Generalized Agent-mediated Procurement Auctions

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ABSTRACT

Procurement auctions (where the auctioneer needs a service and bidders offer it at their own conditions) are an appealing method for on-line service selection. They can improve service features and cost by exploiting the competition between different service providers. Software agents, acting on behalf of human users and organizations, are essential in making such auctions practical and usable. Since conveying user preferences to the agents in a faithful and complete way is virtually impossible, we advocate an approximate approach, where only partial preferences are formalized, and users pick their choice from a short list of options selected by the agents by means of those partial preferences. Another peculiarity of our scenarios is that there may be no contracts with null utility for a given bidder. These features affect the classical, desirable properties of standard auction mechanisms. We prove some impossibility results concerning truthfulness and (a qualitative analogue of) revenue. Then, we investigate a novel auction mechanism that is *almost* truthful in the sense that any strategic deviation from truthfulness has limited impact on the auctioneer's revenue.

Categories and Subject Descriptors

Theory of Computation [**Theory and Algorithms for application domains**]: Algorithmic game theory and mechanism design—*Computational pricing and auctions*

General Terms

Economics, Human Factors, Theory

Keywords

Auctions, partial orders, preference relations

1. INTRODUCTION

In recent years, an intense cross-fertilization between Multiagent Systems and economic paradigms has determined a fruitful area of research. Here we are particularly interested in *auction mechanisms*, with particular regard to *procurement auctions*, that can be used for selecting on-line services in a way that improves the corresponding contracts by exploiting the competition between different service providers. Such scenarios differ from classical auction settings in a few aspects that determine significant changes in the underlying theoretical properties (especially the strategic behavior of participating agents), as discussed in the following.

When we *import* an economic paradigm (e.g. Game Theory and Social Mechanism Design) into our field, we are implicitly importing its model of what an agent is and how it acts proactively. In this respect, Decision Theory and its agent preference axiomatizations play a fundamental role.

In Decision Theory, one of the very basic properties of a preference relation $\preceq$ is its totality: given two potential outcomes a and b, either $a \preceq b$ or $b \preceq a$. From the theory's very start, such an assumption raised some criticisms[1].

Somewhat surprisingly, the founders themselves of Decision Theory considered this point as debatable:

> “It is conceivable and may even in a way be more realistic to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable. How real this possibility is, both for individuals and for organizations, seems to be an extremely interesting question, but it is a question of fact. It certainly deserves further study.” [12].

Several domains would advocate the foregoing considerations. Preferences often result from complex trade-offs between different attributes (functionalities, cost, Quality of Service, information disclosure risks, etc.); in this case possible lack of knowledge as well as introspective sensations of vacillation would naturally move towards a relaxation of the totality property, admitting that some pairs of outcomes may be incomparable.

In this work we support and develop this perspective in the context of procurement auctions. However, before undertaking our study, a preliminarily step requires to deeply understand why decision theorists have substantially preserved totality so far. We clarify this point with an example. Assume to be in a restaurant which offers exotic foods. To make it simple, you can only choose between two dishes, say *pipi cruschi* and *stigghiola*, you do not have any idea of. If someone asks *do you like them the same?* you would probably reply *How can I know?* This seems to suggest that, from an introspective viewpoint, you do not judge them as equivalent. However, if you are starving and we only look at how

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[1]In particular, Simon's kicked off a problem-solving perspective which was very influential in AI [11].

you actually behave, in this situation you would not mind delegating to coin flipping, which is just the same behavior you adopt in case of equivalence. Shortly, Decision Theory does not claim to look inside your mind, it only aims at justifying your behavior *at the moment* a decision is made.

However, unlike Decision Theory where decision makers *own* their preferences, in MAS we have to consider another fundamental aspect: software agents generally act *on behalf of* real users. For instance, in an electronic market, a software agent may apply an auction mechanism on behalf of a consumer, to filter the offers of the providers. This clearly raises the issue of how user preferences can be effectively transferred to software agents, considering that on the web offers can be plenty and diversified. If the delegated agent admits total preferences only, a user would be forced to disambiguate up front all (possibly unexpected) offers, and we have seen that one is introspectively reluctant to identify indecisions – either due to lack of knowledge or trembling desires – with indifference. Furthermore, even if the user had no indecisions and conformed with a classical economic man, her preference relation could be so voluminous and lacking of regularities that it could not be entirely and efficiently injected into the software agent – albeit ceteris paribus techniques could be adopted [4].

The foregoing arguments end up with the following consideration: if Decision Theory considers sufficient three possible options (choose a, choose b, and flip a coin), we propose a fourth one for the delegated agent, namely, deferring the decision and asking the user again. This involves distinguishing between indifference and incomparability.

Preference incompleteness is not the only source of differences between classical auction settings and our scenarios. Traditional settings enjoy the property that it is always possible to make a bid with null utility, by bidding exactly the value v attributed to the auction item. However, not all of our reference scenarios involve payments, and the space of bids may consist of contracts belonging to a discrete domain. In similar settings, it is well possible that no contract has exactly null utility for the bidder. We will show that due to this apparently minor difference, the tie-breaking rules traditionally used to select a winner when multiple bidders submit the same optimal choice make auctions *not truthful*. Recovering truthfulness (at least in part) requires invasive changes to the auction mechanism.

In the following section, we describe a detailed motivating scenario with the above features. Then, Section 3 lays the formal basis for our investigation, by defining auctions and their main desirable properties. In Section 4, we draw a comparison between our framework – characterized by partial preference relations and discrete contract spaces – and the classical one, where preferences are total orders and contracts are monetary and hence continuous. We show that the obvious adaptations of classical truthful auctions to our setting violate one or more of the desirable properties of auctions.

Next, in Section 5, we focus on single-item auctions. After proving that *no* such auction can possibly give strong guarantees on truthfulness and revenue at the same time, we introduce a *relatively optimal* auction and discuss its properties. Roughly speaking, a relatively optimal mechanism is one in which the outcome is guaranteed to be maximally preferred by the auctioneer. Then, by our negative result, this mechanism is necessarily not truthful. In particular, we prove that bidders may have an interest to offer contracts that are not actually convenient to them. However, the mechanism still manages to filter out those over-bids and select an outcome that is convenient to the winning bidder and maximally preferred by the auctioneer. Section 6 shows that multi-item auctions are much less problematic. We define a multi-item mechanism that is both truthful and weakly optimal. Finally, we discuss some related work in Section 7 and draw our conclusions in Section 8.

2. MOTIVATING SCENARIO

Consider the following scenario: Alice is organizing a trip and has to choose a portal for booking tickets and hotels. There is a long list of portals that provide such services, and a first polling shows that the options (across all portals) are: (*i*) traveling time between 8 and 24 hours; (*ii*) only ticket booking, only hotel booking, or complete bookings; (*iii*) prices between €1000 and €2500.

The software agent accepts constraints such as maximum cost and travel time, restrictions on transport means (e.g. no planes), hotel category, portal usability (e.g. no ads), privacy preferences (e.g. no contacts for marketing purposes are allowed), etc. Then, according to the specified preference relation, the agent filters a winning deal. If this is done through an electronic procurement auction, then the competition between the different booking services pushes them to improve their offers.

Clearly, Alice's desires are influenced by many of the above features. For instance, she favors 3 star hotels, prefers fast travels and flights, hates ads and being contacted for marketing purposes. Of course, ceteris paribus, she prefers cheaper solutions. Unfortunately, such preferences do not yield a total order over the options, as Alice cannot establish any fixed priority between the features. For instance, she prefers a package comprising an 8 hours long flight and 3 star hotel to another package comprising a 12 hours long flight and a 5 star hotel, when the latter costs more than €150/night; otherwise she prefers the latter option. Among two packages with the same hotel and similar prices (differing less than €20), Alice prefers the one with the shortest travel time. Alice soon realizes that providing a total order to the software agent is frustrating and requires about the same effort as comparing all the offers by herself.

On the contrary, by using partial preferences, we allow the software agent to use an approximate representation of Alice's desires and return a restricted list of choices from which Alice can select the preferred one. In other words, Alice has the ability to balance between the filtering power and the usability of the software agent. As a further advantage, Alice retains the ability to apply unforeseen, situation-specific preferences that could not be formalized in advance.

Note that the typical portals, in this scenario, are "free"; their revenues come from advertisement and the user's personal information, that can be sold to third parties for marketing purposes. Thus, the parameters that determine the portals' behavior and the quality of their suggestions (kind of trip, cost, duration, hotel category and price, value of advertisement and personal data, etc.) are not under the control of the provider, and it may well be the case that none of the possible options has null utility for the provider. If this is the case, then being selected by the user is never indifferent to not being selected, for any option.

3. PRELIMINARIES

The agents involved in our framework are an auctioneer a and a finite set $N = \{1, \ldots, n\}$ of bidders. We denote by A the set of possible contracts that the auctioneer can stipulate with a bidder. Typically, in procurement auctions – which are our primary reference domain – the auctioneer is a customer asking for a certain service, bidders are the providers of that service and a contract states all relevant properties of the service, such as functionality, quality of service, costs, terms of rescission, etc.

Clearly, the auctioneer may judge some contracts more profitable than others. Let $\leq_a$ be the auctioneer's preference relation. In standard auctions, which are founded on classical decision theory, $\leq_a$ is defined as a weak order, i.e. a total and transitive relation. On the contrary, we assume that two contracts may be incomparable for the auctioneer. Intuitively, two contracts being incomparable means that the auctioneer is not currently able to weigh them up against each other, but she does not want to demand the decision to coin flipping. Formally, we relinquish the totality property and assume $\leq_a$ to be a preorder, that is a reflexive and transitive relation over A. Clearly, weak orders can be viewed as a special class of preorders.

As usual, we write $b <_a b'$ and $b \sim_a b'$ in case b' is *strictly* preferred to b (i.e. $b \leq_a b'$ and $b' \not\leq_a b$) and b is equivalent to b' ($b \leq_a b'$ and $b' \leq_a b$), respectively. With $b \bowtie_a b'$ we mean that b and b' are incomparable ($b \not\leq_a b'$ and $b' \not\leq_a b$). Similarly, for each bidder i, $\leq_i$ is her partial preference relation. The only assumption we make on these preferences is the following: some contracts in A can be considered by some agent as inadmissible. A contract is inadmissible if by stipulating that contract the agent incurs a loss; in standard utility-based auctions inadmissible contracts are associated with negative values. We denote with $A_a \subseteq A$ (resp. $A_i \subseteq A$) the set of contracts that are admissible for the auctioneer (resp. for the bidder i). Then, any contract which does not incur a loss is strictly preferred to a contract that incurs a loss. Formally, for each $b \in A \setminus A_a$ and $b' \in A_a$ (resp. $b \in A \setminus A_i$ and $b' \in A_i$), $b <_a b'$ (resp. $b <_i b'$).

Note that in standard auctions it is generally assumed that the preference of the auctioneer are the inverse of the bidders' preferences. On the contrary, here we do not make any assumption on how preferences are related with each other. In particular this means that for some pairs of contracts, say b_1 and b_2, the preference of the auctioneer could be aligned with the preference of some bidder i, i.e. $b_1 \leq_a b_2$ and $b_1 \leq_i b_2$. This makes sense in several domains, for example in the context of cloud service providing, where *ceteris paribus* both the customer and providers may prefer to use a secure transport layer instead of unsafe protocols. Given a set $B \subseteq A$, we denote by $\max_a B$ (resp., $\max_i B$) the subset of B containing the elements b such that for all other elements $c \in B$, it holds that $b \not<_a c$ (resp., $b \not<_i c$).

At the end of the auction, the user must make a final decision. So, we assume that the auctioneer also has a *choice function* $choice_a : \wp(A) \to A$ which refines her partial preference $\leq_a$. Formally, for all $B \subseteq A$, $choice_a(B) \in \max_a B$. The objective of the following sections is to provide auctions that, using the partial preference $\leq_a$, filter the set of contracts offered by the providers, giving to the user a restricted list of contracts, from which she can choose the final outcome using function $choice_a$.

A *bid vector* is a vector $\langle B_1, \ldots, B_n \rangle$, where $B_i \subseteq A$ is the bid of bidder i. Hereafter, given a bid vector $\mathbf{B} = \langle B_1, \ldots, B_n \rangle$ and a bidder i, we also consider the corresponding *masked* vectors $\mathbf{B}_{-i} = \langle B_1, \ldots, B_{i-1}, ?, B_{i+1}, \ldots, B_n \rangle$, obtained by replacing B_i with the special distinguished symbol "?". As usual, by $(B, \mathbf{B}_{-i})$ we denote the bid vector obtained by replacing '?' with B, hence $(B_i, \mathbf{B}_{-i}) = \mathbf{B}$.

DEFINITION 3.1. *A multi-item A-auction is a function $\mathbf{A}$ from bid vectors to pairs $(\mathbf{x}, \mathbf{c})$, where $\mathbf{x}$ is a boolean allocation vector ($x_i = 1$ iff i is a winner) and $\mathbf{c}$ is the vector of contracts stipulated by each bidder: $\mathbf{c} \in (A \cup \{\bot\})^n$, where $\bot \notin A$ is the null contract.*

A *single-item* A-auction identifies a single winner and a single winning contract, i.e., for all $\mathbf{B}$ there exists at most one i such that $x_i = 1$, where $\mathbf{A}(\mathbf{B}) = (\mathbf{x}, \mathbf{c})$, and for all $j \neq i$, $c_j = \bot$. For simplicity, in case of a single winner i we write $\mathbf{A}(\mathbf{B}) = (i, c_i)$.

To reason about the expected behavior of a bidder, we need to lift her preference $\leq_i$ over contracts to a preference $\preceq_i$ over all possible outcomes $(\mathbf{x}, \mathbf{c})$ of the auction. We assume that $\preceq_i$ satisfies the following natural properties:

- bidders are indifferent losers (in the economic jargon, there are *no externalities*): if $x_i = x'_i = 0$ then $(\mathbf{x}, \mathbf{c}) \preceq_i (\mathbf{x}', \mathbf{c}')$ and $(\mathbf{x}', \mathbf{c}') \preceq_i (\mathbf{x}, \mathbf{c})$;
- preference between outcomes is consistent with preference between contracts: if $x_i = x'_i = 1$ then
$$c_i \leq_i c'_i \Leftrightarrow (\mathbf{x}, \mathbf{c}) \preceq_i (\mathbf{x}', \mathbf{c}');$$
- winning the auction with an admissible contract is at least as good as not winning: if $x_i > x'_i$ and $c_i \in A_i$, then $(\mathbf{x}', \mathbf{c}') \preceq_i (\mathbf{x}, \mathbf{c})$;
- bidders strictly prefer not to win rather than winning with an inadmissible contract: if $x_i > x'_i$ and $c_i \notin A_i$, then $(\mathbf{x}, \mathbf{c}) \prec_i (\mathbf{x}', \mathbf{c}')$.

In general, $\preceq_i$ may satisfy additional properties (i.e. it is not necessarily the least relation satisfying the above properties). Moreover, in general, $\preceq_i$ can be a partial relation as well. This leads to the following two notions of dominance.

DEFINITION 3.2 (DOMINANCE). *Let B_i and B'_i be two bids of provider i. We say that B'_i* dominates *B_i iff for all $\mathbf{B}_{-i}$: $\mathbf{A}(B_i, \mathbf{B}_{-i}) \preceq_i \mathbf{A}(B'_i, \mathbf{B}_{-i})$. Moreover, B'_i is a* dominant strategy *iff it dominates all other bids B_i. We say that a bid vector $\mathbf{B}'$ is a* dominant strategy equilibrium (DSE) *if for all i, B'_i is a dominant strategy.*

DEFINITION 3.3 (WEAK DOMINANCE). *Let B_i and B'_i be two bids of provider i. We say that B'_i* weakly dominates *B_i iff for all $\mathbf{B}_{-i}$: $\mathbf{A}(B'_i, \mathbf{B}_{-i}) \not\prec_i \mathbf{A}(B_i, \mathbf{B}_{-i})$. Moreover, B'_i is a* weakly dominant strategy *iff it weakly dominates all other bids B_i. We say that a bid vector $\mathbf{B}'$ is a* weakly dominant strategy equilibrium (WDSE) *if for all i, B'_i is a weakly dominant strategy.*

Definition 3.3 weakens Definition 3.2 since $\mathbf{A}(B'_i, \mathbf{B}_{-i}) \not\prec_i \mathbf{A}(B_i, \mathbf{B}_{-i})$ means that either $\mathbf{A}(B_i, \mathbf{B}_{-i}) \preceq_i \mathbf{A}(B'_i, \mathbf{B}_{-i})$ or $\mathbf{A}(B_i, \mathbf{B}_{-i})$ and $\mathbf{A}(B'_i, \mathbf{B}_{-i})$ are incomparable.

We can now state several desirable properties of an auction. For all bid vectors $\mathbf{B}$, let $\mathbf{A}(\mathbf{B}) = (\mathbf{x}, \mathbf{c})$:

- *no positive transfer*: for all bidders i, if $x_i = 1$ then $c_i \in B_i$;

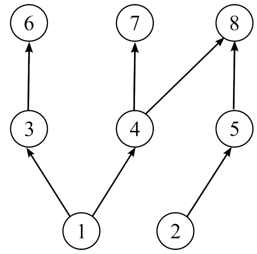

Figure 1: A graph representation of $\leq_{\mathrm{a}}$ in Example 5.1.

- *relative optimality*: for all bidders i, j with $i \neq j$ and $b \in B_j$, if $x_i = 1$ then $c_i \not<_{\mathrm{a}} b$; relative optimality, roughly speaking, is a sort of qualitative revenue maximization guarantee;
- *voluntary participation*: for all i, if $x_i = 0$ then $c_i = \perp$;
- *no failure*: if there exists at least one bidder i such that $B_i \cap A_{\mathrm{a}} \neq \emptyset$ then there exists at least one j such that $x_j = 1$ and for all j such that $x_j = 1$ it holds that $c_j \in A_{\mathrm{a}}$;
- *truthfulness*: the vector $\hat{\mathbf{B}} = \langle A_1, \ldots, A_n \rangle$ is a DSE;
- *weak truthfulness*: $\hat{\mathbf{B}}$ is a WDSE.

We say that an auction satisfies one of the above properties if it does so for all auctioneer preferences, bidder preferences, and bid vectors. Clearly, a single-item auction satisfies voluntary participation by definition.

4. A COMPARISON WITH CLASSICAL AUCTION SETTINGS

In this section we show that the standard properties of auctions do not hold in our reference scenarios.

4.1 Reduction to classical Vickrey auctions

It is tempting to obtain a truthful auction mechanism for agents with partial preferences by reducing the problem to standard second-price auctions, e.g.:

1. Let $\leq$ be any linearization of $\leq_{\mathrm{a}}$ (the preference relation of the auctioneer);
2. Run a second-price auction using $\leq$ to order the bids.

This mechanism violates one of the standard desiderata for auctions: the property of *no positive transfer*. In particular, the second best bid might not be admissible for the winner, which may happen when another bidder makes an offer that is incomparable with i's bid.

Example 4.1. *Let A_{a} be a set of 8 possible contracts and $\leq_{\mathrm{a}}$ be the preference relation depicted in Figure 1, where an edge between two contracts c_i and c_j means that $c_i <_a c_j$. Moreover, we have three bidders that offer $B_1 = \{c_3, c_4, c_6\}$, $B_2 = \{c_1, c_2, c_7\}$, and $B_3 = \{c_5, c_8\}$, respectively.*

Assume the following linearization of $\leq_{\mathrm{a}}$: $c_h \leq c_k$ iff $h \leq k$. Then, a second-price auction assigns the contract c_7 to bidder 3. However, c_7 is not included in B_3 which means that it is not an admissible option for 3.

It is not hard to see that adopting as $\leq$ the linearization of the preferences of any of the bidders yields similar problems. The details are left to the reader.

4.2 Bid-independent and truthful auctions

It is well-known that in classical price-based auctions, where bid vectors consist of tuples $\langle b_1, \ldots, b_n \rangle \in \mathbb{R}^n$, truthfulness is completely characterized by *bid-independent auctions.*

Definition 4.1. *A* bid-independent *auction consists of a* threshold function f *from masked bid vectors to* $\mathbb{R} \cup \{\infty\}$. *For all given bid vectors $\mathbf{B}$, a bidder i is a winner iff $b_i \geq f(\mathbf{B}_{-i})$. All winners get a copy of the item and pay $f(\mathbf{B}_{-i})$.*

Proposition 4.1 ([5]). *An auction mechanism is truthful iff it is equivalent to a bid-independent auction.*

Vickrey's single-item, second-price auctions are usually mentioned as an example of bid-independent auction that, by setting $f(\mathbf{B}_{-i}) = \max\{b_j \mid j \in N, j \neq i\}$, satisfies all the desiderata mentioned in Section 3. However, this is correct only if we assume that one bidder makes an offer that is strictly better than all the other bids, otherwise there would more than one winner and the auction would not be single-item.

Should this happen, the bid independent auction can be refined with a tie-breaking rule to select one winner among the bidders that submit the best offer. Strictly speaking, this is not a bid-independent auction anymore, so in general there is no truthfulness guarantee. However, in Vickrey's auctions, any tie-breaking criterion preserves truthfulness, because in case of multiple best offers the price paid by a truthful winner i is exactly $b_i = v_i$, (the value attributed to the item), therefore winning and not winning are indifferent to i, as $v_i - b_i = 0$ and there is no incentive to *overbid* (i.e. bid some $b_i > v_i$).

Unfortunately, in our scenarios, the space of bids is not continuous, and it is possible (even likely) that winning and not winning are *not* indifferent, for any contract (as if it were only possible to bid $b_i \neq v_i$).[2] Then each "almost truthful" bidder j who bids the maximal possible offer $b_j < v_j$ and is excluded by the tie breaking rule has an incentive to overbid. Indeed, if the bid were replaced by some $b'_j > b_j$, then j would become the only winner, and the price paid would still be $b_j < v_j$.

Remark 4.1. *This proves that* if the bid space does not contain the private value v_i *(for some bidder i),* then Vickrey auctions are not truthful.

Remark 4.2. *Bid vectors with multiple optimal offers are often considered unlikely in the classical settings. However, when preferences are partial, it is not so unlikely to incur in multiple, mutually incomparable optimal offers, therefore fixing this problem is essential in our framework.*

Non-numeric bids require a slight generalization of the notion of bid-independent auction.

Definition 4.2. *A* bid-independent A-auction *consists of a generalized threshold function f from masked vectors of contract sets to $A \cup \{\perp\}$. For all given bid vectors $\mathbf{B}$, a bidder i is a winner iff $f(\mathbf{B}_{-i}) \in B_i$. Each winner stipulates the contract $f(\mathbf{B}_{-i})$ with the auctioneer.*

[2]This happens when the provided service's utility is always nonzero, for all possible contracts.

Note that classical bid-independent auctions are essentially a special case of the above definition, in effect if i is willing to provide the good at the price b_i, it will also accept to give it at a higher price. This means that b_i is just a shorthand for $B_i = \{b \mid b \geq b_i\}$ and hence the condition $b_i \geq f(\mathbf{B}_{-i})$ in Definition 4.1 reduces to $f(\mathbf{B}_{-i}) \in B_i$.

It would be nice to use bid-independent A-auctions to reconstruct in the generalized framework the nice properties of Vickrey's auctions. Unfortunately, in so far as single-item auctions are concerned, relative optimality cannot be achieved at reasonable conditions.

PROPOSITION 4.2. *Assume that A has at least three contracts. Then, no single-item, bid-independent A-auction satisfies both relative optimality and no failure.*

PROOF. Let $N = \{1, 2\}$, $A = \{c, \top_1, \top_2\}$, and let $\leq_a = \{(\top_1, c), (\top_2, c)\}$. Suppose that f induces a single-item, bid-independent A-auction satisfying no-failure and relative optimality; we shall derive a contradiction.

Consider $\mathbf{B} = \langle\{c\}, \{c\}\rangle$. Assume without loss of generality that 1 wins (the other case is symmetrical). Then 2 shall not win as the auction is single-item, and hence

$$f(\mathbf{B}_{-2}) = d \in \{\top_1, \top_2, \bot\}\,. \tag{1}$$

Case $d = \top_1$. Let $\mathbf{B}' = \langle\{c\}, \{\top_2\}\rangle$. By relative optimality, 1 shall lose. Moreover, $\mathbf{B}'_{-2} = \mathbf{B}_{-2}$ so $f(\mathbf{B}'_{-2}) = f(\mathbf{B}_{-2}) = d = \top_1$ (by (1)). But then $f(\mathbf{B}'_{-2}) \notin b'_2$, therefore 2 loses and f violates the no-failure property (a contradiction).

Case $d = \top_2$. Symmetrical (swap $\top_1$ and $\top_2$ in the above proof).

Case $d = \bot$. Similar to the first case (replace $\top_1$ with $\bot$). □

In the light of the above result, we will focus on single-item auction mechanisms that are not bid-independent.

5. SINGLE-ITEM AUCTIONS

In classical price-based settings, Vickrey's auctions satisfy all the desiderata listed in Section 3. Therefore, a natural question is: can we in principle design generalized single-item auctions which reproduce the same good properties of Vickrey's auctions? Unfortunately, the following proposition shows that this is not possible.

PROPOSITION 5.1. *For some A and $\leq^a$, no single-item auction satisfies the properties no positive transfer, relative optimality, and weak truthfulness.*

PROOF. Suppose that $\mathbf{A}$ is always weakly optimal, weakly truthful, and satisfies the no positive transfer property too; we shall derive a contradiction.

Consider any $A \supseteq \{c_1, c_2\}$ such that $c_1 <^a c_2$. Let $\mathbf{B} = \langle\{c_1\}, \ldots, \{c_1\}\rangle$ and $\mathbf{A}(\mathbf{B}) = (w, c_1)$. Finally, suppose that none of c_1 and c_2 has null value for w, so that winning with outcome c_1 or c_2 is preferable to not winning.

Claim 1: $\mathbf{A}((\{c_1, c_2\}, \mathbf{B}_{-w})) = \mathbf{A}(\mathbf{B}) = (w, c_1)$.

First, only two different alternatives make the property of no positive transfer satisfied: $\mathbf{A}((\{c_1, c_2\}, \mathbf{B}_{-w})) = (i, c_1)$ and $\mathbf{A}((\{c_1, c_2\}, \mathbf{B}_{-w})) = (w, c_2)$.

If $\mathbf{A}((\{c_1, c_2\}, \mathbf{B}_{-w})) = (i, c_1)$, with $i \neq w$, then w would have an incentive to underbid when $A_w = \{c_1, c_2\}$ (because w is the winner in $\mathbf{B}$), so $\mathbf{A}$ would not be weakly truthful. Conversely, if $\mathbf{A}((\{c_1, c_2\}, \mathbf{B}_{-w})) = (w, c_2)$, then w would have an incentive to underbid whenever $c_2 <^w c_1$.

Claim 2: $\mathbf{A}((\{c_2\}, \mathbf{B}_{-w})) = (w, c_2)$.

Indeed, by weak optimality, w must be the winner (because $c_1 <^a c_2$). Then, according to the no positive transfer property, c_2 is the only possible contract.

But then, by Claim 1 and Claim 2, whenever $A_w = \{c_1, c_2\}$ and $c_1 <^w c_2$, w has an incentive to underbid, so $\mathbf{A}$ cannot be weakly truthful. □

Concerning the no positive transfer property, it is likely that a bidder is not willing to accept a contract not included in her bid. So, mechanisms not satisfying this property will not be taken into account.

The remaining options are either preserving weak truthfulness or preserving relative optimality. The first option is achieved by a simple mechanism that depends on two parameters: *(i)* the set A_a of contracts that are admissible for the auctioneer, and *(ii)* a linear priority ordering among bidders, which is used as a tie-breaking rule. The mechanism works according to the following definition.

DEFINITION 5.1. *Each bidder i submits a bid B_i. The winner w is the highest priority bidder i such that $A_a \cap B_i \neq \emptyset$; w selects an element b in $A_a \cap B_w$. The output is (w, b).*

Note that the previous definition is nothing but a form of dictatorship: the tie-breaking rule chooses a priori a winner w which can freely select any of her offers in $A_a \cap B_w$. Thus, w is incentivated to maximize the set of admissible choices, which means that A_w is her best strategy, and all the other bidders $j \neq w$ are indifferent to any strategy – indeed, they lose no matter what they offer. Consequently, Definition 5.1 is surely truthful. Moreover, it is equally clear that the mechanism is not relative optimal; more precisely, from the auctioneer's side the substantial lack of competition merely guarantees that the resulting contract is admissible.

So far, it is still an open question whether there exists a (weakly) truthful auction different from a dictatorship.

5.1 A relatively optimal mechanism

In this section we specify a relatively optimal mechanism for partial preferences. Such mechanism depends on the following parameters: *(i)* the set A_a of contracts that are admissible for the auctioneer; *(ii)* the preference relation $\leq_a$; *(iii)* a linear priority ordering among bidders, which is used as a tie-breaking rule. For simplicity, here we assume that i has higher priority than j iff $i < j$. Then, the mechanism works according to the following definition.

DEFINITION 5.2. *Each bidder i submits a bid B_i, giving rise to the bid vector $\mathbf{B} = \langle B_1, \ldots, B_n\rangle$. The auction proceeds as follows:*

1. *For each provider i, let filter$(i, \mathbf{B})$ be the set of contracts offered by i that are admissible for the auctioneer and such that no other bidder has made a strictly preferable offer. Formally,*

$$\mathit{filter}(i, \mathbf{B}) = \{b \in B_i \cap A_a \mid \forall j \neq i, \forall c \in B_j\,, b \not<_a c\}.$$

2. *The mechanism first selects those who submitted one of the top offers, that is, the members of*

$$\mathit{cw}(\mathbf{B}) = \{i \mid \mathit{filter}(i, \mathbf{B}) \neq \emptyset\}\,,$$

(cw stands for candidate winners*). Then, by the tie-breaking rule, the winner w is the highest priority element in $cw(\mathbf{B})$.*

3. *The mechanism transmits to w the set $\mathit{filter}(w, \mathbf{B})$.*

4. *The winner replies with a non-empty subset $B'_w \subseteq \mathit{filter}(w, \mathbf{B})$.*

5. *The mechanism transmits to the auctioneer the set B'_w.*

6. *The auctioneer replies with the contract $b = choice_{\mathrm{a}}(B'_w)$.*

7. *The output of the auction is (w, b) and the other bidders are assigned the null contract $\perp$.*

REMARK 5.1. *Note that the mechanism aims at using minimal information to select a contract. In particular, the bidders do not provide their preferences to the mechanism as the final selection is made privately (steps 3 and 4). This is a valuable desideratum in concrete applications where service providers, public companies or other kinds of enterprises, are scarcely inclined to disclose any information about their own profit model.*

EXAMPLE 5.1. *Consider the same preference $\leq_{\mathrm{a}}$ and bid vector described in Example 4.1. By applying Definition 5.2, in step 1, the resulting filters are:*

$$\begin{aligned} \mathit{filter}(1, \mathbf{B}) &= \{c_3, c_6\} \\ \mathit{filter}(2, \mathbf{B}) &= \{c_7\} \\ \mathit{filter}(3, \mathbf{B}) &= \{c_5, c_8\}. \end{aligned}$$

Therefore, all the bidders are candidate winners. Then, according to the tie-breaking rule, bidder 1 is the final winner. In step 3 the mechanism transmits to bidder 1 the filter $\{c_3, c_6\}$. Assume that she is indifferent between these two possible contracts and replies in step 4 with the same set $\{c_3, c_6\}$. Then, the mechanism returns the contracts c_3 and c_6 to the auctioneer that, according to her preferences, chooses c_6.

Differently, if the preferences of bidder 1 were inverted w.r.t. Figure 1, then in step 4 she would return the single contract c_3 which would be also the final outcome.

The following proposition shows that, apart from truthfulness, Definition 5.2 fulfills all the desiderata mentioned in Section 3.

PROPOSITION 5.2. *The auction in Definition 5.2 satisfies the properties no failure, no positive transfer, relative optimality, and voluntary participation.*

PROOF. Assume that at least one bidder submits a contract which is admissible for the auctioneer, as the other case is trivial. Let $M = \max_{\mathrm{a}} \bigcup_{i=1}^{n} B_i$. Clearly, there exists a bidder j such that $M \cap B_j \neq \emptyset$. By construction, $M \cap B_j \subseteq \mathit{filter}(B_j, \mathbf{B}_{-j})$. Hence, $cw(\mathbf{B}) \neq \emptyset$ and hence the auction returns an output (w, b) with $b \in A_{\mathrm{a}}$ (no failure) and all other bidders receive the null contract (voluntary participation). Note that $b \in \mathit{filter}(B_w, \mathbf{B}_{-w})$ which means that $b \in B_w$ (no positive transfer) and for all $j \neq w$ and $c \in B_j$, $b \not<_{\mathrm{a}} c$ (relative optimality). $\square$

Finally, note that when offers are represented by monetary values, the presented mechanism *collapses* to standard second-price auctions. Keeping procurement auctions as our reference scenario, the auctioneer aims at minimizing prices, that is $b \leq_{\mathrm{a}} c$ iff $c \leq b$. Conversely, for each bidder i, we have that $b \leq_i c$ iff $b \leq c$. The auctioneer fixes a maximum price p_{a} she is willing to pay, hence the set of admissible offers is $A_{\mathrm{a}} = \{b \in A \mid b \leq p_{\mathrm{a}}\}$. Each bidder i makes an offer b_i, which implicitly stands for the bid $B_i = \{b \in A \mid b \geq b_i\}$. Once the mechanism has collected all the bids, it computes the corresponding filters. Since prices are totally ordered, we have that $\mathit{filter}(i, \mathbf{B}) = \{b \in B_i \mid b \leq p_{\mathrm{a}} \text{ and } \forall j \neq i, \forall c \in B_j, b \leq c\}$. Notice that if a bidder makes an offer b_i that is higher than p_{a}, his filter will be empty. For the sake of simplicity, assume that there exists only one candidate winner i_1 (the bidder offering the lowest price) and let i_2 the bidder offering the second lowest price. By definition, it holds that

$$\mathit{filter}(i_1, \mathbf{B}) = \{b \in B_{i_1} \mid b_{i_1} \leq b \leq b_{i_2}\}.$$

Consequently, once the mechanism returns $\mathit{filter}(i_1, \mathbf{B})$ in Step 4, i_1 will select her best possible choice which is indeed the second price b_{i_2}.

5.2 Strategic Analysis

In this section we look at an auction as a game where bidders act strategically.

First of all, in Step 3 the winning bidder w has to select a subset of her filter. Since no other bidder can influence the outcome and the auction satisfies no failure and no positive transfer, it is convenient for w to select the maximal elements of her filter, i.e.,

$$\max_{w} \mathit{filter}(w, \mathbf{B}).$$

The following proposition shows that, differently from classical Vickrey auctions, partial preferences make Definition 5.2 not weakly truthful.

PROPOSITION 5.3. *The auction in Definition 5.2 is not weakly truthful.*

PROOF. Consider the following counterexample: the set of admissible contracts for the auctioneer is $A_{\mathrm{a}} = \{b, c, d\}$, where $c <_{\mathrm{a}} b$, $d <_{\mathrm{a}} b$, and $c \bowtie_{\mathrm{a}} d$. We also have two bidders such that $A_1 = B_1 = \{c\}$ and $A_2 = \{d\}$. If 2 plays truthfully, i.e. $B_2 = \{d\}$, since $c \bowtie_{\mathrm{a}} d$, both 1 and 2 are candidate winners. According to the tie-breaking rule, 2 loses the auction and the output is $(1, c)$. On the contrary, if 2 offers $B'_2 = \{b, d\}$, she is the only candidate winner and her filter is B'_2 itself. Then, in step 4 bidder 2 can select the set $\{d\}$, which means that the output is $(2, d)$. Now, assuming that d is better than losing, we have that $(1, c) \prec_2 (2, d)$. Therefore, playing truthfully is not a weakly dominant strategy for 2. $\square$

Even if the mechanism is not (weakly) truthful, some valuable dominances between bids hold. In particular, each bid of i is dominated by the one obtained by adding to it all contracts that are admissible for i.

PROPOSITION 5.4. *For each bid B_i of bidder i, $B_i \cup A_i$ dominates B_i.*

PROOF. The thesis is trivially true in case $A_i \subseteq B_i$. Therefore, assume that there exists $c \in A_i$ such that $c \notin B_i$. Clearly, $B'_i = B_i \cup A_i$ is an overbid and contains c.

Let $\mathbf{B}_{-i}$ be a bid vector of the other providers. We denote with $\mathbf{B}$ and $\mathbf{B}'$ the bid vectors $(B_i, \mathbf{B}_{-i})$ and $(B'_i, \mathbf{B}_{-i})$, respectively. If $\mathit{filter}(i, \mathbf{B}') \cap A_i = \emptyset$, then $\mathit{filter}(i, \mathbf{B}) = \mathit{filter}(i, \mathbf{B}')$. Moreover, for all j it also holds $\mathit{filter}(j, \mathbf{B}) = \mathit{filter}(j, \mathbf{B}')$, and consequently $\mathbf{A}(\mathbf{B}) = \mathbf{A}(\mathbf{B}')$ and the conclusion follows.

Next suppose that $\mathit{filter}(i, \mathbf{B}') \cap A_i \neq \emptyset$. Since $B_i \subset B'_i$, by construction it follows that $\mathit{filter}(i, \mathbf{B}) \subseteq \mathit{filter}(i, \mathbf{B}')$. Therefore B'_i can only add better optimal offers (w.r.t. $\leq_a$), or leave them unchanged in the worst case. Consequently, B'_i may cause i to enter the set of candidate winners and other bidders to exit from it. We distinguish two cases: if i is the winner in $\mathbf{B}$, according to step 2 of the auction he is also the winner in $\mathbf{B}'$. Since $\mathit{filter}(i, \mathbf{B}) \subseteq \mathit{filter}(i, \mathbf{B}')$, for all $b \in \max_i \mathit{filter}(i, \mathbf{B})$ there exists $b' \in \max_i \mathit{filter}(i, \mathbf{B}')$ such that $b \leq_i b'$. Let $\mathbf{A}(\mathbf{B}) = (i, b)$ and $\mathbf{A}(\mathbf{B}') = (i, b')$. Assume by contradiction that $b' <_i b$ and let $c' \in \max_i \mathit{filter}(i, \mathbf{B}')$ be such that $b \leq_i c'$. By transitivity, $b' <_i c'$, which implies $b' \notin \max_i \mathit{filter}(i, \mathbf{B}')$, a contradiction.

Otherwise, i is not the winner in $\mathbf{B}$. If i is not the winner in $\mathbf{B}'$, then clearly i is indifferent between $\mathbf{B}$ and $\mathbf{B}'$ (indifferent loser assumption) and we are done. If instead i is also the winner in $\mathbf{B}'$, let $\mathbf{A}(\mathbf{B}') = (i, b)$. We have that $\max_i \mathit{filter}(i, \mathbf{B})$ is contained in A_i, because $\mathit{filter}(i, \mathbf{B}') \cap A_i \neq \emptyset$. Consequently, $b \in A_i$ and the thesis follows.

Finally, since we have proved that $\mathbf{A}(\mathbf{B}) \preceq_i \mathbf{A}(\mathbf{B}')$ no matter which $\mathbf{B}_{-i}$ is chosen, B'_i dominates B_i. □

The previous theorem provides a uniform way to improve any bid B_i with a dominant extension that can possibly be an over-bid, namely $B_i \cup A_i$.[3] Therefore, i can safely reduce her space of possible moves to the bids $B \supseteq A_i$ only.

In the rest of this section we give a closer look at the strategic relations between the truthful bid A_i and a generic over-bid s $B \supseteq A_i$ from the points of view of the bidders and the auctioneer. The next theorem shows that if a bidder wins with the truthful bid, then over-bidding does not yield any advantage.

PROPOSITION 5.5. *Let* $\mathbf{B} = (A_i, \mathbf{B}_{-i})$ *and* $\mathbf{B}' = (B'_i, \mathbf{B}_{-i})$, *where* $A_i \subset B'_i$ *and* $\mathbf{B}_{-i}$ *is any bid vector. If* $\mathbf{A}(\mathbf{B}) = (i, p)$, *then* $\mathbf{A}(\mathbf{B}') = (i, p)$.

PROOF. First, from $A_i \subset B'_i$, it is easy to verify that (1) $\mathit{filter}(j, \mathbf{B}') \subseteq \mathit{filter}(j, \mathbf{B})$ for all $j \neq i$, and (2) $\mathit{filter}(i, \mathbf{B}') = \mathit{filter}(i, \mathbf{B}) \cup C$ where C consists of non-admissible contracts only, i.e., for all $c \in C$ and $b \in \mathit{filter}(i, \mathbf{B})$ it holds that $c <_i b$. Since $\mathbf{A}(\mathbf{B}) = (i, p)$, we have that $\mathit{filter}(i, \mathbf{B}) \neq \emptyset$. From (2) and (1), it follows that $i \in cw(\mathbf{B}')$ and $cw(\mathbf{B}') \subseteq cw(\mathbf{B})$. So, the tie-breaking rule selects i as the winner in $\mathbf{B}'$ too.

Moreover, (2) implies that $\max_i \mathit{filter}(i, \mathbf{B}')$ does not contain any element in C and hence

$$\max_i \mathit{filter}(i, \mathbf{B}') = \max_i \mathit{filter}(i, \mathbf{B}) \triangleq D.$$

Consequently, in both $\mathbf{B}$ and $\mathbf{B}'$ the auction sends to the auctioneer the same set D of contracts at Step 5. The auctioneer then replies with the same contract $p = \mathit{choice}_a(D)$. □

Somewhat dually, if a truthful bidder is not a candidate winner, then over-bidding is counterproductive.

PROPOSITION 5.6. *Let* $\mathbf{B} = (A_i, \mathbf{B}_{-i})$ *and* $\mathbf{B}' = (B'_i, \mathbf{B}_{-i})$, *where* $A_i \subset B'_i$ *and* $\mathbf{B}_{-i}$ *is any bid vector. If* $i \notin cw(\mathbf{B})$, *then* $\mathbf{B}$ *dominates* $\mathbf{B}'$.

[3]In other words over-bids, taken *as a whole*, can be seen as a dominant set of strategies.

PROOF. Let $\mathbf{A}(\mathbf{B}) = (h, p)$ and $\mathbf{A}(\mathbf{B}') = (k, p')$. Clearly, $i \notin cw(\mathbf{B})$ implies that $h \neq i$. Now, if $k \neq i$, since i is an indifferent loser, she is indifferent between $\mathbf{B}$ and $\mathbf{B}'$, $(h, p) \sim_i (k, p')$. Then, assume that i wins the auction in the context $\mathbf{B}'$, i.e., $k = i$. Since all admissible contracts are discarded when i plays truthfully, i.e., $\mathit{filter}(i, \mathbf{B}) = \emptyset$, then $\mathit{filter}(i, \mathbf{B}')$ consists of non-admissible contracts only. This implies that $p' \notin A_i$ and hence $(i, p') \prec_i (h, p)$. □

REMARK 5.2. *In the proof of Proposition 5.3 we have shown a case where over-bidding "hacks" the tie-breaking rule in favor of a certain bidder. Propositions 5.5 and 5.6 show that this is the only way a bidder can use over-bidding for her own advantage.*

Finally, we show that from the point of view of the auctioneer the effects of over-bidding on the outcome are limited. In particular, at least at the level of granularity of the partial preference $\leq_a$ provided to the mechanism, the auctioneer has no reason to complain because of an over-bid, as it cannot yield a strictly less preferable contract.

PROPOSITION 5.7. *Let* $\mathbf{B} = (A_i, \mathbf{B}_{-i})$ *and* $\mathbf{B}' = (B'_i, \mathbf{B}_{-i})$, *where* $A_i \subseteq B'_i$ *and* $\mathbf{B}_{-i}$ *is any bid vector. Let* $\mathbf{A}(\mathbf{B}) = (k, q)$ *and* $\mathbf{A}(\mathbf{B}') = (j, p)$. *It holds that* $p \not<_a q$.

PROOF. First, assume $k = i$, i.e., i is the winner in $\mathbf{B}$. By Proposition 5.5, $j = i$ and $p = q$, which implies the thesis. In the following, we assume $k \neq i$ and distinguish three cases.

Case 1. $j = k$, i.e., the winner in $\mathbf{B}$ and $\mathbf{B}'$ is the same (different from i). When moving from $\mathbf{B}$ to $\mathbf{B}'$, some contracts from the filter of j are lost, because they are dominated by some "new" contract in $B'_i \setminus A_i$. The remaining contracts cannot be worse (for the auctioneer) than those that were removed, otherwise they would have been removed as well. Formally, for all $b \in \mathit{filter}(j, \mathbf{B}')$ and $c \in \mathit{filter}(j, \mathbf{B}) \setminus \mathit{filter}(j, \mathbf{B}')$ we have that $b \not\leq_a c$. If $q \notin \mathit{filter}(j, \mathbf{B}')$, since $p \in \mathit{filter}(j, \mathbf{B}')$, by the above observation we have that $p \not\leq_a q$ and in particular $p \not<_a q$. Otherwise, $q \in \mathit{filter}(j, \mathbf{B}')$. Since $\mathit{filter}(j, \mathbf{B}') \subseteq \mathit{filter}(j, \mathbf{B})$ and q is a maximal element of $\mathit{filter}(j, \mathbf{B})$ w.r.t. $\leq_j$, q is a maximal element of $\mathit{filter}(j, \mathbf{B}')$ as well. Formally, $q \in D$, where $D = \max_j \mathit{filter}(j, \mathbf{B}')$. Now, since $p = \mathit{choice}_a(D)$ and choice_a refines $\leq_a$, $p \in \max_a D$ which means that $p \not<_a q$.

Case 2. $j \neq k$ and $j \neq i$, that is the winners in $\mathbf{B}$ and $\mathbf{B}'$ are different from each other and i. First, since $j \neq i$, it holds that $\mathit{filter}(j, \mathbf{B}') \subseteq \mathit{filter}(j, \mathbf{B})$. Now, assume $p <_a q$. By definition, $p \notin \mathit{filter}(j, \mathbf{B})$ and hence $p \notin \mathit{filter}(j, \mathbf{B}')$. This is in contradiction with the hypothesis $\mathbf{A}(\mathbf{B}') = (j, p)$; subsequently, $p \not<_a q$.

Case 3. $j \neq k$ and $j = i$, i.e., i is the winner in $\mathbf{B}'$ but not in $\mathbf{B}$. Assume again that $p <_a q$. Now, since q belongs to B_k and $k \neq i$, p is not relative optimal in $\mathbf{B}'$ and hence it is discarded. As before, $p \notin \mathit{filter}(j, \mathbf{B}')$ clearly leads to a contradiction. □

6. MULTI-ITEM AUCTIONS

Sometimes the auctioneer may be looking for multiple goods or services and then she is disposed to acknowledge multiple contracts. The bid-independent auctions defined in Def. 4.2 are naturally multi-item, as every agent i such that $f(\mathbf{B}_{-i}) \in B_i$ is a winner. Relative optimality is not precluded a priori (Prop. 4.2 refers to single-item auctions only). Nevertheless, Def. 4.2 presents another critical point:

there is no way to assure that $f(\mathbf{B}_{-i}) \in B_i$ holds for at least one bidder. Consequently, the following proposition holds.

PROPOSITION 6.1. *Generalized bid-independent auctions do not satisfy the no-failure property.*

PROOF. Assume there are two possible contracts c_1 and c_2 and two bidders. Since $\mathbf{B}_{-1} = B_2$ and $\mathbf{B}_{-2} = B_1$, f is a mapping from $\{c_1\}$, $\{c_2\}$, and $\{c_1, c_2\}$ into either c_1 or c_2.

Consider the restriction of f to singletons. If such a restriction is not surjective, e.g. $f(\{c_1\}) = f(\{c_2\}) = c_1$, then in case $B_1 = B_2 = \{c_2\}$ no winner is selected. If $f(\{c_1\}) = c_1$ and $f(\{c_2\}) = c_2$, then the mechanism fails when, for instance, $B_1 = \{c_1\}$ and $B_2 = \{c_2\}$. Finally, if $f(\{c_1\}) = c_2$ and $f(\{c_2\}) = c_2$, failure occurs when $B_1 = B_2 = \{c_1\}$.

Consequently, no function f can guarantee the no-failure property. $\square$

To avoid failures, we consider a multi-item variant of the mechanism presented in Definition 5.2. Roughly speaking, for a given bid vector $\mathbf{B}$, the mechanism assigns a contract to *each* candidate winner in $cw(\mathbf{B})$. More formally, the mechanism is defined as follows.

DEFINITION 6.1. *Given an input bid-vector $\mathbf{B}$, for each bidder i, filter$(i, \mathbf{B})$ and $cw(\mathbf{B})$ are computed as in steps 1 and 2 in Definition 5.2. Steps $3-6$ are executed for each $i \in cw(\mathbf{B})$, that is, the mechanism (a) sends filter$(i, \mathbf{B})$ to bidder i, (b) gets $B'_i \subseteq$ filter$(i, \mathbf{B})$ back from i (c) sends B'_i to the auctioneer, (d) receives $b_i = choice_{\mathrm{a}}(B'_i)$. Finally the mechanism outputs $(\mathbf{x}, \mathbf{c})$, where for each $i \in cw(\mathbf{B}), x_i = 1$ and $c_i = b_i$ while for each $i \notin cw(\mathbf{B}), x_i = 0$ and $c_i = \perp$.*

This mechanism is obviously relatively optimal, by definition of filters. On the strategy side, it is easy to verify that Propositions 5.4 and 5.6 still hold for the new mechanism. Similarly, it is possible to rephrase Proposition 5.5 as:

PROPOSITION 6.2. *Let $\mathbf{B} = (A_i, \mathbf{B}_{-i})$ and $\mathbf{B}' = (B'_i, \mathbf{B}_{-i})$, where $A_i \subset B'_i$ and $\mathbf{B}_{-i}$ is any bid vector. Furthermore, denote by $\mathbf{A}(\mathbf{B}) = (\mathbf{x}, \mathbf{c})$ and $\mathbf{A}(\mathbf{B}') = (\mathbf{x}', \mathbf{c}')$. If $x_i = 1$ then $x'_i = 1$ and $c'_i = c_i$.*

On the one hand Proposition 5.6 states that if bidder i is not a candidate winner by playing truthfully, then no over-bid B_i may lead to win the auction with an admissible contract. On the other hand, Proposition 6.2 states that if bidder i is a candidate winner by playing truthfully, then no over-bid will improve her revenue. Therefore we can prove the following:

PROPOSITION 6.3. *Definition 6.1 satisfies truthfulness.*

PROOF. Assume that $\mathbf{B} = (A_i, \mathbf{B}_{-i})$, $\mathbf{B}' = (B'_i, \mathbf{B}_{-i})$, and $\mathbf{A}(\mathbf{B}') \not\preceq_i \mathbf{A}(\mathbf{B})$.

Now, if $i \notin cw(\mathbf{B})$, then according to Proposition 5.6 $\mathbf{A}(\mathbf{B}') \preceq_i \mathbf{A}(\mathbf{B})$. Conversely, if $i \in cw(\mathbf{B})$, by Propositions 5.4 and 6.2 we have that

$$\mathbf{A}(\mathbf{B}') \preceq_i \mathbf{A}((A_i \cup B'_i, \mathbf{B}_{-i})) \sim_i \mathbf{A}(\mathbf{B}) \,.$$

Hence, $\mathbf{A}(\mathbf{B}') \preceq_i \mathbf{A}(\mathbf{B})$. In both the cases we have a contradiction. $\square$

7. RELATED WORK

Traditionally, preferences are essentially linear orders, induced by underlying utility functions (e.g. [1, 6]). Partial preferences are dealt with in mechanisms without money, where most results on incentive compatibility are negative, [9, 10]. In [8] a generic model for matching with contracts (using a doctor-hospital metaphor) is introduced. Definitions and results rely on the assumption that preferences are total orders, if not, then truthfulness does not hold.

Vickrey auctions without payments and qualitative preference relations are dealt with in [7]. Auctioneers and bidders have independent preferences, as in our framework. However, the preference relation of the auctioneer is restricted to *total* preorders (actually, linear orders in the finite case). The tie-breaking over multiple maximal offers is dealt with by assuming the auctioneer's preferences to be *equipeaked* (all local maxima are also global maxima). Our framework supports unrestricted partial preferences, instead.

Bonatti et al. [3] introduce two mechanisms for unrestricted partial preferences and contracts with non-null utility. Among the main differences between [3] and this paper we mention: (i) the mechanisms of [3] are probabilistic; (ii) the theoretical properties of those two mechanisms need additional assumptions about agent preferences (that must be so-called superweak orders, or satisfy 5 axioms that extend some classical decision theory properties, respectively); the theoretical guarantees of the second mechanism of [3] concern the bidders rather than the auctioneer. We are more interested in the auctioneer, according to the applications to privacy enhancement such as those illustrated in [2].

Finally, to the best of our knowledge, Prop. 4.2 is the first negative result on the extension to partial preferences of the traditional techniques for achieving truthfulness.

8. CONCLUSIONS AND FUTURE WORK

In our reference scenarios, preferences are partial and the bid space is discrete, so all bids may have nonzero utility. These two features affect the theoretical properties of classical auctions: We showed that (i) a naive reduction to Vickrey's auctions violates the analogue of the "no positive transfer" principle; (ii) the tie-breaking rules for selecting a winner among multiple candidate winners affect truthfulness; (iii) bid-independent auctions in general violate either robustness (there may be no winner) or relative optimality.

We proved that single-item auctions cannot be both truthful and relatively optimal (the latter is a qualitative revenue guarantee). On the one hand, there exist dictatorial truthful mechanisms. On the other hand, relative optimality is preserved by a mechanism that admits over-bidding. However, from the auctioneer's perspective, overbidding can never make the outcome worse than the outcome of a truthful bid. Interestingly, from a bidder's perspective, overbidding can possibly be advantageous only in case of tie-breaking; otherwise it may yield an equivalent outcome or even cause a loss. This mechanism admits human user intervention in the final phases, when the space of possible outcomes has been restricted to a restricted list by user agents.

Two interesting open questions are whether dictatorship is the only truthful single-item mechanism in our scenarios, and whether some (possibly relaxed) form of truthfulness is compatible with relative optimality.

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