means for the player $\mathbf{P}$ that (s)he has a winning strategy in the *evaluation game* that continues from the exit position of the embedded game.

Definition 4.1. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game and let $\gamma_0 < \Gamma$.*

*1. We say that $\sigma_{\mathbf{P}}$ is a **winning strategy for the player $\mathbf{P}$ in $\mathbf{G}$** if infinite plays are possible with $\sigma_{\mathbf{P}}$ only if $\mathbf{P} \neq \mathbf{C}$ and the equivalence $\mathcal{M}, q \models^g_u \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$ holds for all exit positions $(\mathbf{V}, q, \psi)$ of $\mathbf{G}$ that can be reached with $\sigma_{\mathbf{P}}$.*

*2. If $\mathbf{P} = \mathbf{C}$, we say that the pair $(\sigma_{\mathbf{P}}, t)$ is a **timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$** if $\mathcal{M}, q \models^g_\Gamma \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$ holds for all exit positions $(\mathbf{V}, q, \psi)$ that can be encountered when $\mathbf{P}$ plays using the strategy $\sigma_{\mathbf{P}}$ and timer $t$.*

*If $\mathbf{P} \neq \mathbf{C}$, we say that $\sigma_{\mathbf{P}}$ is a **winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$** if $\mathcal{M}, q \models^g_\Gamma \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$ holds for all exit positions $(\mathbf{V}, q, \psi)$ that can occur when $\mathbf{P}$ plays using $\sigma_{\mathbf{P}}$.*

If the unbounded (respectively, bounded) embedded game in the above definition ends in a position where the equivalence $\mathcal{M}, q \models^g_u \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$ (respectively, $\mathcal{M}, q \models^g_\Gamma \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$) holds, we also say that $\mathbf{P}$ **wins** the embedded game. In the unbounded case, $\overline{\mathbf{C}}$ wins also if the play is infinite.

Consider an embedded game $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$. We next define for $\mathbf{G}$ so called **winning time labels**, $\mathcal{L}_{\mathbf{P}}(q)$, for each $q \in \mathrm{St}$. The labels will indicate how good the state $q$ is for the player $\mathbf{P}$ when different bounded embedded games $\mathbf{G}[q, \gamma_0]$ are played with different time limits $\gamma_0 < \Gamma$. If the label is "win" or "lose", then the state is a winning (respectively, losing) state for $\mathbf{P}$, regardless of the time limit $\gamma_0$. If the label is an ordinal $\gamma < \Gamma$, it means that $\gamma$ is the "critical time limit" for winning or losing the game: if $\mathbf{P} = \mathbf{C}$, $\gamma$ is the least time limit needed for $\mathbf{P}$ to win from $q$, and if $\mathbf{P} \neq \mathbf{C}$, then $\gamma$ is the least time limit such that $\mathbf{P}$ can no longer guarantee that (s)he will not lose the game from $q$.

Definition 4.2. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game and $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$. The **winning time label $\mathcal{L}_{\mathbf{P}}(q)$ for $\mathbf{P}$ in $\mathbf{G}$ at state** $q \in \mathrm{St}$ is defined as follows.*

***Case 1.** Suppose $\mathbf{P} = \mathbf{C}$. Let $\sigma_{\mathbf{P}}$ be a strategy for $\mathbf{P}$. We first define a **strategy label** $l(q, \sigma_{\mathbf{P}})$ as follows.*

- *Set $l(q, \sigma_{\mathbf{P}}) := \mathsf{lose}$ if $(\sigma_{\mathbf{P}}, t)$ is not a timed winning strategy in $\mathbf{G}[q, \gamma]$ for any timer $t$ and $\gamma < \Gamma$.*
- *Else, set $l(q, \sigma_{\mathbf{P}}) := \gamma$, where $\gamma < \Gamma$ is the least time limit for which there is a timer $t$ such that $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$.*

*When there exists at least one strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ such that $l(q, \sigma_{\mathbf{P}}) \neq \mathsf{lose}$, we define $\mathcal{L}_{\mathbf{P}}(q)$ as the least ordinal value of strategy labels $l(q, \sigma_{\mathbf{P}})$. Else, we define $\mathcal{L}_{\mathbf{P}}(q) := \mathsf{lose}$.*

***Case 2.** Suppose $\mathbf{P} \neq \mathbf{C}$. Let $\sigma_{\mathbf{P}}$ be a strategy for $\mathbf{P}$.*

- *If $\sigma_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, \gamma]$ for every time limit $\gamma < \Gamma$, then set $l(q, \sigma_{\mathbf{P}}) := \mathsf{win}$.*
- *Else, set $l(q, \sigma_{\mathbf{P}}) := \gamma$, where $\gamma < \Gamma$ is the least time limit such that $\sigma_{\mathbf{P}}$ is not a winning strategy in $\mathbf{G}[q, \gamma]$.*

*If $l(q, \sigma_{\mathbf{P}}) = \mathsf{win}$ for some $\sigma_{\mathbf{P}}$, then set $\mathcal{L}_{\mathbf{P}}(q) := \mathsf{win}$. Else, set $\mathcal{L}_{\mathbf{P}}(q)$ to be the least upper bound for the values $l(q, \sigma_{\mathbf{P}})$.*

The following proposition relates values of winning time labels to durations of embedded games and existence of winning strategies. A proof, which follows quite directly from the definition of winning time labels, is given in [7].

Proposition 4.3. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ and $q \in \mathrm{St}$.*

1. *Assume $\mathbf{P} = \mathbf{C}$. We have $\mathcal{L}_{\mathbf{P}}(q) = \gamma < \Gamma$ iff there is a pair $(\sigma_{\mathbf{P}}, t)$ that is a timed winning strategy in $\mathbf{G}[q, \gamma']$ for all $\gamma'$ such that $\gamma \leq \gamma' < \Gamma$, but there is no timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma']$ for any $\gamma' < \gamma$.*
   *We have $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{lose}$ iff there is no timed winning strategy $(\sigma_{\mathbf{P}}, t)$ for $\mathbf{P}$ in $\mathbf{G}[q, \gamma]$ for any $\gamma < \Gamma$.*
2. *Assume $\mathbf{P} \neq \mathbf{C}$. We have $\mathcal{L}_{\mathbf{P}}(q) = \gamma < \Gamma$ iff for every $\gamma' < \gamma$, there is some $\sigma_{\mathbf{P}}$ which is a winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma']$, but there is no winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma']$ for any $\gamma'$ such that $\gamma \leq \gamma' < \Gamma$.*
   *We have $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$ iff there is a strategy $\sigma_{\mathbf{P}}$ which is a winning strategy in $\mathbf{G}[q, \gamma]$ for every $\gamma < \Gamma$.*

Winning time labels $\mathcal{L}_{\mathbf{P}}(q)$ of an embedded game are either ordinals less than the time limit bound $\Gamma$ or labels $\mathsf{win}$, $\mathsf{lose}$. If we increased the value of $\Gamma$ to some $\Gamma' > \Gamma$ and considered the values of winning time labels of the corresponding embedded game within the evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma')$, then some of the labels that originally were $\mathsf{win}$ or $\mathsf{lose}$, could now obtain ordinal values $\gamma$ s.t. $\Gamma \leq \gamma < \Gamma'$. Other kinds of changes of labels would also be possible because the truth sets of the goal formulae $\psi_{\mathbf{C}}$ and $\psi_{\overline{\mathbf{C}}}$ could change. However, it is easy to see that if all ordinal valued labels stay strictly below $\Gamma$ in all embedded games when going from $\Gamma$ to $\Gamma'$, then each label in fact remains the same in the transition.

We say that $\Gamma$ is **stable** for an embedded game $\mathbf{G}$ if the winning time labels of the game cannot be altered by increasing $\Gamma$. We say that $\Gamma$ is **globally stable** for a concurrent game model $\mathcal{M}$ if $\Gamma$ is stable for *all* bounded embedded games within all evaluation games $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$. We will see later that there exists a globally stable time limit bound for every concurrent game model. When $\Gamma$ is globally stable, its role is not so relevant anymore, since players would not benefit from the ability to choose arbitrarily high time limits. However, for technical reasons, we always need some time limit bound to avoid strategies becoming proper classes.

## 4.2 Canonical strategies for embedded games

Definition 4.4. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, let $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ and assume that $\mathbf{P} = \mathbf{C}$. We define the **canonical strategy $\tau_{\mathbf{P}}$** and **canonical timer** $t_{can}$ for $\mathbf{P}$ in $\mathbf{G}$ as follows.*

*If $\mathcal{L}_{\mathbf{P}}(q) = \gamma$, then $\tau_{\mathbf{P}}(q) = \sigma_{\mathbf{P}}(\gamma, q)$ for some strategy $\sigma_{\mathbf{P}}$ for which there is a timer $t$ such that $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q, \gamma']$ for all $\gamma'$ such that $\gamma \leq \gamma' < \Gamma$. (Note that such a strategy exists by Proposition 4.3). And if $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{lose}$, then $\tau_{\mathbf{P}}(q) = \mathsf{void}$.*

*We define $t_{can}$ for any pair $(\gamma, q)$ ($\gamma < \Gamma$ is a limit ordinal and $q \in \mathrm{St}$) such that if $\mathcal{L}_{\mathbf{P}}(q) \neq \mathsf{lose}$ and $\mathcal{L}_{\mathbf{P}}(q) < \gamma$, then $t_{can}(\gamma, q) = \mathcal{L}_{\mathbf{P}}(q)$, and otherwise $t_{can}(\gamma, q) = \mathsf{void}$.*

*We call the pair $(\tau_{\mathbf{P}}, t_{can})$ the **canonically timed strategy** (for the controller).*

Note that when $\mathbf{P} = \mathbf{C}$, the canonical strategy $\tau_{\mathbf{P}}$ depends only on states and can thus be used in both unbounded and bounded embedded games. We will see that if $\mathbf{C}$ can win $\mathbf{G}[\gamma_0]$ for some $\gamma_0 < \Gamma$, then $\mathbf{C}$ wins $\mathbf{G}[\gamma_0]$ with $(\tau_{\mathbf{C}}, t_{can})$.

The canonical strategy of $\mathbf{C}$ can also be seen, in some sense, as optimal for winning the game as fast as possible.

DEFINITION 4.5. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, let $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ and assume that $\mathbf{P} \neq \mathbf{C}$. We define the **canonical strategy** $\tau_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$ for all $\gamma_0 < \Gamma$. We define $\tau_{\mathbf{P}}$ at every configuration $(\gamma, q)$, where $\gamma < \Gamma$ and $q \in \mathrm{St}$, as follows.*

*If $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, then $\tau_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(\gamma, q)$ for some $\sigma_{\mathbf{P}}$ that is a winning strategy in $\mathbf{G}[q, \gamma]$ for every $\gamma < \Gamma$ (such a strategy exists by Proposition 4.3). Else, if $\mathcal{L}_{\mathbf{P}}(q) = \gamma'$ and $\gamma' > \gamma$, then $\tau_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(\gamma, q)$ for some $\sigma_{\mathbf{P}}$ that is a winning strategy in $\mathbf{G}[q, \gamma]$ (such a strategy exists by Proposition 4.3). Otherwise we define $\tau_{\mathbf{P}}(\gamma, q) = \mathsf{void}$.*

*We also define, for every $n < \omega$, the $n$-**canonical strategy** $\tau^n_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}$ and the $\infty$-**canonical strategy** $\tau^\infty_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}$. These are defined for each $q \in \mathrm{St}$ as follows.*

*If $\mathcal{L}_{\mathbf{P}}(q) \geq \omega$ or $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, then $\tau^n_{\mathbf{P}}(q) = \tau_{\mathbf{P}}(n, q)$. Else, if $\mathcal{L}_{\mathbf{P}}(q) = m > 0$, then $\tau^n_{\mathbf{P}}(q) = \sigma_{\mathbf{P}}(m-1, q)$ for some $\sigma_{\mathbf{P}}$ that is a winning strategy in $\mathbf{G}[q, m-1]$ (such a strategy exists by Proposition 4.3). Otherwise $\tau^n_{\mathbf{P}}(q) = \mathsf{void}$.*

*If $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, then $\tau^\infty_{\mathbf{P}}(q) = \tau_{\mathbf{P}}(\Gamma-1, q)$, and otherwise $\tau^\infty_{\mathbf{P}}(q) = \mathsf{void}$. Note that to be able to define $\tau^\infty_{\mathbf{P}}$, we have to assume that $\Gamma$ is a successor ordinal.*

When $\mathbf{P} \neq \mathbf{C}$, the canonical strategy $\tau_{\mathbf{P}}$ depends on time limits, and thus it cannot be used in unbounded embedded games. However, both $n$-canonical and $\infty$-canonical strategies depend only on states. We fix the notation such that from now on $\tau_{\mathbf{P}}$, $\tau^n_{\mathbf{P}}$ and $\tau^\infty_{\mathbf{P}}$ will always denote canonical strategies (of the respective type) for the player $\mathbf{P}$.

DEFINITION 4.6. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game. Let $\sigma_{\mathbf{P}}$ be a strategy in $\mathbf{G}[\gamma_0]$ ($\gamma_0 < \Gamma$). Suppose that $(\gamma, q)$ is such a configuration that $\sigma_{\mathbf{P}}(\gamma, q)$ is either a tuple of actions for $A$ or some response function for $\overline{A}$. We say that set $Q \subseteq \mathrm{St}$ is **forced** by $\sigma_{\mathbf{P}}(\gamma, q)$ if for each $q' \in \mathrm{St}$, it holds that $q' \in Q$ if and only if there is some play with $\sigma_{\mathbf{P}}$ from $(\gamma, q)$ such that the next configuration is $(\gamma', q')$ for some $\gamma'$. We use the same terminology for the set forced by $\sigma_{\mathbf{P}}(q)$ when $\sigma_{\mathbf{P}}$ depends only on states.*

The following lemma shows that the canonical strategy (with the canonical timer) is guaranteed to be a (timed) winning strategy always when such exists. A proof, which follows quite directly from Definitions 4.4 and 4.5, is given in [7].

LEMMA 4.7. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game. Let $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ and $\gamma_0 < \Gamma$.*

1. *Suppose that $\mathbf{P} = \mathbf{C}$. If $\mathbf{P}$ has a timed winning strategy $(\sigma_{\mathbf{P}}, t)$ in $\mathbf{G}[\gamma_0]$, then $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$.*
2. *Suppose that $\mathbf{P} \neq \mathbf{C}$. If $\mathbf{P}$ has a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}[\gamma_0]$, then $\tau_{\mathbf{P}}$ is a winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$.*

By the first claim of the previous proposition, we see that it suffices to consider those strategies of player $\mathbf{C}$ which are independent of time limits. The following lemma shows that the same holds for the player $\overline{\mathbf{C}}$ in bounded embedded games with a *finite* time limit. The key here will be the use of $n$-canonical strategies. A proof is given in [7].

LEMMA 4.8. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, let $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ and assume that $\mathbf{P} \neq \mathbf{C}$.*

*Let $n < \omega$. Now, if $\mathbf{P}$ has a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}[m]$ for some $m \leq n$, then $\tau^n_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[m]$.*

EXAMPLE 4.9. *In the cases where $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \omega$, the player $\overline{\mathbf{C}}$ can win the game with any time limit $n < \omega$, but there is no single strategy that would win for every $n$. But if $\overline{\mathbf{C}}$ knows that the initial time limit is (at most) $m$, then (s)he knows that the $m$-canonical strategy will be her/his winning strategy. Therefore $\overline{\mathbf{C}}$ needs to know the time limit when **selecting** the strategy, but not when using it (since $n$-canonical strategies are independent of time limits).*

## 4.3 Determinacy of embedded games

The following correspondence between the winning time labels of $\mathbf{C}$ and $\overline{\mathbf{C}}$ will be the key for proving determinacy of bounded embedded games.

PROPOSITION 4.10. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game. The equivalence $\mathcal{L}_{\mathbf{C}}(q) = \gamma$ iff $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \gamma$ holds for each state $q \in \mathrm{St}$ and each ordinal $\gamma < \Gamma$.*

PROOF. Sketch of proof: We can prove the claim by transfinite induction on $\gamma$. The case $\gamma = 0$ is clear, since if $\mathbf{C}$ cannot win the game with time limit 0, then $\overline{\mathbf{C}}$ will win it automatically. We then suppose that the claim holds for every $\gamma' < \gamma$ and prove the equivalence for $\gamma$. If $\mathcal{L}_{\mathbf{C}}(q) = \gamma$, then $\mathbf{C}$ has a winning strategy in $\mathbf{G}[q, \gamma]$ and thus $\overline{\mathbf{C}}$ cannot have a winning strategy in that game. Hence by Proposition 4.3 we have $\mathcal{L}_{\overline{\mathbf{C}}} \leq \gamma$. By the induction hypothesis, $\mathcal{L}_{\overline{\mathbf{C}}} \neq \gamma'$ for every $\gamma' < \gamma$ and thus $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \gamma$.

Suppose then that $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \gamma$. If there existed some $\sigma_{\overline{\mathbf{C}}}$, $\gamma' < \Gamma$ and $Q \subseteq \mathrm{St}$ forced by $\sigma_{\overline{\mathbf{C}}}(\gamma', q)$ s.t. $\mathcal{L}_{\overline{\mathbf{C}}}(q') \geq \gamma$ for every $q' \in Q$, then we could use $\sigma_{\overline{\mathbf{C}}}$ to formulate a winning strategy for $\overline{\mathbf{C}}$ in $\mathbf{G}[q, \gamma]$. This is not possible since we have $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \gamma$. Thus it can be shown that $\mathbf{C}$ can play such that for all possible successor states $q'$, we have $\mathcal{L}_{\overline{\mathbf{C}}}(q') < \gamma$, whence by the induction hypothesis $\mathcal{L}_{\mathbf{C}}(q') < \gamma$. Hence we can formulate a timed winning strategy for $\mathbf{C}$ in $\mathbf{G}[q, \gamma]$ and thus $\mathcal{L}_{\mathbf{C}}(q) = \gamma$. For a detailed proof, see [7]. $\square$

Apart from ordinal values that are less than the bound $\Gamma$, the only possible winning time label for $\mathbf{C}$ is the label $\mathsf{lose}$. For $\overline{\mathbf{C}}$, the only non-ordinal value is $\mathsf{win}$. Hence by the previous proposition, we also have $\mathcal{L}_{\mathbf{C}}(q) = \mathsf{lose}$ iff $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \mathsf{win}$.

PROPOSITION 4.11. *$\mathbf{C}$ has a timed winning strategy in a bounded embedded game $\mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})[\gamma_0]$ iff $\overline{\mathbf{C}}$ does not have a winning strategy in that game.*

PROOF. If $\mathcal{L}_{\mathbf{C}}(q_0) = \mathsf{lose}$, then $\mathcal{L}_{\overline{\mathbf{C}}}(q_0) = \mathsf{win}$, whence by Proposition 4.3, the player $\overline{\mathbf{C}}$ has a winning strategy and $\mathbf{C}$ does not have a timed winning strategy. Else $\mathcal{L}_{\mathbf{C}}(q_0) = \gamma$ for some $\gamma < \Gamma$. Now, by Proposition 4.10, also $\mathcal{L}_{\overline{\mathbf{C}}}(q_0) = \gamma$. If $\gamma \leq \gamma_0$, then by Proposition 4.3 the player $\mathbf{C}$ has a timed winning strategy, while $\overline{\mathbf{C}}$ does not have a winning strategy. Analogously, if $\gamma > \gamma_0$, then $\overline{\mathbf{C}}$ has a winning strategy, while $\mathbf{C}$ does not have a timed winning strategy. $\square$

## 4.4 Finding stable time limit bounds

DEFINITION 4.12. *Let $\mathcal{M}$ be a $\mathsf{CGM}$ and let $q \in \mathrm{St}$. We define the **branching degree of** $q$, $\mathsf{BD}(q)$, as the cardinality of the set of states accessible from $q$ with a single transition: $\mathsf{BD}(q) := \mathsf{card}(\{o(q, \vec{\alpha}) \mid \vec{\alpha} \in \mathsf{action}(\mathbb{A}\mathrm{gt}, q)\})$. We define the **infinite branching bound of** $\mathcal{M}$, $\mathsf{IBB}(\mathcal{M})$, as the smallest infinite cardinal $\kappa$ such that $\kappa > \mathsf{BD}(q)$ for every $q \in \mathrm{St}$.*

With this definition $\mathsf{IBB}(\mathcal{M}) = \omega$ iff $\mathcal{M}$ is image finite. We will see that the value of $\mathsf{IBB}(\mathcal{M})$ is closely related to the sizes of a globally stable time limit bounds for $\mathcal{M}$.

LEMMA 4.13. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ and $\mathbf{P} = \mathbf{C}$. Now the following holds for every $q \in \mathrm{St}$: If $\mathcal{L}_{\mathbf{P}}(q) = \gamma > 0$ and $Q \subseteq \mathrm{St}$ is forced by $\tau_{\mathbf{P}}(q)$, we have $\mathcal{L}_{\mathbf{P}}(q') < \gamma$ for every $q' \in Q$, and*
- $\max\{\mathcal{L}_{\mathbf{P}}(q') \mid q' \in Q\} = \gamma - 1$ *if $\gamma$ is a successor ordinal,*
- $\sup\{\mathcal{L}_{\mathbf{P}}(q') \mid q' \in Q\} = \gamma$ *if $\gamma$ is a limit ordinal.*

PROOF. When $\mathcal{L}_{\mathbf{P}}(q) = \gamma > 0$, by Lemma 4.7, $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$. Therefore every winning time label in the set $Q$ forced by $\tau_{\mathbf{P}}(q)$ must be an ordinal less than $\gamma$. If $\gamma$ is a successor ordinal, then there must be some state with label $\gamma - 1$ in $Q$, and if $\gamma$ is a limit ordinal, then $\gamma$ must be the supremum of the winning time labels in $Q$ (else there would be a winning strategy for $\mathbf{C}$ in $\mathbf{G}[q, \gamma']$ for some $\gamma' < \gamma$). See [7] for more details. □

The following lemma shows that if a certain ordinal valued winning time label exist for an embedded game, then all the smaller winning time labels must exist for that game as well. The proof that is given in [7] is done by using Lemma 4.13.

LEMMA 4.14. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game and $\gamma < \Gamma$ an ordinal. Assume that $\mathcal{L}_{\mathbf{P}}(q) = \gamma$ for some $q \in \mathrm{St}$ and $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$. Now for every $\delta \leq \gamma$ there is a state $q_\delta$ for which $\mathcal{L}_{\mathbf{C}}(q_\delta) = \delta$.*

EXAMPLE 4.15. *In finite models all winning time labels are strictly smaller than the cardinality of the model, i.e. if $\mathsf{card}(\mathcal{M}) = n < \omega$, then $n$ is a globally stable time limit bound for $\mathcal{M}$. This can be seen by the following reasoning.*

*If there was some state with a winning time label $\gamma \geq n$, then by Lemma 4.14, there would be a state $q \in \mathrm{St}$ for which $\mathcal{L}_{\mathbf{C}}(q) = n$. Further, by Lemma 4.14 we would now find states with winning time labels $n-1, n-2, \ldots, 0$. But since winning time labels are unique for each state, this would mean that $\mathsf{card}(\mathcal{M}) > n$, a contradiction.*

*This result is quite obvious by the observation that the controller can only win the embedded game by reaching a state in the truth set of the formula $\psi_{\mathbf{C}}$. Hence it would not be beneficial for the controller to go in cycles.*

The following proposition shows how we can find an upper bound for the values of possible winning time labels by just looking at the infinite branching bound of a model.

PROPOSITION 4.16. *Let $\mathcal{M}$ be a $\mathsf{CGM}$ such that we have $\mathsf{IBB}(\mathcal{M}) = \kappa$. We define an ordinal $\Gamma$ as follows:*

$$\begin{cases} \Gamma := \kappa \text{ if } \kappa \text{ is a regular cardinal,} \\ \Gamma := \kappa^+ \text{ (the successor } \textbf{cardinal} \text{ of } \kappa \text{) otherwise.} \end{cases}$$

*Now $\Gamma$ is a globally stable time limit bound for $\mathcal{M}$.*

PROOF. For the sake of contradiction, suppose that there is $\Gamma' > \Gamma$ and embedded game $\mathbf{G}$ within a bounded evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma')$ such that in $\mathbf{G}$ either of the players has winning time labels that are greater or equal to $\Gamma$. By Lemma 4.14, there is a state $q \in \mathrm{St}$ for which $\mathcal{L}_{\mathbf{C}}(q) = \Gamma$. Let $Q \subseteq \mathrm{St}$ be the set of states that is forced by $\tau_{\mathbf{C}}(q)$. Since $\mathsf{IBB}(\mathcal{M}) = \kappa$, we have $\mathsf{card}(Q) < \kappa \leq \Gamma$. By Lemma 4.13, $\mathcal{L}_{\mathbf{C}}(q') < \Gamma$ for every state $q' \in Q$, and furthermore, since $\Gamma$ is a limit ordinal, $\Gamma$ must be the supremum of the winning time labels of the states in $Q$.

Now every winning time label in $Q$ is smaller than $\Gamma$ and the cardinality of $Q$ is less than $\Gamma$. Because successor cardinals are regular, $\Gamma$ is necessarily a regular cardinal, and thus it is equal to its own cofinality. Hence we have $\sup\{\mathcal{L}_{\mathbf{C}}(q') \mid q' \in Q\} < \Gamma$. This is a contradiction and thus $\Gamma$ must be a globally stable time limit bound. □

In [7] we show that $\Gamma$ in Proposition 4.16 is the least ordinal that is guaranteed to be globally stable. From Proposition 4.16 we obtain the following corollary on the stability of the time limit bounds used with the bounded and finitely bounded $\mathsf{GTS}$. For a detailed proof of the corollary, see [7].

COROLLARY 4.17. *Let $\mathcal{M}$ be a $\mathsf{CGM}$. If $\mathsf{card}(\mathcal{M}) = \kappa$, then $2^\kappa$ is a globally stable time limit bound for $\mathcal{M}$. If $\mathcal{M}$ is image finite, $\omega$ is a globally stable time limit bound for $\mathcal{M}$.*

## 4.5 Relationship between the unbounded and bounded embedded games

The following lemma shows that when $\Gamma$ satisfies certain conditions, then, if $\mathbf{P}$ uses $\infty$-canonical strategy $\tau^\infty_{\mathbf{P}}$ and begins from a state with the label $\mathsf{win}$, $\mathbf{P}$ will always stay in states with the label $\mathsf{win}$. A proof is given in [7].

LEMMA 4.18. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ and $\mathbf{P} \neq \mathbf{C}$. Assume that the time limit bound $\Gamma$ is a successor ordinal and $\Gamma - 1$ is stable for $\mathbf{G}$. Now for every $q \in \mathrm{St}$, if $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$ and $Q \subseteq \mathrm{St}$ is forced by $\tau^\infty_{\mathbf{P}}$, then $\mathcal{L}_{\mathbf{P}}(q') = \mathsf{win}$ for every $q' \in Q$.*

The following proposition shows that when the time limit bound $\Gamma$ is stable, then bounded embedded games become essentially equivalent with unbounded embedded games.

PROPOSITION 4.19. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}},)$ be an embedded game and $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$. Suppose also that $\Gamma$ is stable for $\mathbf{G}$. Now the following equivalences hold:*
- *If $\mathbf{P} = \mathbf{C}$, there is a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}$ iff there is $\gamma_0 < \Gamma$ and a timed winning strategy $(\sigma'_{\mathbf{P}}, t)$ in $\mathbf{G}[\gamma_0]$.*
- *If $\mathbf{P} \neq \mathbf{C}$, there is a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}$ iff there is $\sigma'_{\mathbf{P}}$ which is a winning strategy in $\mathbf{G}[\gamma_0]$ for every $\gamma_0 < \Gamma$.*

PROOF. Sketch of proof: Assume first that $\mathbf{P} = \mathbf{C}$. For the implication from right to left, we see that if there is a timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$ for some $\gamma_0 < \Gamma$, then $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[\gamma_0]$. Since $\tau_{\mathbf{P}}$ depends on states only, it can be used in $\mathbf{G}$, where it will be a winning strategy. For the other direction, we first notice that if $\mathcal{L}_{\mathbf{P}}(q_0) = \gamma$ for some $\gamma < \Gamma$, then $(\tau_{\mathbf{P}}, t_{can})$ is a winning strategy in $\mathbf{G}[\gamma]$. We can also show that if $\mathcal{L}_{\mathbf{P}}(q_0) = \mathsf{lose}$, then $\overline{\mathbf{P}}$ can force the game to stay at states with the label $\mathsf{lose}$, whence $\mathbf{P}$ cannot win $\mathbf{G}$.

When $\mathbf{P} \neq \mathbf{C}$, the implication from left to right is immediate. For the other direction, we notice that the assumption immediately implies that we have $\mathcal{L}_{\mathbf{P}}(q_0) = \mathsf{win}$. Now we can construct the $\infty$-canonical strategy $\tau^\infty_{\mathbf{P}}$ for the time limit bound $\Gamma' := \Gamma + 1$. Since $\Gamma$ is stable, raising the value of the time limit bound to $\Gamma'$ will not affect the winning time labels of states. We can then use Lemma 4.18 to show that any play with $\tau^\infty_{\mathbf{P}}$ will stay on states with the label $\mathsf{win}$. But $\mathbf{P}$ can lose $\mathbf{G}$ only if (s)he ends up at a state with label 0 in finite time. Since $\tau^\infty_{\mathbf{P}}$ also depends on states only, it is thus a winning strategy in $\mathbf{G}$. See [7] for more details. □

As bounded embedded games are determined, the previous proposition implies that also unbounded embedded games are determined. By this result, we see that even if we defined memory based strategies for bounded or unbounded embedded games, the semantics so obtained would remain equivalent to the current one. We can now prove the equivalence of unbounded and bounded game-theoretic semantics.

THEOREM 4.20. *Let $\mathcal{M}$ be a $\mathsf{CGM}$, $q \in \mathrm{St}$ and $\varphi$ an $\mathsf{ATL}$-formula. We have $\mathcal{M}, q \models^g_u \varphi$ iff $\mathcal{M}, q \models^g_b \varphi$.*

PROOF. Assume that $\mathsf{card}(\mathcal{M}) = \kappa$. By Corollary 4.17, $2^\kappa$ is globally stable for $\mathcal{M}$. Consider an embedded game $\mathbf{G}$. If Eloise is the controller in $\mathbf{G}$, then by Proposition 4.19 she has a winning strategy in $\mathbf{G}$ iff there is some $\gamma < 2^\kappa$ such that she has a winning strategy in $\mathbf{G}[\gamma]$. If Eloise is not the controller in $\mathbf{G}$, then by Proposition 4.19, she has a winning strategy in $\mathbf{G}$ iff she has a winning strategy in $\mathbf{G}[\gamma]$ for every $\gamma < 2^\kappa$. Hence we can prove by a straightforward induction on $\varphi$ that Eloise has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi)$ iff she has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, 2^\kappa)$. $\square$

Even though the finitely bounded semantics is not equivalent to the bounded semantics (see Example 3.7), the two systems become equivalent on a natural class models:

THEOREM 4.21. *Let $\mathcal{M}$ be an image finite* CGM, $q \in \mathrm{St}$ *and $\varphi$ an* ATL*-formula. Now $\mathcal{M}, q \models^g_f \varphi$ iff $\mathcal{M}, q \models^g_b \varphi$.*

PROOF. By Corollary 4.17, in image finite models all ordinal valued winning time labels are finite. Thus the controller would gain nothing from being able to use infinite ordinals in embedded games. Hence we can prove the claim by a straightforward induction on $\varphi$. $\square$

# 5. COMPARING GAME-THEORETIC AND COMPOSITIONAL SEMANTICS

We next establish that the unbounded GTS is equivalent to the standard compositional semantics of ATL.

THEOREM 5.1. *Let $\mathcal{M}$ be a* CGM, $q_{in} \in \mathrm{St}$ *and $\varphi$ an* ATL*-formula. Now $\mathcal{M}, q_{in} \models \varphi$ iff $\mathcal{M}, q_{in} \models^g_u \varphi$.*

PROOF. Sketch of proof: For the right-to-left direction, we suppose that Eloise has a winning strategy $\Sigma_{\mathbf{E}}$ in the evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$. We then show that the following condition must hold for every position $\mathsf{Pos} = (\mathbf{P}, q, \psi)$ that is reached in the game when Eloise uses $\Sigma_{\mathbf{E}}$:

$$\mathcal{M}, q \models \psi \quad \text{iff} \quad \mathbf{P} = \mathbf{E}. \qquad (\star)$$

We can prove this by induction on $\varphi$. Consequently, this will hold for the initial position $(\mathbf{E}, q_{in}, \varphi)$, which concludes this direction of the claim.

For the other direction, we suppose that $\mathcal{M}, q_{in} \models \varphi$. Hence $(\star)$ holds for the initial position $(\mathbf{E}, q_{in}, \varphi)$. We then construct a strategy $\Sigma_{\mathbf{E}}$ for Eloise in such a way that $(\star)$ will hold for every position $\mathsf{Pos} = (\mathbf{P}, q, \psi)$ that is reached in the game. By using $\Sigma_{\mathbf{E}}$, Eloise will then finally win the game in every reachable ending position. For this direction of the proof, we also need to use the fact that the embedded games are determined. For a detailed proof, see [7]. $\square$

By our earlier observations, the finitely bounded game-theoretic semantics is not equivalent to the standard compositional semantics of ATL. However, it can be shown equivalent to a natural semantics, to be defined next, which we call **finitely bounded compositional semantics**.

DEFINITION 5.2. *Let $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a* CGM, $q \in \mathrm{St}$ *and $\varphi$ an* ATL*-formula. The truth of $\varphi$ in $\mathcal{M}$ at $q$ according to **finitely bounded semantics**, denoted by $\mathcal{M}, q \models_f \varphi$, is defined recursively as follows:*

- *The semantics for $p \in \Pi$, $\neg\psi$, $\psi \vee \theta$ and $\langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi$ are as in the standard compositional semantics of* ATL *(Def 2.3).*
- *$\mathcal{M}, q \models_f \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{U}\, \theta$ iff there exists $n < \omega$ and $S_A$ such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, there is $i \leq n$ such that $\mathcal{M}, \Lambda[i] \models_f \theta$ and $\mathcal{M}, \Lambda[j] \models_f \psi$ for every $j < i$.*
- *$\mathcal{M}, q \models_f \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta$ iff for every $n < \omega$, there exists $S_{A,n}$ such that for each $\Lambda \in \mathsf{paths}(q, S_{A,n})$ and $i \leq n$, either $\mathcal{M}, \Lambda[i] \models_f \theta$ or there is $j < i$ such that $\mathcal{M}, \Lambda[j] \models_f \psi$.*

To prove the equivalence between the finitely bounded compositional semantics and the finitely bounded GTS, we need to show that it is sufficient to consider only such strategies in the embedded games that depend on states only. This property will be needed because the collective strategies for coalitions in the compositional semantics are of this form.

LEMMA 5.3. *If Eloise has a winning strategy $\Sigma_{\mathbf{E}}$ in a finitely bounded evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \omega)$, then she has a winning strategy $\Sigma'_{\mathbf{E}}$ which uses in the embedded games exclusively strategies $\sigma_{\mathbf{E}}$ that depend only on states.*

PROOF. Sketch: If $\mathbf{E} = \mathbf{C}$ in some embedded game, Eloise may play with the canonical strategy $\tau_{\mathbf{E}}$ that depends only on states. If $\mathbf{E} \neq \mathbf{C}$ and Abelard chooses $n < \omega$ as the time limit, Eloise may play with the $n$-canonical strategy $\tau^n_{\mathbf{E}}$ that depends only on states. For a detailed proof, see [7]. $\square$

With the help of the previous lemma, we can now prove the equivalence between the finitely bounded compositional and game-theoretic semantics using a similar induction as the one in the proof of Theorem 5.1. For a proof, see [7].

THEOREM 5.4. *Let $\mathcal{M}$ be a* CGM, $q_{in} \in \mathrm{St}$ *and $\varphi$ an* ATL*-formula. We have $\mathcal{M}, q_{in} \models_f \varphi$ iff $\mathcal{M}, q_{in} \models^g_f \varphi$.*

## Concluding remarks

We argue that the systems of GTS for ATL introduced in this article are conceptually and technically natural from both logical and game-theoretic perspective. They offer novel complementary approaches to the semantics of ATL. In particular, our bounded GTS provides a framework where truth of ATL-formulae can be determined in finite time. In the future we will develop game-theoretic approaches to $\mathsf{ATL}^+$, $\mathsf{ATL}^*$ and beyond. As argued above, approaches via GTS have proved their worth in multiple fields of logic.

As mentioned in the introduction, *some* of our technical results could have alternatively been derived relatively directly using results for coalgebraic modal logic. This is because concurrent game models can be viewed as coalgebras for a *game functor* defined in [5], and the fixed-point extension of the coalitional coalgebraic modal logic for this functor links to ATL in a natural way. Game-theoretic semantics has been developed for coalgebraic fixed-point logics, e.g. in [14, 4, 6] and can be used to obtain *some* of our results concerning the unbounded game-theoretic semantics. However, that approach would be unhelpful for readers not familiar with coalgebras and coalgebraic modal logic, so the more direct and self-contained approach in this article has its benefits. Moreover, our work on the bounded and finitely bounded semantics is not directly related to existing work in coalgebraic modal logic. However, even there some natural shortcuts based on background theory could have been used. For example, using König's Lemma, it is possible to prove that the finitely bounded and unbounded game-theoretic semantics are equivalent on image-finite models.


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## REFERENCES


[1] N. Alechina, N. Bulling, B. Logan, and H. N. Nguyen. On the boundary of (un)decidability: Decidable model-checking for a fragment of resource agent logic. In *IJCAI 2015*, pages 1494–1501, 2015.

[2] N. Alechina, B. Logan, N. H. Nga, and A. Rakib. Logic for coalitions with bounded resources. *J. Log. Comput.*, 21(6):907–937, 2011.

[3] R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *J. ACM*, 49(5):672–713, 2002.

[4] C. Cîrstea, C. Kupke, and D. Pattinson. EXPTIME tableaux for the coalgebraic $\mu$-calculus. In *CSL 2009*, pages 179–193, 2009.

[5] C. Cîrstea, A. Kurz, D. Pattinson, L. Schröder, and Y. Venema. Modal logics are coalgebraic. *Comput. J.*, 54(1):31–41, 2011.

[6] G. Fontaine, R. A. Leal, and Y. Venema. Automata for coalgebras: An approach using predicate liftings. In *ICALP 2010*, pages 381–392, 2010.

[7] V. Goranko, A. Kuusisto, and R. Rönnholm. Game-theoretic semantics for alternating-time temporal logic. *arXiv:1602.07667 [math.LO]*, 2015.

[8] J. Hintikka. *Logic, Language-games and Information: Kantian Themes in the Philosophy of Logic.* Clarendon Press, 1973.

[9] J. Hintikka and G. Sandu. Informational independence as a semantical phenomenon. In J. E. Fenstad, editor, *Logic, Methodology and Philosophy of Science VIII*, pages 571–589. North-Holland, Amsterdam, 1989.

[10] J. Hintikka and G. Sandu. Game-theoretical semantics. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 361–410. Elsevier, 1997.

[11] P. Lorenzen. Ein dialogisches konstruktivitätskriterium. In A. Mostowski, editor, *Proceedings of the Symposium on Foundations of Mathematics, Warsaw 1959*, pages 193–200. Panstwowe wydawnictwo naukowe, 1961.

[12] D. D. Monica, M. Napoli, and M. Parente. On a logic for coalitional games with priced-resource agents. *Electr. Notes Theor. Comput. Sci.*, 278:215–228, 2011.

[13] C. Smith. Graphs and composite games. *J. Combin. Theory*, 1:51–81, 1966.

[14] Y. Venema. Automata and fixed point logic: A coalgebraic perspective. *Inf. Comput.*, 204(4):637–678, 2006.