Supermodular Games on Social Networks

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ABSTRACT

Cooperative games offer an elegant framework to model cooperation among self-interested agents. A central question of these games is how to distribute the payoff to each player when all players cooperate and derive some benefits. In this work, we consider cooperative transferable utility games where a subset of players can form a coalition if and only if they are connected in the underlying communication structure. We propose a relaxed notion of supermodularity, called *quasi-supermodularity*, for such games, and identify a class of networks where many of these problems are polynomial-time solvable for relaxed-supermodular games. We complement these results by showing that without supermodularity, these problems become hard even if the underlying graph is a tree.

Keywords

Coalitional games; cycle-completeness; supermodularity

1. INTRODUCTION

Cooperation forms an essential part of our society. Consider a group of companies that mine various resources from mountains. Different combinations of companies yield different quantities of resources, and consequently lead to different profits. For the cooperation to be successful, one needs to consider how the benefits should be divided among them. This scenario can be modeled as a cooperative transferable game. Specifically, we are given a finite set N of players and a characteristic function $v : 2^N \to \mathbb{R}$ where $v(S)$ corresponds to a value of a coalition S. The aim of the model is to divide the whole value $v(N)$ among the participants of the game. Over the past decades the cooperative game theory has succeeded in achieving this goal by proposing a number of important solution concepts including the core and the nucleolus.

In particular, the class of supermodular (convex) games has attracted a great deal of attention in the literature. Supermodularity of a characteristic function captures an important economic notion of "increasing marginal return," meaning that entering a larger coalition results in a higher marginal profit as compared to when joining a smaller coalition. Formally, a characteristic function $v : 2^N \to \mathbb{R}$ is supermodular if for each $S, T \subseteq N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \qquad (1.1)$$

It is well known that for supermodular cooperative games, the core is always non-empty, and an element of the core can be computed efficiently [11, 34].

The standard framework of cooperative games assumes that all coalitions are allowed to form. In many real-life settings, however, people often encounter restrictions on cooperation: for instance, a group of companies may not be able to cooperate due to the lack of geographical connection. To model such situations, a cooperative game restricted by an undirected graph, or simply a *graph game*, was proposed by Myerson [30] where a subset of players can form a coalition if and only if they are connected in the graph.

Now, one of the issues that arise when considering graph-restricted games is that the standard definition of supermodular games is no longer applicable, as the set of connected coalitions is not necessarily closed under union and intersection. Intuitively speaking, supermodularity under graph-restricted settings should occur only when players join their neighbouring coalitions.

In this paper, we hence introduce a relaxed notion of supermodularity, called *quasi-supermodularity*, by imposing supermodularity only on subsets of the family that are closed under union and intersection. We consider computational complexity of such games with tree-like restrictions; specifically, we focus on undirected graphs in which the maximal cliques of the graph form a tree. Such graphs are called *cycle-complete*[1] graphs.

In the existing literature on combinatorial optimization, it is known that if a characteristic function satisfies a relaxed form of supermodularity and the graph is cycle-complete, the so-called Myerson game also has supermodularity [15, 12]. Utilizing this property, we derive polynomial-time solvability of computational problems for several solution concepts in quasi-supermodular games on cycle-complete graphs. We then prove that the hereditary property of supermodularity is unlikely to hold unless the graph is cycle-complete.

It turns out that quasi-supermodularity is necessary for games on cycle-complete graphs; we prove that without quasi-supermodularity, many complexity questions become intractable even if the underlying graph is a tree. Indeed, a similar result concerning the core was obtained by Chalkiadakis et al. [5], who showed that many core-related questions are hard for games on trees or graphs having bounded tree-width. We show that the computational problems related to the least core, the kernel, and the Myerson value are co-NP-hard even if the characteristic function is cohesive, and the underlying graph is a star. We also prove that the problems regarding the least core, the nucleolus, and the kernel become Δ_2^p-hard for games on trees if we allow arbitrary values for characteristic functions (Δ_2^p is the class of decision problems solvable in polynomial time by using an **NP** oracle). This strengthens the hardness results by Greco et al. [17, 18], who showed that membership problems

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[1]The term "cycle-complete" is used in [36]; such graphs are also called *block graphs* [24].

for the nucleolus and the kernel are Δ_2^p-hard for games on complete graphs. We summarize our computational complexity results in Table 1.

Related work Several notions of relaxed-supermodularity have been considered in economics and operations research literatures. In fact, quasi-supermodularity in this paper is an extension of quasi-convexity introduced by Bilbao et al. [2], who showed that for games on convex geometries, the convex hull of marginal contribution vectors is contained in the core if and only if the characteristic function is quasi-convex. For graph-restricted games, different types of relaxed-supermodularity were also introduced by Herings et al. [20] and Khmelnitskaya et al. [25]. They showed that such restrictions are sufficient in order for their proposed solutions to lie in the core. We note, however, that few attempts have been made to explore algorithmic implications of relaxed supermodular games with limited communication structure.

There is a rich body of the literature on the computational aspects of cooperative games; see [4] for an overview. Although the literature focuses more on settings where all coalitions are allowed to form [8, 10, 13, 14, 17, 18, 28], several attempts have been made to study the complexity of solutions under graph-restricted settings [5, 9, 23, 35]. Some authors [5, 9, 35] have observed that many solutions for graph games can be computed efficiently if the number of connected coalitions in the underlying graph is bounded by a polynomial in the number of players, as is the case for paths and cycles. Elkind [9] unified these approaches, by giving a characterization of graph families with polynomially bounded number of connected coalitions as well as by showing that various solutions can be computed in time polynomial in the number of connected coalitions. The most closely related to ours is perhaps the work by Chalkiadakis et al [5]. However, there are a number of differences between their work and ours: first, in their work, the membership and search questions for other solution concepts such as the Myerson value are not addressed; second, they explore games constrained by graphs with bounded tree-width, whereas we focus on a different class of tree-like graphs; and finally, they do not explore the input of supermodularity.

2. PRELIMINARIES

We start by introducing basic notation and definitions of set systems.

Subfamilies and set functions Let N be a finite set and $\mathcal{F}$ be a family of subsets of N. We write $\mathcal{F}(a) = \{\, S \in \mathcal{F} \mid a \in S \,\}$ for $a \in N$, and $\mathcal{F}(a \setminus b) = \mathcal{F}(a) \setminus \mathcal{F}(b)$ for $a, b \in N$. We say that $\mathcal{F}$ is an *intersecting family* if for all $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, $S \cup T \in \mathcal{F}$ and $S \cap T \in \mathcal{F}$. A family $\mathcal{F}$ is called a *distributive lattice* if it is closed under union and intersection, i.e., for all $S, T \in \mathcal{F}$, $S \cup T \in \mathcal{F}$ and $S \cap T \in \mathcal{F}$. Throughout this paper, we only consider set functions $f : \mathcal{F} \to \mathbb{R}$ where $f(\emptyset) = 0$ whenever $\emptyset \in \mathcal{F}$. We say that $f : \mathcal{F} \to \mathbb{R}$ is *superadditive* if for all $S, T \in \mathcal{F}$ such that $S \cap T = \emptyset$ and $S \cup T \in \mathcal{F}$, $f(S) + f(T) \leq f(S \cup T)$. For a nonempty subset $S \subseteq N$, a partition $\{X_i\}_{i \in I}$ of S is said to be an $\mathcal{F}$*-partition* if $X_i \in \mathcal{F}$ for all $i \in I$. We define $\hat{\mathcal{F}}$ as the collection of disjoint unions of sets in $\mathcal{F}$, namely,

$$\hat{\mathcal{F}} = \{\, S \subseteq N \mid S \neq \emptyset, \text{there exists an } \mathcal{F}\text{-partition of } S \,\} \cup \{\emptyset\}.$$

Notice that $\hat{\mathcal{F}} = 2^N$ whenever $\mathcal{F}$ includes all the singletons. Also, it is not difficult to show that if $\mathcal{F}$ is an intersecting family, then $\hat{\mathcal{F}}$ is a distributive lattice [12]. For a set function $f : \mathcal{F} \to \mathbb{R}$, the *Dilworth truncation* [33], or simply the *truncation* of f is the function $\hat{f} : \hat{\mathcal{F}} \to \mathbb{R}$ such that for any nonempty $S \in \hat{\mathcal{F}}$,

$$\hat{f}(S) := \max \left\{ \sum_{i \in I} f(X_i) \,\middle|\, \{X_i\}_{i \in I} \text{ is an } \mathcal{F}\text{-partition of } S \right\},$$

and $\hat{f}(\emptyset) = 0$. For a rational-valued set function f on a family $\mathcal{F} \subseteq 2^N$, we define $\langle f \rangle$ as an upper bound on the encoding lengths of outputs of f.

3. GRAPH GAMES

Now, we define a cooperative game constrained by graphs. Given an undirected graph (N, E), let $\mathcal{F}_E$ be the set of connected subsets of N; we assume $\emptyset \in \mathcal{F}_E$ for convention.

DEFINITION 3.1. *A cooperative transferable utility game with a graph structure, or simply a graph game, is a triple (N, v, E) where N is a finite set of players, $v : \mathcal{F}_E \to \mathbb{R}$ is a characteristic function, and $E \subseteq \{\, \{a, b\} \mid a, b \in N \wedge a \neq b \,\}$ is the set of communication edges between players.*

In case where $E = \{\, \{a, b\} \mid a, b \in N \wedge a \neq b \,\}$, the game (N, v, E) is said to have *full communication structure* and is simply denoted by (N, v). The subsets S of N are referred to as *coalitions*. A coalition $S \subseteq N$ is said to be *feasible* if $S \in \mathcal{F}_E$. We only consider allocation scenarios where the grand coalition forms. Namely, our goal is to distribute the total value derived from the grand coalition. We thus assume that the grand coalition N is connected, i.e., $N \in \mathcal{F}_E$. We also assume $v(\emptyset) = 0$. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, we use notation $x(S) = \sum_{a \in S} x_a$ for any $S \subseteq N$. Here, $x(\emptyset) = 0$ for convention. A characteristic function $v : \mathcal{F}_E \to \mathbb{R}$ is said to be *cohesive* if the grand coalition is optimal, namely, $v(N) = \hat{v}(N)$. We call a graph game (N, v, E) superadditive (respectively, cohesive) if the characteristic function is superadditive (respectively, cohesive).

Solution concepts An *imputation* for a graph game (N, v, E) is a vector $\boldsymbol{x} \in \mathbb{R}^N$ satisfying *efficiency* : $x(N) = v(N)$, and *individual rationality* : $x_a \geq v(\{a\})$, for all $a \in N$. For a graph game (N, v, E), let $\mathcal{I}(N, v, E)$ denote the set of imputations of (N, v, E). The *core* is one of the most important solution concepts that are immune to any feasible coalitional deviations. The core $\mathcal{C}(N, v, E)$ of a graph game (N, v, E) is the set of all imputations $\boldsymbol{x}$ such that no feasible coalitions have an incentive to defect from $\boldsymbol{x}$, i.e., $x(S) \geq v(S)$, for all $S \in \mathcal{F}_E$. The *least core* and the *nucleolus* are the refined concepts of the core that take into account *fairness* among coalitions. We first define the degree of unhappiness of a feasible coalition $S \in \mathcal{F}_E \setminus \{N, \emptyset\}$ at an imputation $\boldsymbol{x} \in \mathcal{I}(N, v, E)$: the *excess* $e(\boldsymbol{x}, S)$ of S at $\boldsymbol{x}$, given by

$$e(\boldsymbol{x}, S) := v(S) - x(S).$$

We denote by $e_{\max}(\boldsymbol{x})$ the maximum excess with respect to $\boldsymbol{x}$, i.e., $e_{\max}(\boldsymbol{x}) = \max_{S \in \mathcal{F}_E \setminus \{N, \emptyset\}} e(\boldsymbol{x}, S)$. The *least core* $\mathcal{LC}(N, v, E)$ of a graph game (N, v, E) is the set of all imputations $\boldsymbol{x}$ that minimizes the maximum excess, i.e., $e_{\max}(\boldsymbol{x}) \leq e_{\max}(\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathcal{I}(N, v, E)$. By imposing a lexicographic order on the excess, we can even strengthen the notion of the least core. For each imputation $\boldsymbol{x} \in \mathcal{I}(N, v, E)$, we denote by $\theta(\boldsymbol{x})$ the sequence of the components $e(\boldsymbol{x}, S)$ $(S \in \mathcal{F}_E \setminus \{N, \emptyset\})$ of $\boldsymbol{x}$ arranged in non-increasing order, i.e., $\theta(\boldsymbol{x}) = (e(\boldsymbol{x}, S_1), e(\boldsymbol{x}, S_2), \cdots, e(\boldsymbol{x}, S_k))$ with

$$e(\boldsymbol{x}, S_1) \geq e(\boldsymbol{x}, S_2) \geq \cdots \geq e(\boldsymbol{x}, S_k),$$

where $\mathcal{F}_E \setminus \{N, \emptyset\} = \{S_1, S_2, \cdots, S_k\}$. For real k-sequences $\boldsymbol{u} = (u_1, u_2, \cdots, u_k)$ and $\boldsymbol{v} = (v_1, v_2, \cdots, v_k)$, $\boldsymbol{u}$ is *lexicographically smaller than or equal to* $\boldsymbol{v}$ (denoted by $\boldsymbol{u} \leq_L \boldsymbol{v}$) if and only if $\boldsymbol{u} =$

	Complete (unrestricted cases)			Cycle-complete	Stars	Trees
	supermodular	cohesive	general	q-supermodular	cohesive	general
CORE-MEMBERSHIP	P ([32])	co-NP-c ([5])	co-NP-c ([5])	P (4.5)	co-NP-c ([5])	co-NP-c ([5])
CORE-NONEMPTINESS	$O(1)$ ([11])	co-NP-c ([5])	co-NP-c ([17])	$O(1)$ ([2])	$O(1)$ (5.8)	co-NP-c (5.9)
CORE-FIND	P ([11])	NP-h ([5])	NP-h ([8])	P (4.5)	P (5.7)	NP-h (5.10)
LEASTCORE-MEMBERSHIP	P ([28, 13]+[32])	co-NP-h ([10])	Δ_2^p-c ([17])	P (4.5)	co-NP-h (5.1)	Δ_2^p-c (5.4)
LEASTCORE-FIND	P ([28, 13])	NP-h ([10])	Δ_2^p-h (5.6+5.11)	P (4.5)	NP-h (5.2)	Δ_2^p-h (5.6)
NUCLEOLUS-MEMBERSHIP	P ([28, 13])	co-NP-h ([10])	Δ_2^p-c ([18])	P (4.5)	co-NP-h (5.1)	Δ_2^p-c (5.4)
NUCLEOLUS-FIND	P ([28, 13])	NP-h([10])	Δ_2^p-h ([18])	P (4.5)	NP-h (5.2)	Δ_2^p-h (5.6)
KERNEL-MEMBERSHIP	P ([32])	co-NP-h (5.1+5.11)	Δ_2^p-c ([17])	P (4.5)	co-NP-h (5.1)	Δ_2^p-c (5.4)
KERNEL-FIND	P ([28, 13])	NP-h (5.2+5.11)	Δ_2^p-h (5.6+5.11)	P (4.5)	NP-h (5.2)	Δ_2^p-h (5.6)

Table 1: Computational complexity for transferable utility games on cycle-complete graphs. The top row corresponds to restrictions on graphs; the second row from the top indicates restrictions on characteristic functions. The hardness results for search problems hold with respect to Turing reductions. The non-emptiness questions for the least core, the nucleolus and the kernel are trivial, since such solutions exist if and only if the imputation set is nonempty. We note that although the core in [5] is a generalization of ours, the reductions used in the proofs of the paper apply to our settings.

$\boldsymbol{v}$, or $\boldsymbol{u} \neq \boldsymbol{v}$ and for the minimum index j such that $u_j \neq v_j$ we have $u_j < v_j$. The *nucleolus* $\mathcal{N}(N, v, E)$ of a graph game (N, v, E) is the set of all imputations $\boldsymbol{x}$ that minimizes the maximum excess in a lexicographic sense, i.e., $\theta(\boldsymbol{x}) \leq_L \theta(\boldsymbol{y})$, for all $\boldsymbol{y} \in \mathcal{I}(N, v, E)$.

Another classical solution concept we consider is the *kernel*. Such imputations can be regarded as the outcomes of bargaining among players in the sense that no player is threatened by other players. Formally, given $\boldsymbol{x} \in \mathcal{I}(N, v, E)$ and $a, b \in N$ $(a \neq b)$, we define the *surplus* $s_{ab}(\boldsymbol{x})$ of player a against player b at $\boldsymbol{x}$ as $s_{ab}(\boldsymbol{x}) = \max\{\, e(\boldsymbol{x}, S) \mid S \in \mathcal{F}_E(a \setminus b)\,\}$. We say that player a has *more bargaining power* than b at $\boldsymbol{x}$ if $s_{ab}(\boldsymbol{x}) > s_{ba}(\boldsymbol{x})$. In such situations, player a can claim some of b's payoff, but the amount he can claim is limited by individual rationality. The *kernel* $\mathcal{K}(N, v, E)$ of a graph game (N, v, E) is the set of all imputations $\boldsymbol{x}$ such that for each player $b \in N$, if there exists another player $a \in N \setminus \{b\}$ who has more bargaining power than b, then $x_b = v(\{b\})$.

We have the following containment relations among these classes of outcomes: $\mathcal{N}(N, v, E) \subseteq \mathcal{K}(N, v, E) \cap \mathcal{LC}(N, v, E)$, and whenever $\mathcal{C}(N, v, E) \neq \emptyset$, $\mathcal{N}(N, v, E) \subseteq \mathcal{LC}(N, v, E) \subseteq \mathcal{C}(N, v, E)$. The containment of the nucleolus in the kernel can be shown by a simple adaptation of the proof of Theorem 3 in [31], whereas the other relations follow from the definitions. The imputation set (respectively, the core, the least core, the nucleolus, and the kernel) of a graph game with full communication structure is denoted by $\mathcal{I}(N, v)$ (respectively, $\mathcal{C}(N, v)$, $\mathcal{LC}(N, v)$, $\mathcal{N}(N, v)$, and $\mathcal{K}(N, v)$).

Besides coalitional stability, we consider another important solution concept capturing fairness among players, the *Myerson value* [30]. For a graph game (N, v, E), we define its *Myerson game* as a pair (N, v^E), that is, the game with full communication structure where each $v^E(S)$ for $S \subseteq N$ is given by the sum of values of the connected components of S. The *Myerson value* $m_a(N, v, E)$ of a graph game (N, v, E) for player $a \in N$ is the average of a's marginal contributions at v^E over all permutations of the players, that is,

$$m_a(N, v, E) = \sum_{S \subseteq N \setminus \{a\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \Delta_a(S),$$

where $\Delta_a(S) = v^E(S \cup \{a\}) - v^E(S)$ for $S \subseteq N \setminus \{a\}$.

Cycle-complete graphs In this paper, we will be essentially interested in cooperative games where the graph is *cycle-complete*. Such topology often represents networks having distinct community structure. Specifically, a graph (N, E) is said to be *cycle-complete* if every cycle of the graph forms a clique, i.e., for every cycle $\{a_1, a_2, \ldots, a_m\}$ in the graph (N, E) and every pair of distinct $i, j \in \{1, 2, \ldots, m\}$, we have $\{a_i, a_j\} \in E$. Since forests do not contain any cycle, they are trivially cycle-complete. Another class of cycle-complete graphs is the class of complete graphs. In Figure 1, we represent an example of cycle-complete graphs. Cycle-complete graphs are known to have useful combinatorial features. The following lemma states that a graph is cycle-complete if and only if the family of connected subsets of the graph is closed under union and intersection for every pair of connected subsets whose intersection is nonempty.

LEMMA 3.2 (JAMISON [24]). *A graph $G = (N, E)$ is cycle-complete if and only if $\mathcal{F}_E$ is an intersecting family.*

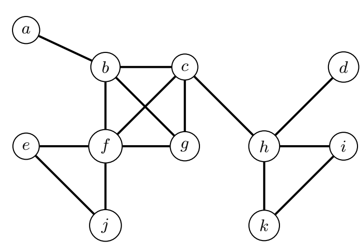

Figure 1: An example of cycle-complete graphs

Computational setting Throughout the paper, we only consider a graph game (N, v, E) whose characteristic function is computable in time polynomial in the number of players $|N|$. Formally, we assume that our game encoding $||(N, v, E)||$ includes the graph $G = (N, E)$, and that each value $v(S)$ for $S \in \mathcal{F}_E$ can be computed in time polynomial in $|N|$.

4. QUASI-SUPERMODULAR GAMES

It is well known that supermodularity often allows us to calculate various solutions in polynomial time. Notable results include the greedy algorithm to find an element of the core by Edmonds [11] and

Shapley [34] and the polynomial-time algorithm for the nucleolus by Kuipers [28] (see also [13]). The aim of this section is to introduce a relaxed notion of supermodularity, called *quasi-supermodularity*, and show that a variety of computational problems are tractable for the class of quasi-supermodular games on cycle-complete graphs.

DEFINITION 4.1 (QUASI-SUPERMODULARITY). *Let $\mathcal{F}$ be a family of subsets of a finite set N. A set function $f : \mathcal{F} \to \mathbb{R}$ is called quasi-supermodular if for each $S, T \in \mathcal{F}$ with $S \cup T \in \mathcal{F}$ and $S \cap T \in \mathcal{F}$, the supermodular inequality* (1.1) *holds*

Notice that by definition, if a set function $f : \mathcal{F} \to \mathbb{R}$ such that $\emptyset \in \mathcal{F}$ and $f(\emptyset) = 0$ is quasi-supermodular, it is also superadditive. A graph game (N, v, E) is said to be *quasi-supermodular* if the characteristic function $v : \mathcal{F}_E \to \mathbb{R}$ is quasi-supermodular. The relaxed-supermodularity arises when we have to take into account connection between players.

EXAMPLE 4.2 (TREASURE HUNTING GAME). Consider a treasure hunting game on a social network. In this game, the players are located in a 2-dimensional Euclidean space. They form a team to search for treasure and the players who are geographically closer are more likely to form a coalition.The probability for a team to successfully find a treasure increases quadratically as the team becomes larger. We describe this scenario as a quasi-supermodular graph game (N, v, E) on a cycle-complete graph where the (N, E) is given by Figure 1 and the characteristic function v is given by $v(S) = |S|^2$ for each feasible coalition $S \in \mathcal{F}_E$.

We shall prove our tractability results in the following two steps. First, we reduce some computational problems on quasi-supermodular graph games on cycle-complete graphs (N, v, E), to those on supermodular games with full communication structure $(N, \hat{v})$. Notice that $\hat{\mathcal{F}}_E = 2^N$ since each $S \subseteq N$ is guaranteed to have an $\mathcal{F}_E$-partition that consists of the singletons in S. Second, we derive polynomial solvability for several solution concepts, by using results previously proven for the class of supermodular games in the unrestricted settings.

To begin with, we observe that if a a characterisitc function is superadditive, the core and the nucleolus of the game $(N, \hat{v})$ coincide with those of the original game.

THEOREM 4.3. *Given a graph game (N, v, E), the following statements hold.*

(*i*) *If v is cohesive, then $\mathcal{C}(N, v, E) = \mathcal{C}(N, \hat{v})$.*

(*ii*) *If v is superadditive and $\mathcal{C}(N, v, E) \neq \emptyset$, then $\mathcal{N}(N, v, E) = \mathcal{N}(N, \hat{v})$.*

PROOF. It is well known that the statement (i) holds (see for instance Theorem 2 in [1]). The statement (ii) was essentially proven by Huberman [21]. For a game with full communication structure (N, v), it was shown in [21] that removing coalitions $S \subseteq N$ such that $v(S) \leq \sum_{i \in I} v(X_i)$ for some proper partition $\{X_i\}_{i \in I}$ of S does not change the nucleolus. It follows that the subsets in $2^N \setminus \mathcal{F}_E$ are redundant in defining the nucleolus $\mathcal{N}(N, \hat{v})$. Moreover, if (N, v, E) is a superadditive graph game, $\hat{v}(S) = v(S)$ for all $S \in \mathcal{F}_E$ by superadditivity, and hence, $\mathcal{N}(N, v, E) = \mathcal{N}(N, \hat{v})$. □

Moreover, supermodularity of a characteristic function is also preserved to the game $(N, \hat{v})$ if the underlying graph is cycle-complete.

THEOREM 4.4. *Suppose that (N, v, E) is a quasi-supermodular graph game where the underlying graph (N, E) is cycle-complete. Then, the following statements hold.*

(*i*) *The game $(N, \hat{v})$ is supermodular.*

(*ii*) *Each $\hat{v}(S)$ for $S \subseteq N$ can be computed in strongly polynomial time in $|N|$.*

PROOF. It is known that if $f : \mathcal{F} \to \mathbb{R}$ is *intersecting supermodular*, i.e., $\mathcal{F}$ is an intersecting family and the supermodular inequality (1.1) holds for every pair $S, T \in \mathcal{F}$ such that $S \cap T \neq \emptyset$, then its truncation function is supermodular [15, 12]. Another nice property of an intersecting supermodular function $f : \mathcal{F} \to \mathbb{R}$ is that each value $\hat{f}(S)$ for $S \in \hat{\mathcal{F}}$ can be computed in strongly polynomial time, having a value-giving oracle for f and a compact representation for $\mathcal{F}$ [16]. Since if $v : \mathcal{F}_E \to \mathbb{R}$ is quasi-supermodular and (N, E) is cycle-complete, v is intersecting supermodular; hence, the claims hold. □

We are now ready to give polynomial-time solvability results in the following theorem.

THEOREM 4.5. *Suppose that (N, v, E) is a quasi-supermodular graph game where (N, E) is cycle-complete. Then, the following statements hold.*

(*i*) *An element of the core can be found in strongly polynomial time in $|N|$.*

(*ii*) *One can check whether a given imputation $\boldsymbol{x}$ belongs to the core or the kernel in strongly polynomial time in $|N|$.*

(*iii*) *If v is a rational-valued function, an imputation of the nucleolus can be found in time polynomial in $|N|$ and $\langle v \rangle$.*

(*iv*) *If v is a rational-valued function, one can check whether a given imputation $\boldsymbol{x}$ belongs to the nucleolus or the least core in time polynomial in $|N|$ and $\langle v \rangle$.*

PROOF. (i): By Theorem 4.3, it suffices to find an element $\boldsymbol{x} \in \mathcal{C}(N, \hat{v})$ for the supermodular game $(N, \hat{v})$. By Theorem 4.4, each $\hat{v}(S)$ for $S \subseteq N$ can be computed in strongly polynomial time in $|N|$. Edmonds [11] and Shapley [34] presented a strongly polynomial-time algorithm to find an element of the core for supermodular games with full communication structure. Therefore, one can also find $\boldsymbol{x} \in \mathcal{C}(N, v, E)$ in strongly polynomial time in $|N|$.

(ii): Checking whether $\boldsymbol{x} \in \mathcal{C}(N, v, E)$ can be reduced to the maximization of a supermodular function f_a on a distributive lattice $\mathcal{F}_E(a)$ where $a \in N$ and $f_a(S) = v(S) - x(S)$ for $S \in \mathcal{F}_E(a)$: the imputation $\boldsymbol{x}$ is in $\mathcal{C}(N, v, E)$ if and only if for each $a \in N$, the maximum of f_a is less than or equal to 0. This can be checked in strongly polynomial time in $|N|$ due to the work of [32, 22]. Notice that checking if $\boldsymbol{x} \in \mathcal{K}(N, v, E)$ is easy if one can efficiently calculate each surplus $s_{ab}(\boldsymbol{x})$ for each pair of distinct players $a, b \in N$. Let (N, v, E) be a quasi-supermodular game on a cycle-complete graph, and $\boldsymbol{x} \in \mathcal{I}(N, v, E)$. For $a, b \in N$ $(a \neq b)$, computing the surplus $s_{ab}(\boldsymbol{x})$ is equivalent to the maximization of a supermodular function f_{ab} on a distributive lattice $\mathcal{F}_E(a \setminus b)$ where $f_{ab}(S) = v(S) - x(S)$ for $S \in \mathcal{F}_E(a \setminus b)$, which can be done in strongly polynomial time in $|N|$ [32, 22].

(iii): Notice that $\mathcal{C}(N, v, E) = \mathcal{C}(N, \hat{v})$ is nonempty since $(N, \hat{v})$ is a supermodular game [11, 34]. Since v is superadditive and $\mathcal{C}(N, v, E)$ is nonempty, it suffices to find $\boldsymbol{x} \in \mathcal{N}(N, \hat{v}) = \mathcal{N}(N, v, E)$ by Theorem 4.3. We have seen that $\hat{v}$ is supermodular, and each $\hat{v}(S)$ for $S \subseteq N$ can be computed in strongly polynomial time in $|N|$. Faigle et al. [13] showed that the nucleolus can be computed in time polynomial in $|N|$ and $\langle v \rangle$ for supermodular games (N, v) with full communication structure. Hence, one can find $\boldsymbol{x} \in \mathcal{N}(N, \hat{v})$ in time polynomial in $|N|$ and $\langle v \rangle$. As noted in

Section 3, the nucleolus always belongs to the least core and the kernel; hence, Theorem 4.5 also implies that finding these solutions can be done in time polynomial in $|N|$ and $\langle v \rangle$.

(iv): By the above argument, one can compute $\boldsymbol{x}^* \in \mathcal{N}(N,v,E)$ in time polynomial in $|N|$ and $\langle v \rangle$. Recall that $|\mathcal{N}(N,v,E)| = |\mathcal{N}(N,\hat{v})| = 1$ [31]. Hence, one can check whether a given imputation $\boldsymbol{x}$ belongs to the nucleolus by comparing each element of the vectors $\boldsymbol{x}$ and $\boldsymbol{x}^*$. Further, an imputation $\boldsymbol{x} \in \mathcal{LC}(N,v,E)$ if and only if $\max\{\, s_{ab}(\boldsymbol{x}) \mid a,b \in N, a \neq b \,\} = \max\{\, s_{ab}(\boldsymbol{x}^*) \mid a,b \in N, a \neq b \,\}$. These values can be computed in strongly polynomial time in $|N|$, since each surplus can be computed in strongly polynomial time. □

Necessity of cycle-completeness

We have seen that the problems with respect to quasi-supermodular games (N,v,E) can be reduced to those for supermodular games with full communication structure $(N,\hat{v})$ since the supermodularity is successfully transmitted to the truncated function if the underlying communication structure forms an intersecting family. Then, it is natural to ask whether the same technique can be applied to other classes of families of feasible sets, i.e., whether the condition for $\mathcal{F} \subseteq 2^N$ to be an intersecting family is necessary to preserve supermodulartiy of f. The next results will show that it is indeed necessary.

LEMMA 4.6. *Let N be a finite set and $\mathcal{F}$ be a family of subsets of N with $N, \emptyset \in \mathcal{F}$ such that $\hat{\mathcal{F}}$ is a distributive lattice. If the truncation $\hat{f} : \hat{\mathcal{F}} \to \mathbb{R}$ is supermodular for every quasi-supermodular function $f : \mathcal{F} \to \mathbb{R}$, then $\mathcal{F}$ is an intersecting family.*

PROOF. Let $\mathcal{F}$ be a subset of 2^N such that $N, \emptyset \in \mathcal{F}$ and $\hat{\mathcal{F}}$ is a distributive lattice. Suppose that $\mathcal{F}$ is not an intersecting family. Then, there exists a pair of nonempty subsets $S, T \in \mathcal{F}$ such that $S \cap T \neq \emptyset$, and $S \cup T \notin \mathcal{F}$ or $S \cap T \notin \mathcal{F}$. We will show that there exists a quasi-supermodular set function on $\mathcal{F}$ whose truncation is not supermodular. To see this, we define a set function $f : \mathcal{F} \to \mathbb{R}$ by $f(X) = |X| - 1$ for each $X \in \mathcal{F} \setminus \{\emptyset\}$ and $f(\emptyset) = 0$. Then, it can be easily seen that the function f is quasi-supermodular. By the definition of f, it holds that $\hat{f}(S) = f(S) = |S| - 1$ and $\hat{f}(T) = f(T) = |T| - 1$. Furthermore, we have $S \cup T, S \cap T \in \hat{\mathcal{F}} \setminus \{\emptyset\}$ since $\hat{\mathcal{F}}$ is a distributive lattice. Let $\{X_j\}_{j \in J}$ be an $\mathcal{F}$-partition of $S \cup T$ such that $\hat{f}(S \cup T) = \sum_{j \in J} f(X_j)$ and let $\{Y_k\}_{k \in K}$ be an $\mathcal{F}$-partition of $S \cap T$ such that $\hat{f}(S \cap T) = \sum_{k \in K} f(Y_k)$. Then, it holds that $\hat{f}(S \cup T) + \hat{f}(S \cap T) = (|S| + |T|) - (|J| + |K|)$. Observe that $|J| \geq 2$ or $|K| \geq 2$ since $S \cup T \notin \mathcal{F}$ or $S \cap T \notin \mathcal{F}$. Hence, $\hat{f}(S \cup T) + \hat{f}(S \cap T) \leq (|S| + |T|) - 3 < (|S| - 1) + (|T| - 1) = \hat{f}(S) + \hat{f}(T)$. Thus, $\hat{f}$ is not supermodular. □

This yields the following corollary.

COROLLARY 4.7. *Let N be a finite set and $\mathcal{F}$ be a family of subsets of N with $N, \emptyset \in \mathcal{F}$ such that $\hat{\mathcal{F}}$ is a distributive lattice. Then, the following two statements are equivalent.*

(*i*) *$\mathcal{F}$ is an intersecting family.*

(*ii*) *For every quasi-supermodular function $f : \mathcal{F} \to \mathbb{R}$, its truncation $\hat{f} : \hat{\mathcal{F}} \to \mathbb{R}$ is supermodular.*

PROOF. The direction (i) $\Rightarrow$ (ii) was proved in [15, 12]. The direction (ii) $\Rightarrow$ (i) follows from Lemma 4.6. □

5. HARDNESS RESULTS

So far, we discussed how supermodularity of the characteristic function enables an efficient computation of cooperative solutions for games on cycle-complete graphs. In this section, let us turn our attention to the complexity questions for non-supermodular games on trees, which are a subclass of cycle-complete graphs. Typically, tree structure appears to be an attractive restriction that can decrease complexity; indeed, in the seminal paper by Demange [7], an efficient procedure to obtain a specific core element was presented for supperadditive games on trees. However, we will see in this section that this may not be the case in general.

Least core, nucleolus, kernel, Myerson value

We first study the computational complexity for the least core, the nucleolus, the kernel, and the Myerson value of games on trees. The following theorems state that even if the characteristic function is cohesive and the graph is a star, it is hard to check whether a given imputation is the Myerson value, or belongs to the least core, the nucleolus, or the kernel.

THEOREM 5.1. *Given a graph game (N,v,E) where v is non-negative and (N,E) is a star, it is co-NP-hard to determine whether an imputation $\boldsymbol{x}$ is the Myerson value or belongs to the least core, the nucleolus, or the kernel.*

PROOF. We will reduce from SAT. Given a Boolean formula ϕ over the set of variables $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, we construct a star (N,E) with the center player c and leaves a_h for each variable α_h $(h = 1, 2, \ldots, k)$ and another player d. For each $T \subseteq A$, we write $\sigma(T) \models \phi$ if ϕ is satisfied by setting the variables $\{\, \alpha_h \mid a_h \in T \,\}$ to true and the variables $\{\, \alpha_h \mid a_h \in A \setminus T \,\}$ to false, and $\sigma(T) \not\models \phi$ otherwise. The value $v(S)$ for $S \in \mathcal{F}_E$ is given as follows.

- $v(S) = 1$ when $S = N$.
- $v(S) = 0$ when $|S| = 1$.
- $v(S) = 0$ when $d \in S \neq N$.
- $v(S) = 1$ when $S = T \cup \{c\}$ for a nonempty subset $T \subseteq A$ such that $\sigma(T) \models \phi$.
- $v(S) = 0$ when $S = T \cup \{c\}$ for a nonempty subset $T \subseteq A$ such that $\sigma(T) \not\models \phi$.

Observe that the characteristic function v is non-negative and cohesive. Further, given a subset $S \in \mathcal{F}_E$, the value $v(S)$ can be computed in polynomial time. Let $n = |N|$ and $x^*_a = 1/n$ for every $a \in N$. Clearly, $\boldsymbol{x}^* \in \mathcal{I}(N,v,E)$. We will now argue that the following statements hold:

(i) If ϕ is unsatisfiable, then $\boldsymbol{x}^* = \boldsymbol{m}(N,v,E)$ and moreover $\{\boldsymbol{x}^*\} = \mathcal{LC}(N,v,E) = \mathcal{K}(N,v,E)$.

(ii) If ϕ is satisfiable, then $\boldsymbol{x}^* \neq \boldsymbol{m}(N,v,E)$ and moreover $\boldsymbol{x}^* \notin \mathcal{LC}(N,v,E) \cup \mathcal{K}(N,v,E)$.

(i) : Suppose that ϕ is unsatisfiable. Observe that $v(S) = 0$, for all $S \in \mathcal{F}_E \setminus \{N\}$ by definition of v. Thus, the marginal contribution of each player a to a coalition S is 1 if $S \cup \{a\} = N$ and $a \notin S$, and 0 otherwise, implying that $m_a(N,v,E) = 1/n$. Hence, we have $\boldsymbol{m}(N,v,E) = \boldsymbol{x}^*$. Now it remains to show that any imputation different from $\boldsymbol{x}^*$ belongs to neither the kernel nor the least core. To see this, take any $\boldsymbol{x} \in \mathcal{I}(N,v,E)$ where $\boldsymbol{x} \neq \boldsymbol{x}^*$. Then, it can be easily seen that $x_a < 1/n < x_b$ for some $a, b \in N$. Further, $\boldsymbol{x} \geq \boldsymbol{0}$ by individual rationality. Hence, we have

$s_{ab}(\boldsymbol{x}) = -x_a > -1/n > -x_b = s_{ba}(\boldsymbol{x})$. Combining this with the fact that $x_b > 0 = v(\{b\})$ implies that $\boldsymbol{x} \notin \mathcal{K}(N, v, E)$. In addition, $\boldsymbol{x} \notin \mathcal{LC}(N, v, E)$ since

$$\max_{S \in \mathcal{F}_E \setminus \{N, \emptyset\}} e(\boldsymbol{x}, S) \geq e(\boldsymbol{x}, \{a\}) > -1/n = \max_{S \in \mathcal{F}_E \setminus \{N, \emptyset\}} e(\boldsymbol{x}^*, S).$$

This also shows that $\boldsymbol{x}^*$ is in the nucleolus of the game. Hence, the claim follows.

(ii) : Suppose that ϕ is satisfiable. Then, there exists $S^* \in \mathcal{F}_E \setminus \{N\}$ such that $v(S^*) = 1$ and $S^* = T \cup \{c\}$ for some $T \subseteq A$. Notice that $\Delta_c(T) = v^E(T \cup \{c\}) - v^E(T) = 1$. Moreover, $\Delta_c(N \setminus \{c\}) = 1$ and $\Delta_c(S) \geq 0$ for all $S \subseteq N \setminus \{c\}$. Hence the Myerson value for the center c is strictly greater than $1/n$. So we have $\boldsymbol{x}^* \neq \boldsymbol{m}(N, v, E)$. To show that $\boldsymbol{x}^*$ is not an element of the least core, let $\boldsymbol{y}$ be an imputation of (N, v, E) such that the central player c receives the whole value $v(N) = 1$ and any other player receives nothing, i.e., $y_c = 1$ and $y_a = 0$ for all $a \in N \setminus \{c\}$. Then, it can be easily checked that $\max_{S \in \mathcal{F}_E \setminus \{N, \emptyset\}} e(\boldsymbol{y}, S) \leq 0$. However, the maximum excess with respect to $\boldsymbol{x}^*$ is greater than 0 because

$$\max_{S \in \mathcal{F}_E \setminus \{N, \emptyset\}} e(\boldsymbol{x}^*, S) \geq v(S^*) - x(S^*) = 1 - |S^*|/n > 0.$$

Hence, $\boldsymbol{x}^* \notin \mathcal{LC}(N, v, E)$. It remains to show that $\boldsymbol{x}^*$ does not belong to the kernel. Observe that $c \in S^*$ and $d \notin S^*$. Thus, $s_{cd}(\boldsymbol{x}^*) \geq e(\boldsymbol{x}^*, S^*) = 1 - |S^*|/n > 0$. In contrast, $s_{dc}(\boldsymbol{x}^*) = v(\{d\}) - x^*_d = -1/n < 0$ since $\{d\}$ is the unique coalition in $\mathcal{F}_E(d \setminus c)$. Combining these with the inequality $x^*_d = 1/n > 0 = v(\{d\})$ yields that $\boldsymbol{x}^* \notin \mathcal{K}(N, v, E)$.

By (i) and (ii), ϕ is unsatisfiable if and only if $\boldsymbol{x}^*$ coincides with the Myerson value or belongs to the least core, the nucleolus, or the kernel of the game (N, v, E). □

The previous reductions can be adapted to obtain hardness of the search problems for respective solution concepts.

THEOREM 5.2. *If one can find in polynomial time the Myerson value, or an element of the least core or the kernel for cohesive graph games (N, v, E) where v is non-negative and (N, E) is a star, then P=NP.*

PROOF. We will show that polynomial-time algorithms for our problems can be used to decide SAT in polynomial time. Given a Boolean formula ϕ, we construct the game (N, v, E) defined in the proof of Theorem 5.1. Let $n = |N|$ and $x^*_a = 1/n$ for every $a \in N$. Take any $\boldsymbol{x} \in \mathcal{I}(N, v, E)$ such that $\boldsymbol{x} = \boldsymbol{m}(N, v, E)$ or $\boldsymbol{x} \in \mathcal{LC}(N, v, E) \cup \mathcal{K}(N, v, E)$. By the previous proof of Theorem 5.1, ϕ is unsatisfiable if and only if $\boldsymbol{x} = \boldsymbol{x}^*$. It follows that, by computing an imputation $\boldsymbol{x}$ that is the Myerson value or belongs to the least core or the kernel of the game (N, v, E), and checking whether $\boldsymbol{x} = \boldsymbol{x}^*$, we can decide whether ϕ is satisfiable or not. □

Now if we allow arbitrary values for characteristic functions, the membership problems for the least core, the nucleolus, and the kernel of games on trees become Δ^p_2-hard. We prove this by a reduction from the problem of deciding whether the least significant variable is true in the lexicographically maximum satisfying assignment, which was shown to be Δ^p_2-complete [27]. The reduction behind the theorem is inspired by an argument of Greco et al. [17, 18] who showed that checking whether a given imputation belongs to the kernel or the nucleolus is Δ^p_2-complete even for games succinctly represented by weighted graphs. We remark, however, that in their paper, there is no restrictions of coalition structures among the players, namely, their result is for the case where the graph (N, E) is complete, whereas our result holds even if (N, E) is a tree.

Given a Boolean formula ϕ over the variables $\alpha_1, \alpha_2, \ldots, \alpha_k$. Recall that, for truth assignments $\boldsymbol{u} = (u_1, u_2, \cdots, u_k)$ and $\boldsymbol{v} = (v_1, v_2, \cdots, v_k)$ of ϕ where $u_i, v_i \in \{0, 1\}$ for $i = 1, 2, \ldots, k$, $\boldsymbol{u}$ is lexicographically greater than $\boldsymbol{v}$ if and only if for the minimum index j such that $u_j \neq v_j$ we have $u_j = 1$ and $v_j = 0$. An instance of the problem LEASTLEXSAT is a satisfiable Boolean formula ϕ over the variables $\alpha_1, \alpha_2, \ldots, \alpha_k$. It is a "yes"-instance if α_k is true in the lexicographically maximum satisfying assignment of ϕ and "no"-instance otherwise. We note the following straightforward observation, which will be used in the proof of the subsequent theorem.

LEMMA 5.3. *For k-sequences $\boldsymbol{u} = (u_1, u_2, \cdots, u_k)$ and $\boldsymbol{v} = (v_1, v_2, \cdots, v_k)$ where $u_i, v_i \in \{0, 1\}$ for $i = 1, 2, \ldots, k$, $\boldsymbol{u}$ is lexicographically greater than $\boldsymbol{v}$ if and only if $\sum_{u_i=1} 2^{k-i+1} \geq \sum_{v_i=1} 2^{k-i+1} + 2$.*

THEOREM 5.4. *Given a graph game (N, v, E) where v is non-negative and (N, E) is a tree, it is Δ^p_2-complete to determine whether an imputation $\boldsymbol{x}$ belongs to the least core, the nucleolus, or the kernel.*

PROOF. Membership in Δ^p_2 was proved in [17, 18]. We reduce from LEASTLEXSAT.

Let ϕ be a satisfiable Boolean formula over the set of variables $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. We assume that the formula ϕ is not satisfied by setting all the variables to true, or setting all the variables to false. We construct a star with center c and leaves a_h for each variable α_h $(h = 1, 2, \ldots, k-1)$; similarly, we construct another star with center $\overline{c}$ and leaves $\overline{a}_h$ for each variable α_h $(h = 1, 2, \ldots, k-1)$. We introduce the player a_k for α_k and attach her to each of the center players. Specifically, we let $N = A \cup \overline{A} \cup \{c, \overline{c}\}$ and $E = \{\, \{c, a\} \mid a \in A \,\} \cup \{\, \{\overline{c}, a\} \mid a \in \overline{A} \,\}$, where $A = \{a_1, a_2, \ldots, a_k\}$ and $\overline{A} = \{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{k-1}, a_k\}$. For a nonempty subset $T \subseteq A$, we denote by $\overline{T}$ the dual of T, i.e., $\overline{T} = (T \cap \{a_k\}) \cup \{\, \overline{a}_h \mid a_h \in A, h \neq k \,\}$. Now, a feasible coalition of this game is either a singleton, a coalition including the three players c, a_k, and $\overline{c}$, a coalition of the form $T \cup \{c\}$ where $T \subseteq A$, or a coalition of the form $\overline{T} \cup \{\overline{c}\}$ where $\overline{T} \subseteq \overline{A}$.

For each $T \subseteq A$, we write $\sigma(T) \models \phi$ if ϕ is satisfied by setting the variables $\{\, \alpha_h \mid a_h \in T \,\}$ to true and the variables $\{\, \alpha_h \mid a_h \in A \setminus T \,\}$ to false, and $\sigma(T) \not\models \phi$ otherwise. The value $v(S)$ for $S \in \mathcal{F}_E$ is given as follows.

- $v(S) = 1$ when $S = N$.
- $v(S) = 0$ when $|S| = 1$.
- $v(S) = 0$ when $\{c, a_k, \overline{c}\} \subseteq S \neq N$.
- $v(S) = \sum_{a_i \in T} 2^{k-i+1}$ when $S = T \cup \{c\}$ or $\overline{T} \cup \{\overline{c}\}$ for some nonempty subset $T \subseteq A$ such that $\sigma(T) \models \phi$.
- $v(S) = 0$ when $S = T \cup \{c\}$ or $\overline{T} \cup \{\overline{c}\}$ for some nonempty subset $T \subseteq A$ such that $\sigma(T) \not\models \phi$.

Clearly, for each $S \in \mathcal{F}_E$, $v(S)$ is non-negative, and can be computed in polynomial time.

Now, let $T^* \in \text{argmax}\{\, v(T \cup \{c\}) \mid T \subseteq A \,\}$ and $\overline{T^*}$ be the dual coalition of T^* (hence, $v(T^* \cup \{c\}) = v(\overline{T^*} \cup \{\overline{c}\})$). By Lemma 5.3 and the definition of v, the truth assignment that sets the variables corresponding to the players in T^* to true and the rest to false is the lexicographically maximum satisfying assignment of ϕ. Thus, α_k evaluates to true in the lexicographically maximum satisfying assignment for ϕ if and only if $a_k \in T^*$. Let $S^* = T^* \cup \{c\}$ and $\overline{S^*} = \overline{T^*} \cup \{\overline{c}\}$. Before we proceed, we give the following lemma.

LEMMA 5.5. *For any $\boldsymbol{x} \in \mathcal{I}(N, v, E)$ and $S \in \mathcal{F}_E$ where $S \neq N, \emptyset, S^*, \overline{S^*}$,*

$$e(\boldsymbol{x}, S^*) > e(\boldsymbol{x}, S) \text{ and } e(\boldsymbol{x}, \overline{S^*}) > e(\boldsymbol{x}, S). \tag{5.1}$$

PROOF. Take any $\boldsymbol{x} \in \mathcal{I}(N, v, E)$ and $S \in \mathcal{F}_E \setminus \{N, \emptyset, S^*, \overline{S^*}\}$. Observe that $x(S) - x(S^*) \geq -1$, since $\boldsymbol{x} \geq \mathbf{0}$ and since $x(N) = v(N) = 1$. We now claim that $v(S^*) - v(S) \geq 2$. This holds when $S = T \cup \{c\}$ or $S = \overline{T} \cup \{\overline{c}\}$ for some $T \subseteq A$ where $T \neq T^*$ by Lemma 5.3. Consider the case when $|S| = 1$ or $\{c, a_k, \overline{c}\} \subseteq S$. By definition of v, $v(S) = 0$. In addition, $v(S^*) \geq 2$ since ϕ is a satisfiable formula not to be satisfied by setting all the variables to false. Hence, $v(S^*) - v(S) \geq 2$. Thus, we have that

$$e(\boldsymbol{x}, S^*) - e(\boldsymbol{x}, S) = (v(S^*) - v(S)) + (x(S) - x(S^*)) \geq 2 - 1 > 0.$$

One can prove $e(\boldsymbol{x}, \overline{S^*}) > e(\boldsymbol{x}, S)$ similarly. Hence, the inequalities (5.1) hold. □

Let $\boldsymbol{x}^* \in \mathbb{R}^N$ be an imputation such that player a_k receives the whole value $v(N) = 1$ and any other player receives nothing, that is, $x^*(a_k) = 1$ and $x^*_b = 0$ for each $b \in N \setminus \{a_k\}$. We will now argue that the following statements hold:

(i) If $a_k \in T^*$, then $\{\boldsymbol{x}^*\} = \mathcal{LC}(N, v, E) = \mathcal{K}(N, v, E)$.

(ii) If $a_k \notin T^*$, then $\boldsymbol{x}^* \notin \mathcal{LC}(N, v, E) \cup \mathcal{K}(N, v, E)$.

(i) : Suppose that $a_k \in T^*$. We will first prove that $\boldsymbol{x}^*$ is the unique imputation that belongs to the kernel of the game (N, v, E). Observe that a_k is the only player that belongs to both S^* and $\overline{S^*}$. It follows from Lemma 5.5 that player a_k has more bargaining power than any other player, i.e., $s_{a_k b}(\boldsymbol{x}) > s_{b a_k}(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathcal{I}(N, v, E)$ and any $b \in N \setminus \{a_k\}$, implying that an imputation $\boldsymbol{x}$ belongs to the kernel of the game (N, v, E) if and only if $x(a_k) = 1$ $x_b = 0$ for any $b \in N \setminus \{a_k\}$. Hence, $\boldsymbol{x}^*$ is the unique imputation that belongs to the kernel of the game (N, v, E).

Next, we claim that $\boldsymbol{x}^*$ is the unique imputation that lies in $\mathcal{LC}(N, v, E)$. Observe that $e(\boldsymbol{x}^*, S^*) = e(\boldsymbol{x}^*, \overline{S^*}) = v(S^*) - 1$, and hence $\max_{S \in \mathcal{F}_E \setminus \{N, \emptyset\}} e(\boldsymbol{x}^*, S) = v(S^*) - 1$ by Lemma 5.5. It suffices to show that for any imputation $\boldsymbol{x} \neq \boldsymbol{x}^*$,

$$\max\{e(\boldsymbol{x}, S^*), e(\boldsymbol{x}, \overline{S^*})\} > v(S^*) - 1. \tag{5.2}$$

Take any $\boldsymbol{x} \in \mathcal{I}(N, v, E)$ such that $\boldsymbol{x} \neq \boldsymbol{x}^*$. If $x(S^*) = 1$ and $x(\overline{S^*}) = 1$, then this would mean that $x(a_k) = 1$, a contradiction. Hence, we have $x(S^*) < 1$ or $x(\overline{S^*}) < 1$, which implies that $e(\boldsymbol{x}, S^*) > v(S^*) - 1$ or $e(\boldsymbol{x}, \overline{S^*}) > v(S^*) - 1$. This gives (5.2).

(ii) : Suppose that $a_k \notin T^*$. Then, player a_k belongs to neither the coalition S^* nor the coalition $\overline{S^*}$, and hence $x^*(S^*) = x^*(\overline{S^*}) = 0$. It follows from Lemma 5.5 that any imputation $\boldsymbol{x}$ such that $x(S^*) = x(\overline{S^*}) > 0$ gives a smaller maximum excess than that of $\boldsymbol{x}^*$. Thus, $\boldsymbol{x}^* \notin \mathcal{LC}(N, v, E)$. By Lemma 5.5, for any $\boldsymbol{x} \in \mathcal{I}(N, v, E)$,

$$s_{c a_k}(\boldsymbol{x}) \geq e(\boldsymbol{x}, S^*) > \max_{S \in \mathcal{F}_E(a_k \setminus c)} e(\boldsymbol{x}, S) = s_{a_k c}(\boldsymbol{x}),$$

because a_k belongs to neither S^* nor $\overline{S^*}$. Thus, if $\boldsymbol{x} \in \mathcal{K}(N, v, E)$, then we must have $x(a_k) = v(\{a_k\}) = 0$. We conclude that $\boldsymbol{x}^* \notin \mathcal{K}(N, v, E)$.

It follows from (i) and (ii) that the least significant player a_k belongs to T^* if and only if $\boldsymbol{x}^*$ belongs to the least core, the nucleolus, or the kernel of the game (N, v, L).

Similarly to the proof of Corollary 5.2, one can show that it is Δ_2^p-hard to find an imputation of the above three solutions for the class of games on trees.

THEOREM 5.6. *If one can find in polynomial time an element of the least core or the kernel for graph games (N, v, E) where v is non-negative and (N, E) is a tree, then P=Δ_2^p.*

PROOF. We will show that polynomial-time algorithms for our problems can be used to decide LEASTLEXSAT. Let ϕ be a satisfiable Boolean formula over the set of variables $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. We assume that the formula ϕ is not satisfied by setting all the variables to true, or setting all the variables to false.

Introduce one player a_k for α_k; introduce two players a_h and $\overline{a}_h$ for each of other variables α_h $(h = 1, 2, \ldots, k-1)$; finally, introduce two other players c and $\overline{c}$. We construct the same graph game (N, v, E) as defined in the proof for Theorem 5.4. Again, let $T^* \in \text{argmax}\{\, v(T \cup \{c\}) \mid T \subseteq \{a_1, a_2, \ldots, a_k\} \,\}$. Recall that the truth assignment that sets the variables corresponding to T^* to true and the rest to false is the lexicographically maximum satisfying assignment of ϕ. Hence, the lexicographically least significant variable α_k is true in the lexicographically maximum satisfying assignment of ϕ if and only if $a_k \in T^*$. Let $\boldsymbol{x}^* \in \mathbb{R}^N$ be an imputation such that $x^*(a_k) = 1$ and $x^*_b = 0$ for each $b \in N \setminus \{a_k\}$. Take any $\boldsymbol{x} \in \mathcal{LC}(N, v, E) \cup \mathcal{K}(N, v, E)$. By the proof for Theorem 5.4, it holds that the least significant player $a_k \in T^*$ if and only if $\boldsymbol{x} = \boldsymbol{x}^*$.

It follows that, by finding an imputation $\boldsymbol{x}$ in the least core or the kernel of the game (N, v, E), and checking whether $\boldsymbol{x} = \boldsymbol{x}^*$, we can decide whether the lexicographically least significant variable α_k is true in the lexicographically maximum satisfying assignment of ϕ. □

Core

In contrast to the least core, the nucleolus, and the kernel, constructing a core-imputation turns out to be easy for cohesive games on a star.

THEOREM 5.7. *One can find an element of the core for cohesive graph games (N, v, E) whose underlying graph (N, E) is a star in time polynomial in $|N|$.*

PROOF. Let c be the center player of the star, and $\boldsymbol{x} \in \mathbb{R}^N$ be a vector such that $x_a = v(\{a\})$ for each $a \in N \setminus \{c\}$ and $x_c = v(N) - \sum_{a \in N \setminus \{c\}} v(\{a\})$. Clearly, $x(N) = v(N)$ and $x_a \geq v(\{a\})$ for any $a \in N \setminus \{c\}$. Moreover, for any $S \in \mathcal{F}_E$ where $c \in S$,

$$\begin{aligned} x(S) &= (v(N) - \sum_{a \in N \setminus \{c\}} v(\{a\})) + \sum_{a \in S \setminus \{c\}} v(\{a\}) \\ &= v(N) - \sum_{a \in N \setminus S} v(\{a\}) \geq v(S). \end{aligned}$$

The last inequality holds since v is cohesive. Hence, $\boldsymbol{x}$ is immune to any feasible coalitional deviations involving c, concluding that $\boldsymbol{x} \in \mathcal{C}(N, v, E)$. Clearly, the imputation $\boldsymbol{x}$ can be constructed in time polynomial in the number of players. □

However, deciding the non-emptiness of the core turns out to be co-NP-complete for general games on trees. To this end, we provide a necessary and sufficient condition for a game on a tree to have a nonempty core: the core of such games is nonempty if and only if the game is cohesive.

LEMMA 5.8. *For graph games (N, v, E) whose underlying graph (N, E) is a tree, the core is nonempty if and only if $v : \mathcal{F}_E \to \mathbb{R}$ is cohesive.*

PROOF. Suppose that $\mathcal{C}(N, v, E) \neq \emptyset$ but v is not cohesive. Let $\boldsymbol{x} \in \mathcal{C}(N, v, E)$ and $\{X_i\}_{i \in I}$ be a $\mathcal{F}_E$-partition of N such that

$\hat{v}(N) = \sum_{i\in I} v(X_i)$. Since v is not cohesive, $\sum_{i\in I} x(X_i) = v(N) < \sum_{i\in I} v(X_i)$. Hence, there exists X_i such that $x(X_i) < v(X_i)$, contradicting the fact that $\boldsymbol{x} \in \mathcal{C}(N, v, E)$.

Conversely, suppose that v is cohesive. Let $v' : \mathcal{F}_E \to \mathbb{R}$ be the restriction of $\hat{v}$ to $\mathcal{F}_E$, i.e., $v'(S) = \hat{v}(S)$ for each $S \in \mathcal{F}_E$. Notice that $v'(N) = v(N)$ by supposition. Further, it can be easily verified that v' is superadditive. To see this, take any $S, T \in \mathcal{F}_E$ where $S \cap T = \emptyset$. Let $\{X_i\}_{i\in I}$ and $\{Y_j\}_{j\in J}$ be $\mathcal{F}_E$-partitions of S and T such that $v'(S) = \sum_{i\in I} v(X_i)$ and $v'(T) = \sum_{j\in J} v(Y_j)$, respectively. Then, the coalitions $\{X_i\}_{i\in I} \cup \{Y_j\}_{j\in J}$ form an $\mathcal{F}_E$-partition of $S \cup T$, and hence

$$v'(S) + v'(T) = \sum_{i\in I} v(X_i) + \sum_{j\in J} v(Y_j) \le v'(S \cup T).$$

Then, $\mathcal{C}(N, v', E) \neq \emptyset$, because (N, v', E) is a superadditive game on a tree [29, 6, 7]. We will now argue that $\mathcal{C}(N, v, E) = \mathcal{C}(N, v', E)$, which implies the desired claim. By definition of v', it is clear that $\mathcal{C}(N, v, E) \supseteq \mathcal{C}(N, v', E)$. We now show that the opposite also holds. Take any $\boldsymbol{x} \in \mathcal{C}(N, v, E)$. Clearly, $x(N) = v(N) = v'(N)$. Let $S \in \mathcal{F}_E$. Now, for any $\mathcal{F}_E$-partition $\{X_i\}_{i\in I}$ of S, we have $x(S) = \sum_{i\in I} x(X_i) \ge \sum_{i\in I} v(X_i)$, where the second inequality follows from the fact that $\boldsymbol{x} \in \mathcal{C}(N, v, E)$. Thus, $x(S) \ge \hat{v}(S) = v'(S)$. We conclude that $\boldsymbol{x} \in \mathcal{C}(N, v', E)$. □

In the following theorem, we will essentially prove that it is hard to determine whether a given game is cohesive.

THEOREM 5.9. *Given a graph game (N, v, E) where v is non-negative and (N, E) is a star, it is co-NP-complete to determine whether the core is non-empty.*

PROOF. By Lemma 5.8, the core of a game on a tree is empty if and only if its characteristic function is not cohesive. Thus, the problem is in co-NP: we can guess a partition and calculate its sum.

Again, we will prove the hardness via a reduction from SAT. Given a Boolean formula ϕ over the set of variables $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, which we assume not to be satisfied by setting all the variables to true, we construct a star with the center and leaves a_h for each variable α_h $(h = 1, 2, \ldots, k)$; specifically, we set $N = \{c\} \cup A$ where $A = \{a_1, a_2, \ldots, a_k\}$ and $E = \{\, \{c, a_h\} \mid a_h \in A \,\}$.

For each $T \subseteq A$, we write $\sigma(T) \models \phi$ if ϕ is satisfied by setting the variables $\{\, \alpha_h \mid a_h \in T \,\}$ to true and the variables $\{\, \alpha_h \mid a_h \in A \setminus T \,\}$ to false, and $\sigma(T) \not\models \phi$ otherwise. The value $v(S)$ for $S \in \mathcal{F}_E$ is given as follows.

- $v(S) = 1$ when $S = N$.
- $v(S) = 0$ when $|S| = 1$.
- $v(S) = 2$ when $S = T \cup \{c\}$ for a nonempty proper subset $T \subsetneq A$ such that $\sigma(T) \models \phi$.
- $v(S) = 0$ when $S = T \cup \{c\}$ for a nonempty proper subset $T \subsetneq A$ such that $\sigma(T) \not\models \phi$.

Notice that for each $S \in \mathcal{F}_E$, $v(S)$ is non-negative, and can be computed in polynomial time.

We will now argue that ϕ is unsatisfiable if and only if v is cohesive. Indeed, if ϕ is unsatisfiable, it holds that $v(S) = 0$, for all $S \in \mathcal{F}_E \setminus \{N\}$ by construction of v, and hence v is cohesive. Conversely, suppose that ϕ is satisfiable. Then, $v(T^* \cup \{c\}) = 2$ for some nonempty proper subset T^* of A. This implies that $\hat{v}(N) \ge v(T^* \cup \{c\}) + \sum_{j\in A\setminus T^*} v(\{j\}) = 2 > v(N)$, and thereby v is not cohesive. Combining this with Lemma 5.8, ϕ is unsatisfiable if and only if $\mathcal{C}(N, v, E)$ is nonempty. □

COROLLARY 5.10. *If one can find in polynomial time an element of the core for graph games (N, v, E) where $v(S) \ge 0$ for each $S \in \mathcal{F}_E$ and (N, E) is a star, then P=NP.*

We remark that Chalkiadakis et al. [5] define the core as the set of pairs of coalition structures and associated stable payoffs for each coalition. If the underlying graph is a tree, such core is always non-empty [7].

5.1 Games on complete graphs

Not surprisingly, most of the hardness results in the previous subsections hold for games on complete graphs: given a graph game where the characteristic fuction is non-negative, we can build in polynomial time a game with full communication structure whose associated core, least core, nucleolus, and kernel coincide with those of the original game. For a graph game (N, v, E), we define its *Demange game* [9] as the game with the full communication structure (N, v_0), where $v_0 : 2^N \to \mathbb{R}$ is a characteristic function given by $v_0(S) = v(S)$ if $S \in \mathcal{F}_E$ and $v_0(S) = 0$ otherwise.

THEOREM 5.11. *Given a graph game (N, v, E) where v is non-negative, the core, the least core, the nucleolus, and the kernel of the Demange game (N, v_0) coincide with those of the original game. Moreover, if v is cohesive, then v_0 is also cohesive.*

PROOF. Elkind [9] showed the above relationships for the core, the least core, and the nucleolus; hence, it suffices to prove that $\mathcal{K}(N, v_0) = \mathcal{K}(N, v, E)$. Observe first that $\mathcal{I}(N, v_0) = \mathcal{I}(N, v, E)$ since $v_0(\{i\}) = v(\{i\})$ for any $i \in N$, and $v_0(N) = v(N)$. Moreover, it is clear that the surpluses associated with (N, v_0) remain the same as those of the original game for each pair of distinct players and any imputations. Thus, $\mathcal{K}(N, v_0) = \mathcal{K}(N, v, E)$. Clearly, whenever $v(N) = \hat{v}(N)$, we have $\hat{v}_0(N) = v_0(N)$, which implies that v_0 is cohesive. Notice that the Demange game (N, v_0) can be constructed in polynomial time. □

6. CONCLUSION AND FUTURE WORK

This paper has explored several computational questions for cooperative transferable utility games constrained by graph structure. We defined a relaxed form of supermodularity in the graph-restricted settings and showed that such restriction is almost necessary and sufficient to obtain polynomial solvability results for games on cycle-complete graphs. Although we have shown that the truncation technique may not apply to non-cycle-comple graphs, it would be interesting to see whether similar tractability results hold for quasi-supermodular games on general graphs. Also, it remains unknown whether the least core, the nucleolus and the kernel for superadditive games on trees can be computed in polynomial time. We conjecture that superadditivity does not help to decrease the complexity of computing solutions even in graph-restricted settings.

There are other solution concepts we have not addressed in this paper, such as the *average tree solution* [19, 20] and the *average covering tree value* [25], which are relatively unexplored in the literature. Notably, the average tree solution can be computed in polynomial time if the graph is a tree; however, the computation becomes hard for games on complete graphs since in such cases, it coincides with the Shaple value. It would be interesting to see whether the positive result extends to almost acyclic graphs; in particular, the graphs with bounded-treewidth are promising.

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