On the Gap between Outcomes of Voting Rules (Extended Abstract)

Anurita Mathur
Indian Institute of Science
Bangalore, India
anurita.mathur@csa.iisc.ernet.in

Arnab Bhattacharyya
Indian Institute of Science
Bangalore, India
arnabb@csa.iisc.ernet.in

ABSTRACT

Various *voting rules* (or *social choice procedures*) have been proposed to select a winner from the preferences of an entire population: Plurality, veto, Borda, Minimax, Copeland, etc. Although in theory, these rules may yield drastically different outcomes, for real-world datasets, behavioral social choice analyses have found that the rules are often in perfect agreement with each other! This work attempts to give a mathematical explanation of this phenomenon.

We quantify the gap between the outcomes of two voting rules by the pairwise margin between their winners. We show that for many common voting rules, the gap between them can be almost as large as 1 when the votes are unrestricted. As a counter, we study the behavior of voting rules when the vote distribution is a uniform mixture of a small number of multinomial logit distributions. This scenario corresponds to a population consisting of a small number of groups, each voting according to a latent preference ranking. We show that for any such voting profile on g groups, at least $1/2g$ fraction of the population prefers the winner of a Borda election to any other candidate.

Keywords

voting, social choice, probability distribution,latent models, generative models

1. INTRODUCTION

A common and natural way to aggregate preferences of agents is through an election. In a typical election, we have a set of candidates and a set of voters, and each voter reports his preference about the candidates in the form of a vote. We will assume that each vote is a ranking of all the candidates. The purpose of voting, especially in the context of democratic forms of government, is to aggregate the preferences or opinions of individuals and process them to produce a single opinion, which purportedly will be an accurate reflection of the views of the electorate. A *voting rule* selects one candidate[1] as the winner once all voters provide their votes. Determining the "best" possible voting rule is the fundamental question.

The issue of course is what one means by the "best". The trouble was first formally pointed out by Marquis de Condorcet in 1785 through the following example. Suppose there are three candidates A, B, C. One can construct a situation where 2/3 of the voters prefer A over B, 2/3 prefer B over C, and 2/3 prefer C over A. This issue of *intransitivity* is often called a 'social choice paradox', because no matter which candidate is elected, a majority of voters will be disappointed. Thus, in a precise sense, no good voting rule exists!

However, it turns out that there is almost no empirical evidence of the Condorcet paradox in actual survey or ballot data [12, 13]! This is in spite of results from social choice theory [5, 7, 15, 6, 9, 20] which conclude that intransitivity is very likely in the *impartial culture* distribution. The impartial culture distribution is a voting profile where all possible rankings among the candidates are present in equal proportions. In other words, if there are m candidates, $1/m!$ fraction of the population vote π where π is any ranking among the candidates. The aforementioned empirical studies clearly imply that this widely studied assumption in the theoretical literature must not be close to being true in reality.

A different stream of results in social choice theory show that there exist no voting rule (except dictatorship) which simultaneously satisfies a set of desired *axioms*. Examples of such results are Arrow's celebrated Impossibility Theorem [2] and the Gibbard-Satterthwaite Theorem [8, 18]. These are rather pessimistic results which rule out a priori benign assumptions about the voting rules. Naturally, researchers were led to comparative analysis of different voting rules in an axiomatic framework. It is known that different voting rules satisfy different subsets of axioms, and in fact, voting rules can often be characterized by the axioms they satisfy. In particular, there exist preference profiles for which different voting rules produce different winners. Saari has shown that in there is a voting profile over 10 candidates such that over 84 million different rankings can be generated just by using scoring rules! In some of these rankings, a candidate is a winner whereas in others, the same candidate is in the last position.

But again, behavioral social choice analyses have found that in many real-world datasets, all the common rules very often agree with each other in their outcome! As Regenwetter et al discuss in their survey [14], "the theoretical literature may promote overly pessimistic views about the like-

[1]In this work, we focus throughout on *single-winner voting systems*, although aggregation rules that produce an entire ordering is also of course extensively studied.

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lihood of consensus among consensus methods. Axiomatics highlight that competing methods cannot universally agree with each other." They posit that more study needs to be done to understand the properties of real-world distributions of preferences.

2. OUR CONTRIBUTION

We ask how different can the outcomes of two voting rules be when applied on the same preference profile. The natural way to compare two outcomes a and b is to look at their pairwise margin i.e., the fraction of the population who vote for a over b. For randomized voting rules (*lotteries*), we look at the expected margin between the two outcomes.

Our first contribution is to determine that if the preference profile is allowed to be arbitrary, then the margin between the outcomes of common voting rules may reach nearly 1. Almost all of the population can side with one voting rule versus another. Thus, at least among common voting rules, no rule beats another over the set of all profiles.

Our second contribution is to examine the outcomes of voting rules when the voter preferences are generated by a *latent preference* model [17]. The population is viewed as consisting of distinct groups, and an individual in group i votes independently according to a probability distribution $\mathcal{D}_i$ on votes. Subsequently, the preference of the population as a whole can be associated with a distribution over votes. Analogous to the "low rank" assumption made in inferential statistics, we assume that the number of groups is small. More precisely, the votes are drawn i.i.d. from a mixture of *multinomial logit* (MNL) models. An MNL model on m candidates is described by a non-negative real vector parameter $\mathbf{w} = (w_1, \ldots, w_m)$ called the score vector and outputs a profile as follows. Make candidate i the top-ranked with probability $w_i/(\sum_{j=1}^m w_j)$. Let i_1 be the chosen candidate. Then, if $i \neq i_1$, candidate i is the second-ranked with probability $w_i/(\sum_{j\neq i_1} w_j)$, and so on. The MNL model was introduced independently by Thurstone [19], Bradley and Terry [4] and by Zermelo [21] (see also [10]). The mixture of MNL models has been studied previously in the context of inference and learning [1, 11].

We show that if the population is a mixture of two groups, then Condorcet cycles cannot exist, no matter the score vectors. This result is false for 3 groups. We then examine the behavior of common voting rules on mixtures of multinomial logits. We find that if the number of groups is constant, then no matter the score vectors, there will always be a constant fraction of the population who prefers the Borda winner to any other candidate. In particular, if the number of groups is g, then the margin between the Borda winner and any other candidate is at least $1/2g$. The same holds for Plurality, Minimax and Copeland voting rules. In contrast, even over profiles generated according to MNL on 2 groups and 3 candidates, some candidate other than the 2-approval voting rule winner can be preferred by almost all of the population.

We also look at the margin between winners of various voting rules on on real election data. We used a slice of the Netflix Prize [3] dataset containing only those instances that do not contain a Condorcet winner. We obtained this from the ED-00029 dataset at `http:///www.preflib.org`. The non-existence of Condorcet winners means that we are able to obtain non-trivial gaps between Condorcet consistent rules also. We find that in this data, the winner of a deterministic variant of the maximal lottery rule (see [16]) beats all others. Schulze and Minimax always agree, and they beat Borda, Copeland and Plurality. Borda beats Copeland and Plurality, and Copeland beats Plurality.

3. COMPARING VOTING RULES OVER ARBITRARY PROFILES

We define the gap between voting rules R_1 and R_2 with respect to a preference profile Π as:

DEFINITION 1.

$$Gap_\Pi(R_1, R_2) = E[M(R_2(\Pi), R_1(\Pi))]$$

where the expectation is over the randomness of the voting rule and M is the margin matrix. If $\mathcal{C}$ is a collection of voting profiles, the gap between voting rules R_1 and R_2 over $\mathcal{C}$ *is:* $Gap_{\mathcal{C}}(R_1, R_2) = \max_{\Pi\in\mathcal{C}} Gap_\Pi(R_1, R_2)$

Though for any particular profile Π, $Gap_\Pi(\cdot,\cdot)$ is anti-symmetric, there may not be such symmetry in $Gap_{\mathcal{C}}(\cdot,\cdot)$. For a collection of voting profiles $\mathcal{C}$, we informally say that *rule R_2 beats rule R_1 over $\mathcal{C}$* if $Gap_{\mathcal{C}}(R_1, R_2) > Gap_{\mathcal{C}}(R_2, R_1)$.

We determined that the gap value is close to $1 - \frac{2}{m}$ (where m is the number of candidates) between common voting rules like Plurality, Borda, Copeland, Minimax and Schulze.

4. MIXTURE OF MULTINOMIAL LOGITS

4.1 Condorcet Winners

THEOREM 1. *For any uniform mixture of two MNL models with score vectors $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$, there are no Condorcet cycles.*

The uniform mixture among three groups with score vectors (0.42, 0. 161, 0.477),(0.42, 0. 161, 0.477) and (0.308, 0.377, 0.103) has a Condorcet cycle among the three candidates. However, heuristically speaking, cycles seem to be less common (over random score vectors) in the mixture of MNL model than in the impartial culture model.

4.2 Loss of voting rules

For a voting rule R, we quantify the worst possible gap between R and another rule as its *loss*. Precisely:

DEFINITION 2. *The* Loss *of a voting rule R with respect to a preference profile Π is defined as:*

$$Loss_\Pi(R) = \max_x(M(x, R(\Pi))$$

where R is deterministic voting rule and $R(\Pi)$ denotes the winner of voting rule R. The Loss of the voting rule R over $\mathcal{C}$ is $Loss_{\mathcal{C}}(R) = \max_{\Pi\in C} Loss_\Pi(R)$.

In contrast to our findings in Section 3, for voting profiles generated over a uniform distribution of MNL models, the loss of many common voting rules is bounded away from 1.

THEOREM 2. *For voting profiles generated from a uniform mixture of g MNL models, the Loss of the Plurality, Borda, Minimax, Copeland voting rules is at most $1-1/(2g)$.*

However, even with the distributional assumption, the gap can be close to 1 for other voting rules.

LEMMA 1. *For voting profiles generated from a uniform mixture of 2 MNL models, the Loss of the 2-approval voting rule is 1.*

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