Analyzing Games with Ambiguous Player Types Using the MINthenMAX Decision Model[*]

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ABSTRACT

In many common interactive scenarios, participants lack information about other participants, and specifically about the preferences of other participants. In this work, we model an extreme case of incomplete information, which we term *games with type ambiguity*, where a participant lacks even information enabling him to form a belief on the preferences of others. Under type ambiguity, one cannot analyze the scenario using the commonly used Bayesian framework, and therefore one needs to model the participants using a different decision model.

To this end, we present the MINthenMAX decision model under ambiguity. This model is a refinement of Wald's MiniMax principle, which we show to be too coarse for games with type ambiguity. We characterize MINthenMAX as the finest refinement of the MiniMax principle that satisfies three properties we claim are necessary for games with type ambiguity. This prior-less approach we present follows the common practice in computer science of worst-case analysis.

Finally, we define and analyze the corresponding equilibrium concept, when all players follow MINthenMAX. We demonstrate this equilibrium by applying it to bilateral trade, which is a common economic scenario. We show that an equilibrium in pure strategies always exists, and we analyze the equilibria.

Keywords

Decision making under ambiguity; Games with ambiguity; Wald's MiniMax principle; MINthenMAX decision model; MINthenMAX equilibrium

1. INTRODUCTION

In many common interactive scenarios participants lack information about other participants, and specifically about the preferences of other participants. The extreme case of such partial information scenario is termed *ambiguity*,[1] and in our case ambiguity about the preferences of other participants. In these scenarios, not only does a participant not know the preferences of other participants, but he cannot even form a belief on them (that is, he lacks the knowledge to form a probability distribution over preferences). Hence, one cannot analyze the scenario using the Bayesian framework, which is the common practice for analyzing partial-information scenarios, and new tools are needed.[2] Similarly, in the computer science literature, algorithms, agents, and mechanisms are often analyzed without assuming a distribution on the input space or on the environment. In this work, we define and analyze equilibria under ambiguity about the the game. In particular, we concentrate on equilibria of *games with type ambiguity*, i.e., games with ambiguity about the other players' preferences, namely, their *type*. Our equilibrium definition is based on a refinement of Wald's MiniMax principle, which corresponds to the common practice in computer science of worst-case analysis.[3]

In Section 2, we define a general model of games with ambiguity, similar to Harsanyi's model of games with incomplete information [12],[4] and derive from it the special case of *games with type ambiguity*. In this model, the knowledge of player i on player j is represented by a set of types $\mathcal{T}$. Player i knows that the type of player j belongs to $\mathcal{T}$, but has no prior distribution on this set, and no information that can be used to construct one. Our model also enables us to apply the extensive literature on knowledge, knowledge operators, and knowledge hierarchy to ambiguity scenarios.

Next, we present a novel model for decision making under ambiguity: MINthenMAX preferences. We characterize MINthenMAX in the general framework of decision making under partial information, and show MINthenMAX is the unique finest preference that satisfies a few natural properties. Specifically, we claim that these properties are satisfied

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[1]In decision theory literature, the terms "ambiguity," "pure ambiguity," "complete ignorance," "uncertainty" (as opposed to "risk"), and "Knightian uncertainty" are used interchangeably to describe this case of unknown probabilities.

[2]Clearly, if a player has information that he can use to construct a belief about the others, we expect the player to use it. In this work, we study the extreme case in which one has no reason to assume the players hold such a belief.

[3]Another interpretation we offer for this model, is that the participant might be Bayesian but an extremely risk averse one. Note that this interpretation assumes a richer model in which one can define mixture of alternatives, e.g., if the alternatives are monetary.

[4]As described, for example, in [18, Def. 10.37 p. 407].

by rational players in games with type ambiguity, and hence that MINthenMAX is the right tool of analysis.

Finally, we derive the respective equilibrium concept, dubbed MINthenMAX-NE, and present some of its properties, both in the context of the general model of games with type ambiguity and in two common economic scenarios.

Wald's MiniMax principle

A common model for decision making under ambiguity is the MiniMax principle presented by Wald [25], which we refer to as MIN preferences (as distinct from the MINthenMAX preferences that we present later).[5] In the MIN model, similarly to worst-case analysis in computer science, the preference of the decision maker over actions is based solely on the set of possible outcomes. E.g., in games with type ambiguity, the possible outcomes are the consequence of playing the game with the possible types of the other players. An action a is preferred to another action b if the worst possible outcome (for the decision maker) of taking action a is better than the worst outcome of taking action b. This generalizes the classic preference maximization model: if there is no ambiguity, there is a unique outcome for each action, and the MIN decision model coincides with preference maximization. The MIN model has been used for analyzing expected behavior in scenarios of decision making under ambiguity. For example, ambiguity about parameters of the environment, such as the distribution of prizes (the multiprior model) [11] and ambiguity of the decision maker (DM) about his own utility [5]. The MIN model was also applied to define higher-order goals for a DM, like regret minimization (Minimax regret) [22, Ch. 9] which is applying MIN when the utility of the DM is the regret comparing to other possible actions. In addition, MIN has been used for analyzing interactive scenarios with ambiguity, e.g., first-price auctions under ambiguity both of the bidders and of the seller about the ex-ante distribution of bidders' values [4], and for designing mechanisms assuming ambiguity of the players about the ex-ante distribution of the other players' private information [26], or assuming they decide according the regret minimization model [13].

As we show shortly, the MIN decision model is too coarse and offers too little predictive value in some scenarios involving ambiguity about the other players' types. We show a natural scenario (a small perturbation of the Battle of the Sexes game [16, Ch. 5 Sec. 3]) in which almost all action profiles are Nash equilibria according to MIN. Hence, we are looking for a refinement of the MIN model that breaks indifference in some reasonable way in cases in which two actions result in equivalent worst outcomes. In Section 3, we show that naïvely breaking indifference by applying MIN recursively[6] does not suit scenarios with ambiguity about the other players' types either. We present two game scenarios, and we claim that they are equivalent in a very strong sense: a player cannot distinguish between these two scenarios, even if he has enough information to know the outcomes of all of his actions. Hence, we claim that a rational player should act the same way in these two scenarios. Yet, we show that if a player follows the recursive MIN decision model he plays differently in the two scenarios. In general, we claim that a decision model for scenarios with type ambiguity should not be susceptible to this problem, i.e., it should instruct the player to act the same way in scenarios if the player cannot distinguish between them. Otherwise, when a decision rule depends on information which is not visible to the DM, we find it to be ill-defined. This assumption is with accordance to our assumption of ambiguity which in particular assumes there is no information on the world except the information on outcomes.

The MIN*then*MAX *decision model*

In this work we suggest a refinement of Wald's MiniMax principle that is not susceptible to the above-mentioned problems, and which we term MINthenMAX. According to MINthenMAX the decision maker picks an action having an optimal worst outcome (just like under MIN), and breaks indifference according to the best outcome. We characterize MINthenMAX as the unique finest refinement of MIN that satisfies three desired properties (Section 3): monotonicity in the outcomes, state symmetry, and independence of irrelevant information. *Monotonicity in the outcomes* is a natural rationality assumption stating that the DM (weakly) prefers an action a to an action b if in every state of the world (in our framework, a state is a vector of types of the other players), action a results in an outcome that is at least as good as the outcome of action b in this state. *State symmetry* asserts that the decision should not depend on the names of the states and should not change if the names are permuted. *Independence of irrelevant information* asserts that the DM should not be susceptible to the irrelevant information bias, describes above. That is, the DM's decision should depend only on state information that is relevant to his utility. Specifically, it requires that if two states of the world have the same outcomes for each of the actions, the distinction between the two should be irrelevant for the DM, and his preference over actions should not change in case he considers these two states as a single state. We show that these properties characterize the family of preferences that are determined by only the worst and the best outcomes of the actions. Moreover, we show that MINthenMAX is the finest refinement of MIN in this family: for any preference $\mathbf{P}$ that satisfies the three properties, if $\mathbf{P}$ is a refinement of MIN,[7] then MINthenMAX is a refinement of $\mathbf{P}$.

Equilibrium under MIN*then*MAX *preferences*

In Section 2, we define MIN-NE to be the Nash equilibria under MIN preferences, that is, the set of action profiles in which each player best-responds to the actions of other players, and similarly we define MINthenMAX-NE to be the Nash equilibria under MINthenMAX preferences.

We show that for every game with ambiguity, a MIN-NE in mixed strategies always exists (Thm. 4). On the other hand, we show that there are generic games with ambiguity in which the set of MIN-NE is unrealistic and too large to be useful. This holds even for cases in which the ambiguity is symmetric (all players have the same partial knowledge) and is only about the other players' preferences. Here once again

[5]Wald's principle measures actions by their losses rather than by gains like we do here. Hence, Wald dubbed this principle, which aims to maximize the worst (minimal) gain, the MiniMax principle while we dub it MIN.

[6]I.e., when the DM faces two actions that have equivalent worst outcomes, he decides according to the second-worst outcome.

[7]That is, for any two actions a and b, if a is strongly preferred to b according to MIN, then a is strongly preferred to b according to $\mathbf{P}$ too.

is our motivation for studying the equilibria under MINthenMAX. On the other hand, we present a simple generic two-player game with type ambiguity for which no MINthenMAX-NE exists. We show that the problem of finding a MIN-NE is a *PPAD-complete* problem [20, 21], just as finding a Nash equilibrium when there is no ambiguity.

We note that since MINthenMAX is the unique finest refinement of MIN (which satisfies some properties), the equilibria of a game with ambiguity under any other refinement of MIN is a super-set of the set of MINthenMAX-NE. Hence, one can think of MINthenMAX-NE as the set of equilibria that do not depend on assumptions on the tie-breaking rule over MIN applied by the players.

Applying MIN*then*MAX*-NE to economic scenarios*

To understand the benefits of analysis using the MINthenMAX model, we apply the equilibrium concept MINthenMAX-NE to a well studied economic scenario – bilateral trade games – while introducing ambiguity. We show that in this scenario, a MINthenMAX-NE in pure strategies always exists, and we analyze these equilibria. In the full version of this paper [19], we fully characterize a second economic scenario – n-player coordination games – both in general and under a natural constraint on the type ambiguity – single-peaked consistency w.r.t. a line. In both examples the utility of a player does not depend directly on the types of other players, as in the the general game-theoretic model we analyze, but only via their actions.

Bilateral trade

The main scenario we analyze is bilateral trade. These are two-player games between a seller owning an item and a buyer who would like to purchase the item. Both players are characterized by the value they attribute to the item (their respective willingness to accept and willingness to pay). In the mechanism that we analyze, both players simultaneously announce a price and if the price announced by the buyer is higher than the one announced by the seller, then a transaction takes place and the price is the average of the two.[8] For simplicity, we assume that a player has the option not to participate in the trade.[9]

When there is no ambiguity, an equilibrium that includes a transaction consists of a single price, which is announced by both players. When there is ambiguity about the values, we show that, in addition to the single-price equilibrium, a new kind of equilibrium emerges. For instance, consider the case in which the value of the buyer can be any value between 20 and 40 and the value of the seller can be any value between 10 and 30. First, we notice that there are equilibria that are based on one price as above,[10] but in any such equilibrium there will be types (either of the buyer or of the seller) that will prefer not to participate. If, for example, the price is 25 or higher, then there are types of the buyer that value the item at less than this price and will prefer not to participate; similarly, for prices below 25 there are possible sellers who value the item at more than 25 and hence will prefer not to participate. We can show further a MINthenMAX-NE with two prices, 15 and 35, in which both players participate regardless of their type: the seller announces 35 if his value is higher than 15 and 15 otherwise, and the buyer announces 35 if his value is higher than 35 and 15 otherwise.[11] In this profile, a buyer who values the item at more than 35 prefers buying the item at either price to not buying it, and hence he best-responds by announcing 35 and buying the item for sure. A buyer who values the item at less than 35 prefers buying the item at 15 to not buying it, and not buying the item to buying it at 35. The best worst-case outcome he can guarantee is not buying the item (e.g., by announcing any value between 15 and his value). Based on the worst outcome the buyer is indifferent between these announcements. Hence, in choosing between these announcements according to the best-outcome (i.e., meeting a seller who announces 15), he best-responds by announcing 15. A similar analysis shows that also the seller best-responds in this profile.

We characterize the set of MINthenMAX-NE for bilateral trade games, and in particular we show that for every bilateral trade game, an equilibrium consists of at most two prices. As a corollary we characterize the cases for which there exists a full-participation MINthenMAX-NE, i.e., equilibria in which both players choose to participate regardless of their value (but their bid in these equilibria might depend on the value).

Due to space constraints several proof details are deferred to a full version of the paper [19].

2. MODEL

We derive our model for games with type ambiguity as a special case of a more general model of games with ambiguity. A *game with ambiguity* is a vector

$$\left\langle \mathcal{N}, \left(\mathcal{A}^i\right)_{i\in\mathcal{N}}, \Omega, \left(u^i\right)_{i\in\mathcal{N}}, \left(\mathcal{T}^i\right)_{i\in\mathcal{N}} \right\rangle$$

where:

- $\left\langle \mathcal{N}, \left(\mathcal{A}^i\right)_{i\in\mathcal{N}} \right\rangle$ is an n-player game form. That is,
  $\mathcal{N}$ is a finite set of *players* $N = \{1, \ldots, n\}$;
  $\mathcal{A}^i$ is a finite set of *actions* of player i, and we denote by $\mathcal{A}$ the set of *action profiles* $\times_{i\in\mathcal{N}}\mathcal{A}^i$.
- Ω is a finite set of *states of the world*.
- $u^i \colon \Omega \times \mathcal{A} \to \Re$ is a *utility function* for player i that specifies his utility from every state of the world and profile of actions. We identify u^i with its linear extension to mixed actions, $u^i \colon \Omega \times \Delta\left(\mathcal{A}^i\right) \to \Re$, where $\Delta\left(\mathcal{A}^i\right)$ is the set of mixed actions over $\mathcal{A}^i$.[12]
- $\left\langle \mathcal{N}, \Omega, \left(\mathcal{T}^i\right)_{i\in\mathcal{N}} \right\rangle$ is an Aumann model of incomplete information. That is, $\mathcal{T}^i$ is a partition of Ω to a finite

[8] Our result also holds for a more general case than setting the price to be the mean.

[9] This option is equivalent to the option of the seller declaring an extremely high price that will not be matched (and similarly for the buyer).

[10] An equilibrium in which both players choose (as a function of their value) either to announce a price common to both or not to participate.

[11] I.e., the seller announces the lower of the two if both are acceptable to him, and the higher otherwise; and the buyer announces the higher of the two if both are acceptable to him, and the lower otherwise.

[12] An implicit assumption here is that the players hold vNM preferences, that is, they evaluate a mixed action profile by its expectation. This does not restrict the modeling of preferences under ambiguity. Using the terminology of Anscombe and Aumann [1], we distinguish between roulettes and horse races.

number of partition elements ($\Omega = \dot{\cup}_{t^i \in \mathcal{T}^i} t^i$). We refer to $t^i \in \mathcal{T}^i$ as a *type* of player i.[13]

The above is commonly known by the players. A game proceeds as follows.

- Nature chooses (arbitrarily) a state of the world $\omega \in \Omega$.
- Each player is informed (only) about his own partition element $t^i \in \mathcal{T}^i$ satisfying $\omega \in t^i$.
- The players play their actions simultaneously: Player i, knowing his type t^i, selects a (mixed) action $a^i \in \Delta\left(\mathcal{A}^i\right)$.
- Every player gets a payoff according to u: Player i gets $u^i(t, a)$, where $a = \left(a^1, a^2, \ldots, a^2\right)$ is the action profile and $t = \left(t^1, t^2, \ldots, t^2\right)$ is the type profile.

Notice that the difference between this model and the standard model of games with incomplete information [12] (e.g., as described in [18, Def. 10.37, p. 407]) is that in the latter it is assumed that the players also have posterior distributions on t^i (or equivalently, they have subjective prior distributions on Ω).

In this work we are interested in *games with type ambiguity.* In these games the states of the world are types vectors $\Omega \subseteq \times_{i=1}^n \mathcal{T}^i$, i.e., the unknown information can be represented as information on the types., and in particular any two states of the world are distinguishable by at least one player.[14] For this restricted model, we justify our choice of MINthenMAX preferences. Note that we prove the existence of a mixed MIN-NE (Thm. 4) for every game with ambiguity.

A strategy of a player states his action for each of his types $\sigma^i : \mathcal{T}^i \to \Delta\left(\mathcal{A}^i\right)$. Given a type profile $t = \left(t^1, \ldots, t^n\right)$ and a strategy profile $\sigma = \left(\sigma^1, \ldots, \sigma^n\right)$, we denote by t^{-i} the types of the players besides player i and by $\sigma^{-i}\left(t^{-i}\right)$ their actions under t and σ. We note that the utility of a given type of player i is only affected by those actions taken by other players and not by the actions of the other types player i. Hence, we assume a player chooses his action after knowing his type and not ex-ante beforehand, and model player i's choice of action (best-responding to the others) as a series of independent problems, one for each of his types, of choosing an action. We refer to these problems as the decision process carried out by a type.

Preferences under ambiguity

Decision theory ([16, Ch. 13], [10]) deals with scenarios in which a single decision maker (DM) needs to choose an action from a given set $\mathcal{A}$ when his utility from an action $a \in \mathcal{A}$ depends also on an unknown state of the world $\omega \in \Omega$, and so his preference is represented by a utility function $u : \mathcal{A} \times \Omega \to \Re$. Player i (of type t^i) looks for a response (an action) to a profile σ^{-i}. This response problem is of the same format as the DM problem: he needs to choose an action while not knowing the state of the world ω (the types of his opponents t^{-i} and their actions $\sigma^{-i}\left(t^{-i}\right)$ are derived from ω).

We define the two preference orders over actions, MIN and MINthenMAX, in the framework of Decision Theory. We define them by defining the pair-wise comparison relation, and it is easy to see that this relation is indeed an order. The first preference we define corresponds to Wald's MiniMax decision rule [25].

Definition 1 (MIN preference).
A DM strongly prefers an action a to an action a' according to MIN*, if the worst outcome when playing a is preferred to the worst outcome of playing a':*[15]

$$\min_{\omega \in \Omega} u(a, \omega) > \min_{\omega \in \Omega} u\left(a', \omega\right).$$

These preferences follow the same motivation as worst-case analysis of computer science (where a designer needs to choose an algorithm or a system to use and the expected environment is unknown in advance[16]). MIN preference can also be justified as an extreme ambiguity aversion; judging an action by the worst possible outcome ignoring the probability of this outcome.

The second preference we introduce is a refinement of the MIN preference, as it breaks ties in cases where MIN states indifference between actions.

Definition 2 (MINthenMAX preference).
A DM strongly prefers an action a to an action a' according to MIN*then*MAX*, if either $\min_{\omega\in\Omega} u(a, \omega)$ is strictly greater than $\min_{\omega\in\Omega} u(a', \omega)$ or he is indifferent between the two respective worst outcomes and he prefers the best outcome of playing a to the best outcome of playing a':*[17]

$$\begin{cases} \min\limits_{\omega \in \Omega} u(a, \omega) = \min\limits_{\omega \in \Omega} u\left(a', \omega\right) \\ \max\limits_{\omega \in \Omega} u(a, \omega) > \max\limits_{\omega \in \Omega} u\left(a', \omega\right). \end{cases}$$

Returning to our framework of games with ambiguity, we define the corresponding *best response* (BR) correspondences: MIN-BR and MINthenMAX-BR. The best response of a (type of a) player is a function that maps any action profile of the other players to the actions that are optimal according to the preference. It is easy to see that a best response according to MINthenMAX is also a best response according to MIN; that is, MINthenMAX-BR is a refinement of MIN-BR. We show that the two best response notions are well defined and exist for any (finite) game.

Lemma 3. *The following best response correspondences are non-empty: pure* MIN-*BR, mixed* MIN-*BR, pure* MIN*then*MAX-*BR, and mixed* MIN*then*MAX-*BR.*

Equilibria under ambiguity

Next we define the corresponding (interim) Nash equilibrium (NE) concepts as the profiles of strategies in which each type best-responds to the strategies of the other players. From the definition of MINthenMAX it is clear that any equilibrium according to MINthenMAX is also an equilibrium according to MIN. Hence we regard MINthenMAX-NE as

[13] For a full definition of Aumann's model and its descriptive power, see, e.g., [2, 3] and [18, Def. 9.4 p. 323]. As described in [3] this model is equivalent to defining $\mathcal{T}^i$ using signal functions and to defining them using knowledge operators (i.e., the systematic approach).

[14] For Bayesian settings, this assumption is without loss of generality, because we can unify two indistinguishable states and replace them by the respective lottery, without changing the preferences. Since here we assume no posterior distribution, this assumption is indeed constraining.

[15] The MIN preference is representable by a utility function $U(a) = \min_{\omega\in\Omega} u(a, \omega)$.

[16] Note that the commonly used competitive-ratio is defined as applying MIN preference when we define $u(a, \omega)$ to be the ratio between the performance of an algorithm a on an input ω and the performance of an optimal all-knowing algorithm.

[17] The MINthenMAX preference is not representable by a utility function, for the same reason that the lexicographic preference over $\Re^2$ is not representable by a utility function [17, Ch. 3.C, p. 46].

an equilibrium-selection notion or a refinement of MIN-NE, in cases in which we find MIN-NE to be unreasonable. Our main theorem for this section is showing that any game with ambiguity has an equilibrium according to MIN (MIN-NE) in mixed strategies.[18]

Theorem 4.
Every game with ambiguity has a MIN-*NE in mixed actions.*

Proof sketch: We take $\mathbb{S}$ to be the set of all profiles of mixed strategies of the types and define the following set-valued function $F : \mathbb{S} \to \mathbb{S}$. Given a strategy profile s, $F(s)$ is the product of the best responses to s (according to MIN) of the different types. We prove the existence of a mixed MIN-NE by applying Kakutani's fixed point theorem [15] to F. A fixed point of F is a profile s satisfying $s \in F(s)$; I.e., each type best-responds to the others in the profile s, and hence s is a MIN-NE. □

Since the existence of MIN-NE is the result of applying Kakutani's fixed point theorem to the best response function,[19] we get as a corollary the complexity of the problem of finding MIN-NE.

Corollary 5. *The problem of finding a* MIN-*NE is in PPAD [20, 21]. Moreover, it is a PPAD-complete problem since a special case of it, namely, finding a Nash equilibrium, is a PPAD-hard problem [7].*

Next we show that there are games that with no equilibrium according to MINthenMAX. We show that this is true even for a simple generic game: a two-player game with *type ambiguity* on *one side only*.[20]

Lemma 6. *There are games for which there is no* MIN-*then*MAX-*NE.*

Proof. Let G be the following two-player game with two actions for each of the players. The row player's utility is

	L	R
T	0	0
B	−1	1

. The column player is one of two types: either having utility

	L	R
T	0	1
B	0	−2

or

	L	R
T	0	2
B	0	−1

(and the row player does not know which).

Then, in the unique MIN-NE the first type of the column player mixes $\frac{1}{2}L + \frac{1}{2}R$, the second type of the column player plays R, and the row player mixes $\frac{2}{3}T + \frac{1}{3}B$ (all his mixed actions give him a worst-case payoff of 0). But this is not a MINthenMAX-NE since the row player prefers to deviate to playing B for the possibility of getting 1, and hence the game does not have MINthenMAX-NE in mixed strategies. □

3. AXIOMATIZATION OF MINthenMAX

In this section we justify using equilibria under MINthenMAX preferences for the analysis of games with type ambiguity. To do so, we present three properties for decision making under ambiguity and characterize MINthenMAX as the finest refinement of MIN that satisfies them (Thm. 10). We claim that these properties are necessary for modeling decision making under ambiguity about the other players' types. In doing so, we justify our application of MINthenMAX-NE.

The decision-theoretic framework

Let Ω be a finite set of states of the world. We characterize a preference, i.e., a total order, of a decision maker (DM) over the action set $\mathcal{A}$ where an action is a function $a\colon \Omega \to \Re$ that yields a utility for each state of the world.

Our first two properties are natural and we claim that any reasonable preference under ambiguity should satisfy them. The first property we present is a basic rationality assumption: monotonicity. It requires that if an action a results in a higher or equal utility than an action b in all states of the world, then the DM should weakly prefer a to b.

Axiom 7 (Monotonicity). *For any two actions a and b, if $a(\omega) \geqslant b(\omega)$ for all $\omega \in \Omega$, then either the DM is indifferent between the two or he prefers a to b.*

The second property, state symmetry, states that the DM should choose between actions based on properties of the actions and not of the states. I.e., if we permute the states' names, his preference should not change. Since we can assume that the states themselves have no intrinsic utility beyond the definition of the actions, this property formalizes the property that the DM, due to the ambiguity about the state, should satisfy the *Principle of Insufficient Reason* and treat the states symmetrically.[21]

Axiom 8 (State symmetry). *For any two actions a and b and a bijection $\psi\colon \Omega \to \Omega$, if a is preferred to b, then*

[18]Throughout this paper, unless stated otherwise, when we refer to MIN-NE and MINthenMAX-NE we mean equilibria in pure actions.

[19]The best response correspondence can be computed in polynomial time.

$$\begin{aligned}\mathrm{BR}(s) &= \operatorname{argmax}_{\sigma^i} \min_{\omega\in\Omega} u\left(\omega, \sigma^i, s^{-i}\left(t^{-i}(\omega)\right)\right)\\ &= \operatorname{argmax}_{\sigma^i} \min_{\omega\in\Omega} \mathbb{E}_{a\sim\sigma^i} C_{a,\omega}\end{aligned}$$

for $C_{a,\omega} = u\left(\omega, a, s^{-i}\left(t^{-i}(\omega)\right)\right)$.
The maximal value a player can guarantee himself, $v^* = \max_{\sigma^i} \min_{\omega\in\Omega} \mathbb{E}_{a\sim\sigma^i} C_{a,\omega}$, is the solution to the following program which is linear in v and σ^i

$$\max v \text{ s.t. } \forall w \underset{a\sim\sigma^i}{\mathbb{E}} C_{a,\omega} \geqslant v,$$

that can be solved in polynomial time. Given $v^\star$, $\mathrm{BR}(s)$ is the intersection of $|\Omega|$ hyperplanes of the form $\underset{a\sim\sigma^i}{\mathbb{E}} C_{a,\omega} \geqslant v^*$.

[20]Technical comment: The reason Kakutani's theorem cannot be applied here (besides its result being wrong) is twofold:

- The best-response set is not convex:
  Consider a player who has two possible pure actions, T and B, and his utility (as a function of the action of the opponent) is

	L	M	R
T	0	1	2
B	0	2	1

  . Next, consider he faces one of three types of his opponent who play the three actions, respectively. He is indifferent between his two actions (both give him 0 in the worst case and 2 in the best case), but strictly prefers the two pure actions to any mixture of the two (giving him less than 2 in the best case).
- The best-response function is not upper semi-continuous:
  In the example in the lemma, when the row player faces one type that plays the pure strategy R and another type that mixes $\left(\frac{1}{2} + \epsilon\right) L + \left(\frac{1}{2} - \epsilon\right) R$, his unique best response is to play T for any $\epsilon > 0$, but to play B for $\epsilon = 0$.

[21]Notice that this property rules out expectation maximization, except for expectation under the uniform distribution.

$a \circ \psi$ is preferred to $b \circ \psi$ ($a \circ \psi(\omega)$ is defined to be $a(\psi(\omega))$, i.e., the outcome of the action a in the state $\psi(\omega)$).

The last property we present is independence of irrelevant information. This property requires that if the DM considers one of the states of the world as being two states, by way of considering some new parameter, his preference should not change. We illustrate the desirability of this property for games with type ambiguity using the following example. Consider the following variant of the Battle of the Sexes game between Alice and Bob, who need to decide on a joint activity: either a Bach concert (B) or a Stravinsky concert (S). Taking the perspective of Alice, assume that she faces one of two types of Bob: Bob^B whom she expects to choose B, or Bob^S whom she expects to choose S. Assume that Alice prefers B, and so her valuation of actions is

	Bob^B	Bob^S
$B:$	2	0
$S:$	0	1

(0 if they do not meet and 2 or 1 if they jointly go to a concert). But there might be other information Alice does not know about Bob. For example, it might be that in case Bob prefers (and chooses) S, Alice also does not know his favorite soccer team.[22] So she might actually conceive the situation as

	Bob^B	$Bob^{S,\star}$	$Bob^{S,\dagger}$
$B:$	2	0	0
$S:$	0	1	1

. Since this new soccer information is irrelevant to the game, it should not change the action of a rational player. Notice that if Alice chooses according to the recursive MIN rule we described in the introduction, she will choose according to the second-worst outcome and hence choose B in the first scenario and S in the second scenario. We find a decision model of a rational player which is susceptible to this problem to be an ill-defined model.

Axiom 9 (Independence of irrelevant information). *Let a and b be two actions on Ω and let $\widehat{\omega} \in \Omega$ be a state of the world. Define a new state space $\Omega' = \Omega \dot{\cup} \{\widehat{\widehat{\omega}}\}$ and let a' and b' be two actions on Ω' satisfying $a'(\omega) = a(\omega)$ and $b'(\omega) = b(\omega)$ for all states $\omega \in \Omega \setminus \{\hat{\omega}\}$, $a'(\widehat{\widehat{\omega}}) = a'(\hat{\omega}) = a(\hat{\omega})$, and $b'(\widehat{\widehat{\omega}}) = b'(\hat{\omega}) = b(\hat{\omega})$. Then, a' is preferred over b' whenever s.t. a is preferred over b.*

We show that MINthenMAX is the finest refinement of MIN that satisfies the above three axioms.[23]

Theorem 10.
MIN*then*MAX *is the unique preference that satisfies*
- *Monotonicity.* • *State symmetry.*
- *Independence of irrelevant information.*
- *It is a refinement of* MIN.[24]
- *It is the finest preference that satisfies the above three properties. That is, it is a refinement of any preference that satisfies the above properties.*

We claim that the three axioms are necessary for modeling a rational decision making under type ambiguity: Monotonicity is a basic rationality axiom, and the other two capture that the DM does not have any additional information distinguishing between the states of the world besides the outcomes of his actions. Following Wald, one could define the family of all refinements of MIN that satisfy the axioms, and analyze the equilibria when all players follow models from this family. Showing that MINthenMAX is a refinement of any of the preferences in this family, says that the set of equilibria when all players follow models from this family, must include all equilibria for the case when the players follow the MINthenMAX model (i.e., all MINthenMAX-NE). Moreover, MINthenMAX-NE are the only profiles which are equilibria whenever all players follow models from this family. We interpret this result as robustness of the MINthenMAX-NE notion: these are the equilibria an outside party can expect (e.g., the self-enforcing contracts he can offer to the players), while not knowing the exact preferences of the players.

4. BILATERAL TRADE

In order to demonstrate this new notion of equilibrium, MINthenMAX-NE, we apply it to bilateral trade games with type ambiguity. We show a MINthenMAX-NE in pure strategies always exists and analyze these equilibria. Bilateral trade is one of the most basic economic models, which captures many common scenarios.

It describes an interaction between two players, a *seller* and a *buyer*. The seller has in his possession a single indivisible item that he values at v_s (e.g., the cost of producing the item), and the buyer values the item at v_b. We assume that both values are private information, i.e., each player knows only his own value, and we would like to study the cases in which the item changes hands in return for money, i.e., a transaction occurs.[25] Chatterjee and Samuelson [6] presented bilateral trade as a model for negotiations between two strategic agents, such as settlement of a claim out of court, union-management negotiations, and of course a model for negotiation on transaction between two individuals and a model for trade in financial products. The important feature the authors note is that an agent, while certain of the potential value he places on a transaction, has only partial information concerning its value for the other player.[26] Bilateral trade is also of a theoretical importance, and moreover a multi-player generalization of it, *double auction.*,[27] These models have been used as a tool to get insights into how to organize trade between buyers and sellers, as well as to study how prices in markets are determined.

In this section we assume that there is ambiguity about the players' values (their types), and we study trading mechanisms, i.e., procedures for deciding whether the item changes hands, and how much the buyer pays for it. We assume that the players are strategic, and hence a mechanism should be analyzed according to its expected outcomes in equilibrium.

We concentrate on a family of simple mechanisms (a gen-

[22] Of course, his favorite team is clear in case he prefers Bach.
[23] Notice that the MIN preference satisfies these properties.
[24] I.e., for any two actions a and b, if a is strongly preferred to b by a DM holding a MIN preference, then a is strongly preferred to b by a DM holding a MINthenMAX preference.

[25] Another branch of the literature on bilateral trade studies the process of bargaining (getting to a successful transaction). Since we would like to study the impact of ambiguity, we restrict our attention to the outcome.
[26] For instance, in haggling over the price of a used car, neither buyer nor seller knows the other's walk-away price.
[27] In a double auction [9], there are several sellers and buyers, and we study mechanisms and interactions matching them to trading pairs.

eralization of the bargaining rules of Chatterjee and Samuelson [6]): the seller and the buyer post simultaneously their respective bids, a_s and a_b, and if $a_s \leqslant a_b$ the item is sold for $x(a_s, a_b)$, for x being a known monotone function satisfying $x(a_s, a_b) \in [a_s, a_b]$. For ease of presentation, we add to the action sets of both players a "no participation" action $\perp$, which models the option of a player not to participate in the mechanism; i.e., there is no transaction whenever one of the players plays $\perp$. This simplifies the presentation by grouping together profiles in which a player chooses extreme bids that would not be matched by the other player. Hence, the utilities of a seller of type v_s and a buyer of type v_b from an action profile (a_s, a_b) are (w.l.o.g., we normalize the utilities of both players to zero in the case where there is no transaction):

$$u_s(v_s; a_s, a_b) = \begin{cases} a_s \leqslant a_b & x(a_s, a_b) - v_s \\ a_s > a_b & 0 \\ a_s = \perp \vee a_t = \perp & 0 \end{cases}$$

$$u_b(v_b; a_s, a_b) = \begin{cases} a_s \leqslant a_b & v_b - x(a_s, a_b) \\ a_s > a_b & 0 \\ a_s = \perp \vee a_t = \perp & 0. \end{cases}$$

Note that, unlike in the general model we described before, the utility of a player does not depend directly on the types of the other player, but only on his action (that might depend on the type).

Under full information (i.e., the values v_s and v_b are commonly known), there is essentially only one kind of equilibrium: the *one-price equilibrium*. If $v_s \leqslant v_b$, the equilibria in which there is a transaction are all the profiles (a_s, a_b) s.t. $a_s = a_b \in [v_s, v_b]$ (i.e., the players agree on a price), and the equilibria in which there is no transaction are all profiles in which both players choose not to participate, regardless of their type.

Introducing type ambiguity, we define the seller type set V_s and the buyer type set V_b, where each set holds the possible valuations of the player for the item. We show that under type ambiguity, there are at most three kinds of equilibria, and we fully characterize the equilibria set. We show that in addition to the above no-transaction equilibria and one-price equilibria, we get a new kind of equilibrium: the *two-price equilibri*um. In such an equilibrium, both the seller and the buyer participate regardless of their valuations, and bid one of two possible prices: p_L and p_H. For some type sets, namely V_s and V_b, these two prices are the only full-participation equilibria, i.e., equilibria in which both players choose to announce a price and participate, regardless of their value.

Theorem 11.[28]

Let G be a bilateral trade game defined by a price function $x(a_s, a_b)$ and two type sets V_s and V_t, both having a minimum and maximum.[29] *Then all the* MIN*then*MAX-*NE of G are of one of the following classes:*

1. ***No-transaction equilibria*** *(These equilibria exist for any two sets V_s and V_b)*
   In these equilibria, neither the buyer nor the seller participates (i.e., they play $\perp$, or bid a too extreme bid for all types of the other player), regardless of their valuations.
2. ***One-price equilibria*** *(These equilibria are defined only when $\min V_s \leqslant \max V_b$, i.e., when an ex-post transaction is possible.)*
   In a one-price equilibrium, both the seller and the buyer choose to participate for some of their types. It is defined by a price $p \in [\min V_s, \max V_b]$ s.t. the equilibrium strategies are:
   The seller bids p for $v_s \leqslant p$, and $\perp$ otherwise.
   The buyer bids p for $v_b \geqslant p$, and $\perp$ otherwise.
   Hence, the outcome is

Buyer \ Seller	*Low: $v_s \leqslant p$*	*High: $v_s > p$*
Low: $v_b < p$	*no transaction*	*no transaction*
High: $v_b \geqslant p$	p	*no transaction*

.

3. ***Two-price equilibria*** *(These equilibria are defined only when $\min V_s \leqslant \min V_b$ and $\max V_s \leqslant \max V_b$, i.e., when there is a value for the seller s.t. an ex-post transaction is possible for any value of the buyer, and vice versa.)*
   In a two-price equilibrium, all types of both the seller and the buyer choose to participate, and their bids depend on their valuations. It is defined by two prices $p_L < p_H$ s.t. $\begin{cases} \min V_s \leqslant p_L < \max V_s \leqslant p_H \\ p_L \leqslant \min V_b < p_H \leqslant \max V_b \end{cases}$ and the equilibrium strategies are:
   The seller bids p_L for $v_s \leqslant p_L$, and p_H otherwise.
   The buyer bids p_H for $v_b \geqslant p_H$, and p_L otherwise.
   Hence, the outcome is

Buyer \ Seller	*Low: $v_s \leqslant p_L$*	*High: $v_s > p_L$*
Low: $v_b < p_H$	p_L	*no transaction*
High: $v_b \geqslant p_H$	$x(p_L, p_H) \in (p_L, p_H)$	p_H

.

We find it interesting that the set of equilibria depends on the possible types of the players, and not on the price mechanism $x(a_s, a_b)$. In addition to the two classic equilibrium kinds, no-transaction equilibrium and one-price equilibrium, we get a new kind of equilibrium. We see that in this equilibrium each of the players announces one of two bids, which is tantamount to announcing whether his value is above some threshold or not. This decision captures the (non-probabilistic) trade-off a player is facing: whether to trade for sure, i.e., with all types of the other player, or to get a better price. For example., the buyer decides whether to bid the high price and buy the item for sure, taking the risk of paying more than his value; or whether to bid the low price and buy at a lower price, taking the risk of not buying at all. Since MINthenMAX is a function of the worst-case and best-case outcomes only, it does not seem surprising that we get this dichotomous trade-off and at most two bids (messages) for each player in equilibrium.

This result might explain the emergence of market scenarios in which a participant needs to choose which one of two markets to attend, e.g., florists who choose whether to sell in a highly competitive auction or in an outside market, and he needs to choose between the two while not knowing the demand for that day.

[28]This result is also valid, and even more natural, for infinite type sets.

[29]We state the result here for the case where both sets have a minimal and a maximal valuations. Dropping this assumption does not change the result in any essential way: some of the inequalities are changed to strict inequalities.

5. SUMMARY AND FUTURE DIRECTIONS

How people choose an action to take when possessing only partial information, and how they should choose their action, are basic questions in economics, and on them the definition of equilibrium is built (both as a prediction tool and as a self-enforcing contract). The main stream of game-theoretic literature assumes that economic agents are expectation maximizers (according to some objective or subjective prior) and, moreover, that there is some consistency between the players' priors (commonly, the common prior assumption).

In this work, we chose the inverse scenario and studied cases in which the players have no information on the state of the world. We defined a general framework of games with ambiguity following Harsanyi's model of games with incomplete information [12], and in particular, games with type ambiguity. We axiomatized a family of decision models under ambiguity that we claim a rational agent is expected to follow, and characterized the finest refinement of this family, MINthenMAX. This family can be interpreted as all *rational* models of decision in cases of (extreme) ambiguity that follow Wald's MiniMax principle, defined by the different ways to decide in cases in which the MiniMax principle is mute. In many scenarios with type ambiguity Wald's MiniMax principle is too coarse, and so the corresponding equilibrium (MIN-NE) has almost no predictive power. The way to justify a selection process from these equilibria is to refine the players' preference, that is, assume they act according to a decision model in this family. We showed MINthenMAX is the unique model that follows Wald's MiniMax principle and breaks all possible instances of indifference (without violating the rationality axioms we assumed). Finally, we studied the respective equilibrium notion, MINthenMAX-NE, and applied it to bilateral trade games.

One might ask himself why to choose MINthenMAX as the analysis tool, and not a different decision model in the family. First, we note that, just like the MIN model, the MINthenMAX model has a simple and intelligible cognitive interpretation, and so it does not require a complex epistemological assumption on the players, which other models might impose. In addition, we note that MINthenMAX-NE (G) are in fact also equilibria of G under any profile of rational refinements of Wald's MiniMax principle. Moreover, MINthenMAX-NE (G) can be equivalently defined as the set of all such robust equilibria of G. For instance, these are the profiles an outside actor can suggest as a self-enforcing contract, even if he does not know the exact preferences of the players.

This scenario of extreme ambiguity might seem unrealistic. Yet, we claim this model approximates many partial-information real-life scenarios better than the subjective expectation maximization model. Clearly, if players have information which they can use to construct a belief about the other players, and we expect they will use it, then the expectation maximization decision model is a better analysis tool. In intermediate scenarios, when players have some information but it is unreasonable to expect them to form a distribution over the world, it is reasonable to model the players as following one of the intermediate models for decision making under ambiguity,[30] e.g., the multi-prior model (for an overview of such models, the interested reader is referred to [10]). One could also justify the model we presented, via justifying the axioms, for other scenarios. E.g., for scenarios in which the players might have some information on the preferences of others, but due to extreme risk aversion or bounded rationality constraints, they follow Wald's MiniMax model. Moreover, in cases in which one justified the axioms we presented (and mostly the invariance to irrelevant information axiom), e.g., on cognitive grounds, we get that the MINthenMAX-NE is the right analysis tool for the same reasons we presented above.

In order to study further the notion of equilibrium under MINthenMAX, in an extended version of this work [19] we analyzed several types of coordination games with type ambiguity, and in particular the impact on the set of equilibria of knowledge on other players (limiting the ambiguity) and of intra-player homogeneity.

A recently common model in the AI literature is the Minimax regret model (regret minimization). As we state in the introduction, it is equivalent to the MIN model when the utility of the DM is the regret comparing to other possible actions. We think that it should be interesting to extend the analysis to MINthenMAX-Regret in cases in which regret minimization is not decisive enough.

The main drawback of modeling the knowledge of a player using a set of types for other players (or, in the general case, of states of the world) instead of a richer structure, is that there is no reasonable way to define information update in this model. This prevents us from extending this work to two natural directions: analysis of extensive form games (e.g., when the players play in turns, learning the type of each other as the evolve), and value of information (the question of how much a player should invest in order to decrease his ambiguity). In the decision theory literature, there are several non-Bayesian information update rules, e.g., Dempster-Shafer [8, 24] and Jeffrey update rule [14]. These rules usually assume a finer representation of knowledge than the representation we had in this work, but we think that after basing a rational decision model in our simplified knowledge representation, it should not be hard to extend the decision model to these finer knowledge models.

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[30] We refer to them as intermediate since they do not satisfy the principle of indifference, and hence they differentiate between the states of the world.

REFERENCES

[1] F. J. Anscombe and Robert J. Aumann. A definition of subjective probability. *The Annals of Mathematical Statistics*, 34(1):199–205, 1963.

[2] Robert J. Aumann. Agreeing to disagree. *The Annals of Statistics*, 4(6):1236–1239, 1976.

[3] Robert J. Aumann. Interactive epistemology i: Knowledge. *International Journal of Game Theory*, 28(3):263–300, 1999.

[4] Subir Bose, Emre Ozdenoren, and Andreas Pape. Optimal auctions with ambiguity. *Theoretical Economics*, 1(4):411–438, 2006.

[5] Simone Cerreia-Vioglio, David Dillenberger, and Pietro Ortoleva. Cautious expected utility and the certainty effect. *Econometrica*, 83(2):693–728, 2015.

[6] Kalyan Chatterjee and William Samuelson. Bargaining under incomplete information. *Operations Research*, 31(5):835–851, 1983.

[7] Xi Chen, Xiaotie Deng, and Shang-Hua Teng. Settling the complexity of computing two-player Nash equilibria. *Journal of the ACM (JACM)*, 56(3), 2009.

[8] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *The Annals of Mathematical Statistics*, 38(2):325–339, 1967.

[9] Daniel Friedman and John Rust. *The Double Auction Market: Institutions, Theories, and Evidence*. Addison-Wesley Publishing Company, 1993.

[10] Itzhak Gilboa and Massimo Marinacci. Ambiguity and the Bayesian paradigm. In Daron Acemoglu, Manuel Arellano, and Eddie Dekel, editors, *Advances in Economics and Econometrics*, volume 1, pages 179–242. Cambridge University Press, 2013.

[11] Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

[12] John C. Harsanyi. Games with incomplete information played by "Bayesian" players, I-III. part I. the basic model. *Management Science*, 14(3):159–182, 1967.

[13] Nathanael Hyafil and Craig Boutilier. Regret minimizing equilibria and mechanisms for games with strict type uncertainty. In *Proceedings of* $20^{\underline{th}}$ *Conference in Uncertainty in Artificial Intelligence (UAI)*, pages 268–277, 2004.

[14] Richard C. Jeffrey. *The logic of decision*. McGraw-Hill, 1965.

[15] Shizuo Kakutani. A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, 8(3):457–459, 1941.

[16] Robert Duncan Luce and Howard Raiffa. *Games and Decisions: Introduction and Critical Survey*. Dover Publications, 1957.

[17] Andreu Mas-Colell, Michael Whinston, and Jerry Green. *Microeconomic Theory*. Oxford University Press, 1995.

[18] Michael Maschler, Eilon Solan, and Shmuel Zamir. *Game Theory*. Cambridge University Press, 2013.

[19] Ilan Nehama. Analyzing games with ambiguous player types using the MINthenMAX decision model. *arXiv*, `http://arxiv.org/abs/1603.01524`.

[20] Christos Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *Journal of Computer and System Sciences*, 48(3):498–532, 1994.

[21] Christos Papadimitriou. The complexity of computing equilibria. In H. Peyton Young and Shmuel Zamir, editors, *Handbook of Game Theory with Economic Applications*, volume 4, chapter 14, pages 779–810. Elsevier, 2015.

[22] Leonard J. Savage. *The Foundations of Statistics*. Dover Publications, 1972.

[23] Thomas Crombie Schelling. *The Strategy of Conflict*. Harvard University Press, 2 edition, 1980.

[24] Glenn Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[25] Abraham Wald. *Statistical decision functions*. Wiley, 1950.

[26] Alexander Wolitzky. Mechanism design with maxmin agents: Theory and an application to bilateral trade. *Theoretical Economics*, 11(3):971–1004, 2016.