Monotonicity Axioms in Approval-based Multi-winner Voting Rules

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ABSTRACT

In this paper we study several monotonicity axioms in approval-based multi-winner voting rules. We consider monotonicity with respect to the support received by the winners and also monotonicity in the size of the committee. Monotonicity with respect to the support is studied when the set of voters does not change and when new voters enter the election. For each of these two cases we consider a strong and a weak version of the axiom. We observe certain incompatibilities between the monotonicity axioms and well-known representation axioms (extended/proportional justified representation) for the voting rules that we analyze, and provide formal proofs of incompatibility between some monotonicity axioms and perfect representation.

KEYWORDS

Multi-winner voting rules; approval ballots; monotonicity axioms

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1 INTRODUCTION

There are many situations in which it is necessary to aggregate the preferences of a group of agents to select a finite set of alternatives. Typical examples are the election of representatives in indirect democracy, shortlisting candidates for a position [6, 12], selection by a company of the group of products that it is going to offer to its customers [19], selection of the web pages that should be shown to a user in response to a given query [11, 32], peer grading in Massive Open Online Courses (MOOCs) [8] or recommender systems [12, 22]. The typical mechanism for such preference aggregations is the use of multi-winner voting rules.

The use of axioms for analyzing voting rules is well established in social choice and dates back to the work of Arrow [1]. However, multi-winner voting rules have not been studied much so far from an axiomatic perspective. In particular, we can cite the work of Dummet [10], Elkind et al. [12], Faliszewski et al. [14], and Woodall [34] for multi-winner elections that use ranked ballots. For approval-based multi-winner elections the concept of representation has been recently axiomatized by Aziz et al. [2], who proposed two axioms called justified representation and extended justified representation, and Sánchez-Fernández et al. [29], who proposed a weakening of extended justified representation that they called proportional justified representation.

In this paper we complement these previous works with the study of monotonicity axioms for approval-based multi-winner voting rules. First of all, we consider monotonicity in the support received by the winners. Informally, the idea of monotonicity in the support is that if a subset of the winners in an election sees their support increased and the support of all the other candidates remains the same, then it seems reasonable that such candidates should remain in the set of winners. Monotonicity with respect to the support is studied when the set of voters does not change and when new voters enter the election. Our first contribution is to propose an axiom for each of these two cases and, for each of these two axioms, to define a strong and a weak version of the axiom. We also consider monotonicity in the size of the committee, although in this case we will reuse an axiom that has already been proposed by Elkind et al. [12].

Following the work of Elkind et al. [12] and Faliszewski et al. [13] we will discuss the relevance of these axioms in three different types of scenarios:

- **Excellence**. The goal is to select the best k candidates for a given purpose. It is supposed that in a second step (out of the scope of the multi-winner election) one of the selected candidates is finally selected. Examples of this type of elections are choosing the finalists of a competition or shortlisting of candidates for a position.
- **Diversity**. In this case the goal is that as many voters as possible have one of their preferred candidates in the committee. Several examples of this type are discussed by Elkind et al. [12] and Faliszewski et al. [13]. One such example is to select the set of movies that are going to be offered to the passengers during an air flight (the airline company is interested in that all passengers find something that they like).
- **Proportional representation**. In this case the goal is to select a committee that represents as precisely as possible the opinions of the society. The typical example of this scenario are parliamentary elections.

Then, we analyze several well-known voting rules with these axioms. We observe certain incompatibilities between the monotonicity axioms and extended/proportional justified representation for the voting rules that we analyze and provide formal proofs of incompatibility between some of these axioms and perfect representation (another axiom proposed by Sánchez-Fernández et al. [29]). At the end of this paper we review briefly some previous works

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that study monotonicity axioms in approval-based multi-winner elections, draw some conclusions and outline some lines of continuation of this work.

2 PRELIMINARIES

We consider elections in which a fixed number k of candidates or alternatives must be chosen from a set of candidates C. We assume that $|C| \geq k \geq 1$. The set of voters is represented as $N = \{1, \ldots, n\}$. Each voter i that participates in the election casts a ballot A_i that consists of the subset of the candidates that the voter approves of (that is, $A_i \subseteq C$). We refer to the ballots cast by the voters that participate in the election as the ballot profile $\mathcal{A} = (A_1, \ldots, A_n)$. An approval-based multi-winner election $\mathcal{E}$ is therefore represented by $\mathcal{E} = (N, C, \mathcal{A}, k)$. The set of voters N and the set of candidates C will be omitted when they are clear from the context.

Given a voting rule R, for each election $\mathcal{E} = (\mathcal{A}, k)$, we say that $R(\mathcal{E})$ is the output of the voting rule R for such election. Ties may happen in the voting rules that we are going to consider. To take this into account, given an election $\mathcal{E}$ and a voting rule R we say that the value of $R(\mathcal{E})$ is the set of size at least one composed of all the possible sets of winners outputted by rule R for election $\mathcal{E}$. We say that a candidates subset W of size k is a set of winners for election $\mathcal{E}$ and rule R if W belongs to $R(\mathcal{E})$. We stress that our results are to a large extent independent of how ties are broken.

Given an election $\mathcal{E} = (N, C, \mathcal{A}, k)$ and a non-empty candidates subset G of C, we define $\mathcal{E}_{\Delta G}$, as the election obtained by adding to election $\mathcal{E}$ one voter that approves of only the candidates in G. That is, $\mathcal{E}_{\Delta G} = (N_{\Delta G} = \{1, \ldots, n, n+1\}, C, \mathcal{A}_{\Delta G} = (A_1, \ldots, A_n, G), k)$. Given a non-empty candidates subset G and a voter $i \in N$ such that she does not approve of any of the candidates in G we define $\mathcal{E}_{i+G}$, as the election obtained if voter i decides to approve of all the candidates in G in addition to the candidates in A_i. That is, $\mathcal{E}_{i+G} = (N, C, \mathcal{A}_{i+G} = (A_1, \ldots, A_{i-1}, A_i \cup G, A_{i+1}, \ldots, A_n), k)$.

We recall now the notions of justified representation and extended justified representation due to Aziz et al. [2], and of proportional justified representation due to Sánchez-Fernández et al. [29].

Definition 2.1. Consider an election $\mathcal{E} = (N, C, \mathcal{A}, k)$. Given a positive integer $\ell \in \{1, \ldots, k\}$, we say that a set of voters $N^* \subseteq N$ is *ℓ-cohesive* if $|N^*| \geq \ell \frac{n}{k}$ and $|\bigcap_{i \in N^*} A_i| \geq \ell$. We say that a set of candidates W, $|W| = k$, provides *justified representation* (JR) for $\mathcal{E}$ if for every 1-cohesive set of voters $N^* \subseteq N$ it holds that there exists a voter i in N^* such that $A_i \cap W \neq \emptyset$. We say that a set of candidates W, $|W| = k$, provides *extended justified representation* (EJR) (respectively, *proportional justified representation* (PJR)) for $\mathcal{E}$ if for every $\ell \in \{1, \ldots, k\}$ and every ℓ-cohesive set of voters $N^* \subseteq N$ it holds that there exists a voter i in N^* such that $|A_i \cap W| \geq \ell$ (respectively, $|W \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$). We say that an approval-based voting rule satisfies *justified representation* (JR), *extended justified representation* (EJR), or *proportional justified representation* (PJR) if for every election $\mathcal{E} = (N, C, \mathcal{A}, k)$ it outputs a committee that provides JR, EJR, or PJR, respectively, for $\mathcal{E}$.

Aziz et al. [2], and Sánchez-Fernández et al. [29] prove that EJR implies PJR and that PJR implies JR, both for rules and for committees.

Below we introduce the voting rules that we are going to consider in this study. First of all, we present the following voting rules, surveyed by Kilgour [16].

Approval Voting (AV). Under AV, the winners are the k candidates that receive the largest number of votes. Formally, for each approval-based multi-winner election $(\mathcal{A}, k)$, the approval score of a candidate c is $|\{i : c \in A_i\}|$. The k candidates with the highest approval scores are chosen.

Satisfaction Approval Voting (SAV). A voter's *satisfaction score* is the fraction of her approved candidates that are elected. SAV maximizes the sum of the voters' satisfaction scores. Formally, for each approval-based multi-winner election $\mathcal{E} = (\mathcal{A}, k)$:

$$\mathrm{SAV}(\mathcal{E}) = \operatorname*{argmax}_{W \subseteq C : |W| = k} \sum_{i \in N} \frac{|A_i \cap W|}{|A_i|}. \quad (1)$$

Since we are interested in the compatibility between representation axioms and monotonicity axioms we are also going to study several rules that satisfy some of the above mentioned representation axioms.

Chamberlin and Courant rule and Monroe rule. The voting rules proposed by Chamberlin and Courant [9] and Monroe [20] select sets of winners that minimize the misrepresentation of the voters (the number of voters represented by a candidate that they do not approve of). The difference between the rule of Chamberlin and Courant (CC) and the rule of Monroe is that in CC each candidate may represent an arbitrary number of voters while in the Monroe rule each candidate must represent at least $\lfloor \frac{n}{k} \rfloor$ and at most $\lceil \frac{n}{k} \rceil$ voters. For each approval-based multi-winner election $\mathcal{E} = (\mathcal{A}, k)$:

$$\mathrm{CC}(\mathcal{E}) = \operatorname*{argmin}_{W \subseteq C : |W| = k} |\{i : A_i \cap W = \emptyset\}|. \quad (2)$$

Given an election $\mathcal{E} = (\mathcal{A}, k)$ and a candidates subset W of size k let $M_{N,W}$ be the set of all mappings $\pi : N \rightarrow W$ such that for each candidate c in W it holds that $\lfloor \frac{n}{k} \rfloor \leq |\{i : \pi(i) = c\}| \leq \lceil \frac{n}{k} \rceil$. Then,

$$\mathrm{Monroe}(\mathcal{E}) = \operatorname*{argmin}_{W \subseteq C : |W| = k} \min_{\pi \in M_{N,W}} |\{i : \pi(i) \notin A_i\}|. \quad (3)$$

Proportional Approval Voting (PAV) was proposed by the Danish mathematician Thiele [33] in the late 19th century. Given an election $\mathcal{E} = (\mathcal{A}, k)$ and a candidates subset W of size k, the PAV-score of a voter i is 0 if such voter does not approve of any of the candidates in W and $\sum_{j=1}^{|A_i \cap W|} \frac{1}{j}$ if the voter approves of some of the candidates in W. PAV selects the sets of winners that maximize the sum of the PAV-scores of the voters.

$$\mathrm{PAV}(\mathcal{E}) = \operatorname*{argmax}_{W \subseteq C : |W| = k} \sum_{i : A_i \cap W \neq \emptyset} \sum_{j=1}^{|A_i \cap W|} \frac{1}{j}. \quad (4)$$

Phragmén rules Phragmén rules were proposed by the Swedish mathematician Phragmén [24–27] in the late 19th century. In this paper we will focus mainly in one of these rules that we will refer to as max-Phragmén. We refer to the survey by Janson [15] for an extensive discussion of Phragmén rules.

Phragmén voting rules are based on the concept of *load*. Each candidate in the set of winners incurs in one unit of load, that should be distributed among the voters that approve of such candidate. The goal is to choose the set of winners such that the total load is distributed as evenly as possible between the voters.

Formally, given an election $\mathcal{E} = (\mathcal{A}, k)$ and a candidates subset $W \subseteq C$, $|W| = k$, a load distribution is a two dimensional array $\mathbf{x} = (x_{i,c})_{i \in N, c \in W}$, that satisfies the following three conditions:

$$0 \leq x_{i,c} \leq 1 \qquad \text{for all } i \in N \text{ and } c \in W, \tag{5}$$

$$x_{i,c} = 0 \qquad \text{if } c \notin A_i, \text{ and} \tag{6}$$

$$\sum_{i \in N} x_{i,c} = 1 \qquad \text{for all } c \in W. \tag{7}$$

Given a load distribution $\mathbf{x}$, the load of each voter i is defined as $x_i = \sum_{c \in W} x_{i,c}$. Then, given an election $\mathcal{E}$, the rule max-Phragmén outputs the set of winners W that minimizes the maximum voter load.

3 SUPPORT MONOTONICITY

Some previous work (see Section 6) make use of the following idea of support monotonicity: if a candidate that was already in the set of winners is added to the ballot of some voter (without changing anything else in the election), then such candidate must still belong to the set of winners. We will refer to this axiom as *candidate monotonicity*.

Definition 3.1. We say that a rule R satisfies *candidate monotonicity* if for each election $\mathcal{E} = (N, C, \mathcal{A}, k)$, for each candidate $c \in C$, and for each voter i that does not approve of c, the following conditions hold: (i) if c belongs to some winning committee in $R(\mathcal{E})$, then c must also belong to some winning committee in $R(\mathcal{E}_{i+\{c\}})$; and (ii) if c belongs to all winning committees in $R(\mathcal{E})$, then c must also belong to all winning committees in $R(\mathcal{E}_{i+\{c\}})$.

Candidate monotonicity can be seen as the equivalent of the axiom with the same name proposed by Elkind et al. [12] for ranked ballots[1]. They require that, if a winning candidate c is moved forward in some vote, then c must still belong to some winning committee.

Elkind et al. [12] justified this axiom with the following idea:

> "If c belongs to a winning committee W then, generally speaking, we cannot expect W to remain winning when c is moved forward in some vote, as this shift may hurt other members of W."

In this paper we propose to extend the notion of candidate monotonicity for approval-based multi-winner voting rules in several directions. First of all, we study what happens when a subset G of the candidates that was already in the set of winners W is added to the ballot of some voter. Following the idea of Elkind et al. [12] that we have quoted before, we believe that we cannot expect W to remain winning, but we can expect that all the candidates in G (strong version) or, at least, some of the candidates in G (weak version) remain winning.

Secondly, we consider monotonicity when a new voter enters the election and approves of a subset of the candidates that were already in the set of winners. Again, we define a strong and a weak version of this axiom. A similar idea of monotonicity when new voters enter the election has been proposed by Woodall [34] for ranked ballots.

Definition 3.2. We say that a rule R satisfies *strong support monotonicity with population increase* (respectively, *weak support monotonicity with population increase*) if for each election $\mathcal{E} = (N, C, \mathcal{A}, k)$, and for each non-empty subset G of C, such that $|G| \leq k$, the following conditions hold: (i) if $G \subseteq W$ for some $W \in R(\mathcal{E})$, then $G \subseteq W'$ for some $W' \in R(\mathcal{E}_{\Delta G})$ (respectively, $G \cap W' \neq \emptyset$ for some $W' \in R(\mathcal{E}_{\Delta G})$); and (ii) if $G \subseteq W$ for all $W \in R(\mathcal{E})$, then $G \subseteq W'$ for all $W' \in R(\mathcal{E}_{\Delta G})$ (respectively, $G \cap W' \neq \emptyset$ for all $W' \in R(\mathcal{E}_{\Delta G})$).

We say that a rule R satisfies *strong support monotonicity without population increase* (respectively, *weak support monotonicity without population increase*) if for each election $\mathcal{E} = (N, C, \mathcal{A}, k)$, for each non-empty subset G of C, such that $|G| \leq k$, and for each voter i such that $A_i \cap G = \emptyset$, the following conditions hold: (i) if $G \subseteq W$ for some $W \in R(\mathcal{E})$, then $G \subseteq W'$ for some $W' \in R(\mathcal{E}_{i+G})$ (respectively, $G \cap W' \neq \emptyset$ for some $W' \in R(\mathcal{E}_{i+G})$); and (ii) if $G \subseteq W$ for all $W \in R(\mathcal{E})$, then $G \subseteq W'$ for all $W' \in R(\mathcal{E}_{i+G})$ (respectively, $G \cap W' \neq \emptyset$ for all $W' \in R(\mathcal{E}_{i+G})$).

We believe that it is important to know what happens when the support of several of the candidates in the set of winners is incremented simultaneously. Moreover, our results show that for each of the rules that we consider that satisfies any of the support monotonicity axioms (with or without population increase) for $|G| = 1$, such rule also satisfies the corresponding weak support monotonicity axiom (for all values of $|G|$), which is slightly stronger, and therefore provides more information about the behaviour of the rule. Because of this, we do not study candidate monotonicity in this paper. We note, however, that we have been able to build (weird) rules that satisfy support monotonicity with or without population increase for $|G| = 1$ but fail the corresponding weak axiom (examples can be found in the full version of this paper [30]).

We now discuss briefly the relevance of these axioms for the three types of scenarios considered in the Introduction. First of all we note that it is a general property of elections to desire to select winners that receive a high support, and therefore we believe that our weak axioms are generally desirable.

In the case of excellence, we believe that the strong axioms are highly preferable to the weak ones. Since we are looking for the best candidates, adding support to a subset of the candidates that were already considered to be among the best should make all of them stay in the set of winners.

In contrast, in the case of diversity we believe that satisfying the strong axioms is not important for the rule used in the election. We recall that the goal of the election is that every voter has one of her preferred candidates in the set of winners. If the support of a subset G of the winners was increased and at least one of them remained in the set of winners, then the voters that approve of the candidates in G would be satisfied. Removing some of the candidates in G from the set of winners may allow to add other candidates approved by other voters that did not have previously any of their approved candidates in the set of winners. Therefore, for an election of the diversity type, we believe that it would be enough if the rule satisfies the weak axioms.

[1] Elkind et al. [12] also proposed another axiom called non-crossing monotonicity that will not be considered in this paper.

Sánchez-Fernández et al. [29] distinguish two types of proportional representation. In the first type of proportional representation the aim is that each voter is represented by a candidate that she approves of and that each candidate represents the same number of voters. The typical example of this type of scenario are parliamentary elections. As in the case of diversity, in this type of scenario we believe that it is enough to satisfy the weak axioms. Regarding the second type of proportional representation considered by Sánchez-Fernández et al. [29], the goal is that, for each ℓ-cohesive group of voters (see Definition 2.1), as most voters of the group as possible approve of at least ℓ of the candidates in the set of winners. Sánchez-Fernández et al. [29] present as an example of this type of elections the selection of researchers invited to give a seminar in an academic department. We believe that the situation in this case is less clear. It seems that the weak axioms are not enough for this situation because a voter may not be satisfied with having only one of her preferred candidates in the set of winners. However, the strong axioms are maybe too strong if a voter that belongs to an ℓ-cohesive group of voters decides to approve of a subset of the set of winners G of size greater than ℓ.

From now on, we will refer to support monotonicity with population increase as SMWPI and to support monotonicity without population increase as SMWOPI. Table 1 summarizes the results we have obtained in this paper. With respect to the support monotonicity axioms (columns entitled "SMWPI" and "SMWOPI") we use the keys "Str." when the rule satisfies the strong version of the axiom, "Wk." when the rule satisfies the weak version of the axiom and "No" when the rule does not satisfy any of them. The column entitled "Com. Mon." contains the results related to committee monotonicity, which is discussed in Section 4.

For completeness, we also include previous results related to the computational complexity of the rules and the representation axioms that they satisfy, including pointers to the appropriate references. The column entitled "JR/PJR/EJR" shows for each rule the strongest of these axioms satisfied by the rule. The next column says which rules satisfy the perfect representation axiom (PR), that will be discussed in Section 5.

An important type of rules in approval-based multi-winner elections are *approval-based multi-winner counting rules*, which, as discussed by Lackner and Skowron [17, 18], can be seen as analogous to the class of committee scoring rules introduced by Elkind et al. [12] for ranked-based multi-winner elections.

Definition 3.3. A counting function $f : \{1, \dots, k\} \times \{1, \dots, |C|\} \to \mathbb{R}$ is a function that satisfies that $f(x, y) \geq f(x', y)$ whenever $x > x'$. Intuitively, a counting function f defines the score $f(x, y)$ that a certain counting rule r_f assigns to a voter i that approves of x candidates in the set of winners W and y candidates in total. Given a counting function f, and an election $\mathcal{E} = (\mathcal{A}, k)$, the total score of a candidates subset W for counting function f is

$$s_f(W, \mathcal{E}) = \sum_{i \in N} f(|A_i \cap W|, |A_i|),$$

and the counting rule r_f associated to counting function f is defined as follows:

$$r_f(\mathcal{E}) = \underset{W \subseteq C : |W| = k}{\operatorname{argmax}} \; s_f(W, \mathcal{E}).$$

As discussed by Lackner and Skowron [17, 18] several of the voting rules that we have presented in the previous section are counting rules. In particular, we have $f_{\mathrm{AV}}(x, y) = x$ for AV, $f_{\mathrm{SAV}}(x, y) = \frac{x}{y}$ for SAV, $f_{\mathrm{CC}}(x, y) = 1$ if $x > 0$ and $f_{\mathrm{CC}}(0, y) = 0$ for CC, and $f_{\mathrm{PAV}}(x, y) = \sum_{j=1}^{x} \frac{1}{j}$ if $x > 0$ and $f_{\mathrm{PAV}}(0, y) = 0$ for PAV.

For counting rules we have the following results with respect to support monotonicity.

Theorem 3.4. *Every counting rule satisfies strong SMWPI.*

Proof. Consider an election $\mathcal{E} = (\mathcal{A}, k)$, a counting function f and its associated rule r_f, a set of winners W outputted by r_f for election $\mathcal{E}$ and a non-empty subset G of W. We are going to prove that W also belongs to $r_f(\mathcal{E}_{\Delta G})$. The theorem follows from that immediately.

Consider any other candidates subset W' of size k. We simply have to observe that the total score of W for election $\mathcal{E}_{\Delta G}$ under rule r_f is $\sum_{i \in N} f(|A_i \cap W|, |A_i|) + f(|G \cap W|, |G|)$, that $\sum_{i \in N} f(|A_i \cap W|, |A_i|) \geq \sum_{i \in N} f(|A_i \cap W'|, |A_i|)$ (because W is a set of winners for rule r_f and election $\mathcal{E}$), and that $f(|G \cap W|, |G|) = f(|G|, |G|) \geq f(|G \cap W'|, |G|)$ (by the definition of counting function). □

We can also prove weak SMWOPI by introducing a slight restriction to the counting functions that is satisfied by all the counting rules that we consider in this paper.

Theorem 3.5. *Consider a counting function f. If f satisfies that $f(x, y) \geq f(x, y')$ whenever $y \leq y'$, and that for each positive integer z it holds that $f(x + z, y + z) \geq f(x, y)$, then its associated rule r_f satisfies weak SMWOPI.*

Proof. Consider an election $\mathcal{E} = (\mathcal{A}, k)$, a counting function f and its associated rule r_f, a set of winners W outputted by r_f for election $\mathcal{E}$, a non-empty subset G of W, and a voter i such that she does not approve of the candidates in G.

We observe first that because A_i and G are disjoint, for each candidates subset W' it holds that $f(|(A_i \cup G) \cap W'|, |A_i \cup G|) = f(|A_i \cap W'| + |G \cap W'|, |A_i| + |G|)$ and that $s_f(W', \mathcal{E}_{i+G}) - s_f(W', \mathcal{E}) = f(|A_i \cap W'| + |G \cap W'|, |A_i| + |G|) - f(|A_i \cap W'|, |A_i|)$.

Suppose that f satisfies that $f(x, y) \geq f(x, y')$ whenever $y \leq y'$, and that for each positive integer z it holds that $f(x + z, y + z) \geq f(x, y)$, and consider any candidates subset W' of size k such that $W' \cap G = \emptyset$. Then,

$$\begin{aligned}
&s_f(W, \mathcal{E}_{i+G}) - s_f(W', \mathcal{E}_{i+G}) = \\
&\quad (s_f(W, \mathcal{E}) + f(|A_i \cap W| + |G \cap W|, |A_i| + |G|) \\
&\quad - f(|A_i \cap W|, |A_i|)) \\
&\quad - (s_f(W', \mathcal{E}) + f(|A_i \cap W'| + |G \cap W'|, |A_i| + |G|) \\
&\quad - f(|A_i \cap W'|, |A_i|)) \\
&\quad = s_f(W, \mathcal{E}) - s_f(W', \mathcal{E}) \\
&\quad + f(|(A_i \cap W| + |G|, |A_i| + |G|) - f(|(A_i \cap W|, |A_i|) \\
&\quad - (f(|A_i \cap W'|, |A_i| + |G|) - f(|A_i \cap W'|, |A_i|)) \geq 0.
\end{aligned}$$

This proves part (i) of the definition of weak SMWOPI. The proof of part (ii) of the definition of weak SMWOPI follows from the fact that if $G \subseteq W$ for all $W \in r_f(\mathcal{E})$, then $s_f(W, \mathcal{E}) - s_f(W', \mathcal{E}) > 0$, and therefore, the inequality in the equation above is strict. □

Rule	Complexity	JR/PJR/EJR	PR	SMWPI	SMWOPI	Com. Mon.
AV	P [a]	No [d]	No[k]	Str.[Thm. 3.4]	Str.[Thm. 3.6]	Yes
SAV	P [a]	No [d]	No[k]	Str.[Thm. 3.4]	Str.[Thm. 3.6]	Yes
CC	NP-comp.[b]	JR [d]	Yes[g, k]	Str.[Thm. 3.4]	Wk.[Thm. 3.5]	No[Ex. 4.2]
Monroe	NP-comp.[b]	JR [d,e]	Yes [h]	No[Ex. 3.10]	Wk.[Thm. 3.9]	No[Ex. 4.2]
PAV	NP-comp.[a]	EJR [d]	No [h]	Str.[Thm. 3.4]	Wk.[Thm. 3.5]	No [j]
max-Phragmén	NP-comp.[c]	PJR [c,f]	Yes [c]	Wk.[i, Thm. 3.12]	Wk.[i, Thm. 3.12]	No [i]

[a] Results by Aziz et al. [4] and Skowron et al. [31].
[b] Results by Procaccia et al. [28].
[c] Results by Brill et al. [7].
[d] Results by Aziz et al. [2].
[e] Monroe satisfies PJR if k divides n [29].
[f] max-Phragmén satisfies PJR when combined with certain tie-breaking rule [7].
[g] CC satisfies PR if ties are broken always in favour of the candidates subsets that provide PR.
[h] Results by Sánchez-Fernández et al. [29].
[i] Results by Janson [15], Mora and Oliver [21], and Phragmén [26].
[j] Results by Thiele [33].
[k] Results by Sánchez-Fernández and Fisteus [30].

Table 1: Properties of approval-based multi-winner voting rules

However, of the counting rules that we consider in this paper only AV and SAV satisfy strong SMWOPI.

Theorem 3.6. *AV and SAV satisfy strong SMWOPI.*

Proof. The counting functions of AV and SAV hold that $f(x, y) = xf(1, y)$. This makes it possible to assign each candidate c a score $s_f(c, \mathcal{E}) = \sum_{i:c\in A_i} f(1, |A_i|)$ irrespective of which other candidates are in the set of winners W so that $s_f(W, \mathcal{E}) = \sum_{c\in W} s_f(c, \mathcal{E})$. Therefore, the winners in AV and SAV are the k candidates with the highest candidate score. It is now enough to observe that each candidate that belongs to G increases her score in election $\mathcal{E}_{i+G}$ in $f(1, |A_i| + |G|)$ with respect to her score in election $\mathcal{E}$, and that the scores of the candidates that are not in G do not increase. □

The following examples prove that PAV and CC fail strong SMWOPI.

Example 3.7. Let $k = 4$ and $C = \{c_1, \dots, c_7\}$. 131 voters cast the following ballots: for $i, j = 1$ to 3, 3 voters approve of $\{c_i, c_{j+4}\}$, 100 voters approve of $\{c_4\}$, 1 voter approves of $\{c_1, c_2\}$, 1 voter approves of $\{c_1, c_2, c_3\}$, and 2 voters approve of $\{c_5, c_6\}$. For this election PAV outputs one set of winners: $\{c_1, c_2, c_3, c_4\}$, with a PAV score of 391/3. However, if the voter that approves of $\{c_1, c_2\}$ decides to approve of $\{c_1, c_2, c_3, c_4\}$, then PAV outputs only $\{c_4, c_5, c_6, c_7\}$, with a PAV score of 131. Intuitively, this example works as follows. First, the 100 voters that approve of $\{c_4\}$ force that c_4 has to be in the set of winners. Second, the first 27 votes force that either $\{c_1, c_2, c_3\}$ or $\{c_5, c_6, c_7\}$ are in the set of winners. The last 4 votes break the tie between $\{c_1, c_2, c_3, c_4\}$ and $\{c_4, c_5, c_6, c_7\}$ in the two cases considered.

Example 3.8. Let $k = 3$ and $C = \{a, b, c, d, e\}$. 13 voters cast the following ballots: 2 voters approve of $\{a, d\}$, 2 voters approve of $\{a, e\}$, 2 voters approve of $\{c, d\}$, 2 voters approve of $\{c, e\}$, 2 voters approve of $\{b\}$, 2 voters approve of $\{a\}$, and 1 voter approves of $\{d\}$. For this election CC outputs one set of winners: $\{a, b, c\}$ (one voter misrepresented). Now, we consider two consecutive increases of support of $\{b, c\}$, where, in each increase one of the voters that approve of $\{a\}$ decides to approve of $\{a, b, c\}$. Then, after the first increase of support of $\{b, c\}$, CC outputs $\{a, b, c\}$ and $\{b, d, e\}$ (one voter misrepresented), and after the second increase of support of $\{b, c\}$ CC outputs only $\{b, d, e\}$ (0 voters misrepresented). Observe that this example proves that CC fails strong SMWOPI even when it is combined with any tie breaking rule, because if the tie breaking rule selects $\{a, b, c\}$ after the first increase of support, then strong SMWOPI is violated in the second increase of support, and if the tie breaking rule selects $\{b, d, e\}$ after the first increase of support, then strong SMWOPI is violated in the first increase of support.

Let us now turn to analyze the remaining voting rules.

Theorem 3.9. *The Monroe rule satisfies weak SMWOPI.*

Proof. Consider an election $\mathcal{E} = (\mathcal{A}, k)$, a set of winners W outputted by Monroe for election $\mathcal{E}$, a non-empty subset G of W, and a voter i that does not approve of any of the candidates in G. Let π_W be a mapping that minimizes the misrepresentation of W for election $\mathcal{E}$. Clearly the misrepresentation of W with mapping π_W for election $\mathcal{E}_{i+G}$ is the same as for election $\mathcal{E}$ if the candidate $\pi_W(i)$ assigned by π_W to voter i does not belong to G and is equal to the misrepresentation of W with mapping π_W for election $\mathcal{E}$ minus one if $\pi_W(i)$ belongs to G. Furthermore, for each candidates set V such that $V \cap G = \emptyset$, and for each mapping π_V of the voters in N to the candidates in V it holds that the candidate $\pi_V(i)$ assigned by π_V to voter i belongs to $A_i \cup G$ if and only if such candidate belongs to A_i and, therefore, the misrepresentation values of V with mapping π_V are the same for election $\mathcal{E}_{i+G}$ and for election $\mathcal{E}$. □

Examples 3.10 and 3.11 prove that Monroe fails weak SMWPI and strong SMWOPI, respectively. As in the case of CC, these examples prove that Monroe fails weak SMWPI and strong SMWOPI even if combined with any tie breaking rule.

Example 3.10. Let $k = 4$ and $C = \{a, b, c, d, e, f, g, h\}$. 33 voters cast the following ballots: 5 voters approve of $\{a, e\}$, 4 voters approve of $\{a, g\}$, 5 voters approve of $\{b, e\}$, 4 voters approve of

$\{b, h\}$, 5 voters approve of $\{c, f\}$, 4 voters approve of $\{c, g\}$, 3 voters approve of $\{d, f\}$, and 3 voters approve of $\{d, h\}$. For this election Monroe outputs only $\{e, f, g, h\}$ (misrepresentation 1 due to one of the voters that approve of e being represented by h). We now consider two consecutive voters that enter the election, such that each of the new voters approves of $\{e\}$. Then, after the first new voter enters the election, Monroe outputs $\{a, b, c, d\}$ and $\{e, f, g, h\}$ (misrepresentation 2) and, after the second new voter enters the election, Monroe outputs only $\{a, b, c, d\}$ (misrepresentation 2: the new voters would be represented by candidate d).

Example 3.11. Let $k = 3$ and $C = \{a, b, c, d, e\}$. 18 voters cast the following ballots: 2 voters approve of $\{a\}$, 2 voters approve of $\{a, d\}$, 2 voters approve of $\{a, e\}$, 4 voters approve of $\{b\}$, 1 voter approves of $\{b, e\}$, 4 voters approve of $\{c, d\}$, and 3 voters approve of $\{c, e\}$. For this election Monroe outputs only $\{a, b, c\}$ (misrepresentation 1 due to one of the voters that approve of c being represented by candidate b). Now, we consider two consecutive increases of support of $\{b, c\}$, where, in each increase one of the voters that approve of $\{a\}$ decides to approve of $\{a, b, c\}$. Then, after the first increase of support of $\{b, c\}$, Monroe outputs $\{a, b, c\}$ and $\{b, d, e\}$ (misrepresentation 1), and after the second increase of support of $\{b, c\}$ Monroe outputs only $\{b, d, e\}$ (misrepresentation 0).

We study now support monotonicity for max-Phragmén. Phragmén [26] proved that max-Phragmén satisfies support monotonicity when $|G| = 1$. That proof could be easily extended to prove that max-Phragmén satisfies weak SMWPI and weak SMWOPI.

Theorem 3.12. *max-Phragmén satisfies weak SMWPI and weak SMWOPI.*

Proof. We first prove weak SMWOPI. Consider an election $\mathcal{E} = (\mathcal{A}, k)$, a set of winners W output by max-Phragmén for election $\mathcal{E}$, a non-empty subset G of W, and a voter i that does not approve of any of the candidates in G. Let $\mathbf{x}^{\text{opt}} = (x^{\text{opt}}_{i',c})_{i' \in N, c \in W}$ be a load distribution that minimizes the maximum voter load for election $\mathcal{E}$ and candidates subset W, and let $m_{\mathcal{E}}$ be the maximum voter load for load distribution $\mathbf{x}^{\text{opt}}$, that is, $m_{\mathcal{E}} = \max_{i' \in N} x^{\text{opt}}_{i'}$.

Observe that $\mathbf{x}^{\text{opt}}$ is a valid, possibly non-optimal, load distribution for election $\mathcal{E}_{i+G}$ and candidates subset W. In particular, for each candidate c that belongs to G, since voter i does not approve of c in election $\mathcal{E}$, it holds that $x^{\text{opt}}_{i,c} = 0$.

Consider now any candidates subset W' of size k such that $W' \cap G = \emptyset$. Observe that for the candidates subset W' the set of valid load distributions for election $\mathcal{E}_{i+G}$ are the same as the set of valid load distributions for election $\mathcal{E}$. In particular, for voter i, the candidates for which $x_{i,c}$ can be greater than 0 are $A_i \cap W'$ both in election $\mathcal{E}$ and in election $\mathcal{E}_{i+G}$. It follows immediately that the minimum maximum voter load for candidates subset W' is the same in elections $\mathcal{E}$ and $\mathcal{E}_{i+G}$.

Since the minimum maximum voter load for the candidates subset W does not increase in election $\mathcal{E}_{i+G}$ with respect to election $\mathcal{E}$ and, for each candidates subset W' such that $W' \cap G = \emptyset$ the minimum maximum voter load for the candidates subset W' is the same in elections $\mathcal{E}_{i+G}$ and $\mathcal{E}$, it follows that W or some candidates subset that contains some of the candidates in G must be output by max-Phragmén for election $\mathcal{E}_{i+G}$. Further, if for all the set of winners W output by max-Phragmén for election $\mathcal{E}$ it holds that $G \subseteq W$, then for each candidates subset W' such that $W' \cap G = \emptyset$ the minimum maximum voter load for the candidates subset W' is strictly greater than $m_{\mathcal{E}}$, and therefore, it cannot be a set of winners for election $\mathcal{E}_{i+G}$.

The proof for weak SMWPI follows from the facts that max-Phragmén satisfies weak SMWOPI, and that for any election $\mathcal{E}$ the sets of winners output by max-Phragmén do not change if we add a voter to the election that does not approve of any candidate. □

However, the following example proves that max-Phragmén fails both strong SMWPI and strong SMWOPI.

Example 3.13. Let $k = 6$ and $C = \{a, b, c_1, \dots, c_5\}$. 18 voters cast the following ballots: 13 voters approve of $\{c_1, \dots, c_5\}$, 2 voters approve of $\{a, b\}$, 2 voters approve of $\{a\}$, and 1 voter approves of $\{b\}$. For this election max-Phragmén outputs only one set of winners: $\{a, c_1, \dots, c_5\}$. The minimum maximum load for this election is achieved as follows: for each voter i that approves of $\{c_1, \dots, c_5\}$ and each candidate c in $\{c_1, \dots, c_5\}$ we have $x_{i,c} = \frac{1}{13}$, and for each voter i' that approves of a we have $x_{i',a} = \frac{1}{4}$. Then, the load of the voters that approve of $\{c_1, \dots, c_5\}$ is $\frac{5}{13}$ and the load of the voters that approve of a is $\frac{1}{4}$. The maximal voter load for this example is therefore $\frac{5}{13}$. Now, if a new voter enters the election and approves of precisely $\{a, c_1, \dots, c_5\}$, then the sets of winners outputted by max-Phragmén consist of $\{a, b\}$ plus 4 candidates from $\{c_1, \dots, c_5\}$. In this case the minimum maximum voter load is achieved by assigning again $x_{i,c} = \frac{1}{13}$ for each voter i that approves of $\{c_1, \dots, c_5\}$ and each candidate c in $\{c_1, \dots, c_5\}$, assigning $x_{i,a} = \frac{1}{3}$ to the new voter and the voters that approve of $\{a\}$, and assigning $x_{i,b} = \frac{1}{3}$ to all the voters that approve of candidate b. This leads to a maximum voter load of $\frac{1}{3}$. Observe that in this case the minimum maximum voter load for the set $\{a, c_1, \dots, c_5\}$ would be obtained by $x_{i,c} = \frac{1}{14}$ for each voter i that approves of $\{c_1, \dots, c_5\}$, and also for the new voter, which leads to a maximum voter load of $\frac{5}{14}$, greater than $\frac{1}{3}$. This example proves that max-Phragmén fails strong support monotonicity with population increase.

To prove that max-Phragmén fails strong support monotonicity without population increase we simply add an additional candidate d to the original election and a voter that approves of $\{d\}$. This does not make any difference and the set of winners will be again $\{a, c_1, \dots, c_5\}$. Now, if this new voter decides to approve of $\{a, c_1, \dots, c_5, d\}$, then the sets of winners outputted by max-Phragmén consist of $\{a, b\}$ plus 4 candidates from $\{c_1, \dots, c_5\}$.

4 COMMITTEE MONOTONICITY

We turn now to discuss briefly committee monotonicity. The following definition, due to Elkind et al. [12], was given in the context of multi-winner voting rules that make use of ranked ballots, but it can also be directly used for approval-based multi-winner voting rules.

Definition 4.1. We say that a voting rule R satisfies committee monotonicity if for every election $\mathcal{E} = (N, C, \mathcal{A}, k)$, with $k \in \{1, \dots, |C| - 1\}$, the following conditions hold:

(1) for each W in $R(\mathcal{E} = (N, C, \mathcal{A}, k))$ there exists a W' in $R(N, C, \mathcal{A}, k + 1)$ such that $W \subseteq W'$, and

(2) for each W in $R(N, C, \mathcal{A}, k+1)$ there exists a W' in $R(\mathcal{E} = (N, C, \mathcal{A}, k))$ such that $W' \subseteq W$.

It is generally believed that committee monotonicity is a desirable axiom for scenarios of type excellence [6, 12, 13].

It is easy to see that committee monotonicity is satisfied by the rules that consist of an iterative algorithm such that at each iteration the candidate that is added to the set of winners does not depend on the target committee size. This holds for AV, and SAV.

Thiele [33] and Mora and Oliver [21] have already proved that PAV and max-Phragmén, respectively, fail committee monotonicity. We give here an example that shows that both CC and Monroe fail committee monotonicity.

Example 4.2. Let $C = \{a, b, c\}$. 10 voters cast the following ballots: 3 voters approve of $\{a, b\}$, 3 voters approve of $\{a, c\}$, 2 voters approve of $\{b\}$ and 2 voters approve of $\{c\}$. For this set of candidates and this ballot profile, for $k = 1$ both CC and Monroe output only $\{a\}$. For $k = 2$, both CC and Monroe output only $\{b, c\}$.

5 COMPATIBILITY OF AXIOMS

In many applications it would be interesting to use voting rules that satisfy both support monotonicity and representation axioms. While all the voting rules that we have analyzed that satisfy PJR (or EJR) also satisfy the weak support monotonicity axioms, the situation changes when we require the strong axioms. In particular, none of the rules analyzed that satisfy PJR also satisfy strong SMWOPI, and only PAV (which has the additional difficulty of being NP-hard to compute) satisfies strong SMWPI. Whether it is possible to develop a voting rule that satisfies strong SMWOPI and PJR at the same time is left open.

In contrast, we can formally prove that perfect representation (PR) is incompatible both with strong SMWPI and with committee monotonicity. We review first the definition of PR due to Sánchez-Fernández et al. [29].

Definition 5.1. **Perfect representation** (PR) Consider a ballot profile $\mathcal{A}$ over a candidate set C, and a target committee size k, $k \leq |C|$, such that k divides n. We say that a set of candidates W, $|W| = k$, provides perfect representation (PR) for $(\mathcal{A}, k)$ if it is possible to partition the set of voters in k pairwise disjoint subsets $N_1, \dots, N_k$ of size $\frac{n}{k}$ each, such that each candidate w in W can be assigned to one (and only one) different subset N_i so that for all pairs (w, N_i) all the voters in N_i approve of their assigned candidate w. We say that an approval-based voting rule satisfies perfect representation (PR) if for every election $(\mathcal{A}, k)$ it does not output any winning set of candidates W that does not provide PR for $(\mathcal{A}, k)$ if at least one set of candidates W' that provides PR for $(\mathcal{A}, k)$ exists.

Theorem 5.2. *No rule can satisfy PR and strong SMWPI at the same time.*

Proof. Consider the following election. Let $k = 3$ and $C = \{c_1, \dots, c_5\}$. 12 voters cast the following ballots: 2 voters approve of $\{c_1, c_4\}$, 2 voters approve of $\{c_1, c_5\}$, 3 voters approve of $\{c_2, c_4\}$, one voter approves of $\{c_2, c_5\}$, 2 voters approve of $\{c_3, c_5\}$, and 2 voters approve of $\{c_3\}$. For this election any voting rule that satisfies PR has to output $\{c_1, c_2, c_3\}$. Now, suppose that 3 new voters enter the election, and that all these new voters approve of $\{c_1, c_3\}$. For this extended election a voting rule that satisfies PR has to output only $\{c_3, c_4, c_5\}$. □

There is an apparent contradiction between this theorem and Table 1 because Table 1 says that CC satisfies both PR and strong SMWPI. The reason for this apparent contradiction is that, as explained in Footnote g, CC satisfies PR **only** if ties are broken in favour of the sets of candidates that provide PR. The example of Theorem 5.2 illustrates this. For the initial election CC outputs $\{c_1, c_2, c_3\}$ and $\{c_3, c_4, c_5\}$. However, if ties are broken in favour of the sets of candidates that provide PR, then CC (with this tie-breaking rule) will output only $\{c_1, c_2, c_3\}$. Now, after adding 3 new voters that approve of $\{c_1, c_3\}$, strong SMWPI requires that both c_1 and c_3 are in the set of winners while PR requires that the set of winners is $\{c_3, c_4, c_5\}$.

Theorem 5.3. *No rule can satisfy PR and committee monotonicity at the same time.*

Proof. Consider the following election. Let $C = \{c_1, \dots, c_5\}$. 6 voters cast the following ballots. For $i = 1$ to 3, and for $j = 1$ to 2, one voter approves of $\{c_i, c_{3+j}\}$. If the target committee size is 2, a voting rule that satisfies PR has to output only $\{c_4, c_5\}$, but if the target committee size is 3, a voting rule that satisfies PR has to output only $\{c_1, c_2, c_3\}$. □

	SMWPI	**SMWOPI**	**Com. Mon.**
JR	Str.	Wk.(Str.?)	Yes
PJR	Str.	Wk.(Str.?)	Yes
EJR	Str.	Wk.(Str.?)	?
PR	Wk.	Wk.(Str.?)	No

Table 2: Summary of results on compatibility between representation and monotonicity axioms

Table 2 summarises the results that we have found in this paper with respect to the compatibility between representation and monotonicity axioms. Of course, the rules that we have found before that satisfy a certain monotonicity axiom and a certain representation axiom at the same time prove that such axioms are compatible. In the table, "Str." means that the strong version of the support monotonicity axiom is compatible with the corresponding representation axiom, "Wk." means that the weak version of the support monotonicity axiom is compatible with the corresponding representation axiom but that the strong version of the support monotonicity axiom and the corresponding representation axiom are incompatible, and "Wk.(Str.?)" means that the weak version of the support monotonicity axiom is compatible with the corresponding representation axiom but that we do not know whether the the strong version of the support monotonicity axiom and the corresponding representation axiom are compatible.

6 RELATED WORK

There exists some previous work that consider support monotonicity in the context of approval-based multi-winner elections. Aziz and Lee [5] propose several notions of monotonicity for weak preferences (ties between candidates are allowed), and then they consider the restriction of such notions to approval ballots. The strongest monotonicity axiom that they propose when restricted to approval ballots corresponds essentially to the candidate monotonicity axiom that we have presented before (they also call this axiom candidate monotonicity).

Lackner and Skowron [17] consider a different (and much more stronger) notion of support monotonicity, defined for a subclass of the approval-based multi-winner voting rules called ABC ranking rules. They use this axiom to characterize a subset of the ABC ranking rules that they call Dissatisfaction Counting Rules (AV belongs to this class of rules). In particular, a necessary (but not sufficient) condition to satisfy such axiom is that if a voter i that is already in the election adds a candidate w that was already in the set of winners W, then W must still be a set of winners (possibly tied with other sets of winners). It follows that this axiom is strictly stronger than strong SMWOPI.

According to Janson [15], Phragmén also studied support monotonicity in the approval-based rules that he proposed. In particular, Phragmén proved that max-Phragmén satisfies support monotonicity when only one candidate increases her support, either because a voter already in the election adds such candidate to her ballot (candidate monotonicity) or because a new voter enters the election and approves only of such candidate.

Phragmén also proposed another voting rule that we refer to as seq-Phragmén, that can be computed iteratively. Phragmén also proved that seq-Phragmén satisfies support monotonicity when only one candidate increases her support.

Mora and Oliver [21] and Janson [15] have recently extended the study of the monotonicity properties of seq-Phragmén. In particular, they give examples that prove that seq-Phragmén fails both strong SMWPI and strong SMWOPI. We stress the following differences between the works of Mora and Oliver [21] and Janson [15] and ours: 1) they do not consider weak SMWPI and weak SMWOPI; and 2) they do not formalize the strong axioms (they only give examples that show that seq-Phragmén fails them).

There is also some relation between our work and the work of Peters [23]. Peters [23] defines an axiom that they call strategyproofness. For a rule f that satisfies this axiom it cannot happen that $W' \cap (A_i \cup G) \subsetneq W \cap (A_i \cup G)$, were W is the output of rule f for election $\mathcal{E} = (\mathcal{A}, k)$ and W' is the output of rule f for election $\mathcal{E}_{i+G}$ (they assume that the rules are resolute). Although this axiom is similar to SMWOPI neither strong SMWOPI implies strategyproofness nor strategyproofness implies weak SMWOPI. Peters [23] prove that strategyproofness is not compatible with a representation axiom that is even weaker than JR using SAT solvers.

7 CONCLUSIONS AND FUTURE WORK

In this paper we have complemented previous work on the axiomatic study of multi-winner voting rules with the study of monotonicity axioms for rules that use approval ballots. Our results show that support monotonicity in approval-based multi-winner voting rules is trickier than it may seem at first glance. While the weak support monotonicity axioms are satisfied in almost all the cases analyzed in this study (only Monroe fails one of these) the situation changes completely when we look to the strong axioms. Of the 6 rules analyzed only 4 satisfy strong SMWPI and only 2 satisfy strong SMWOPI.

We have also presented some results related to the compatibility between representation and monotonicity axioms. First, we have proved that PR is incompatible both with strong SMWPI and with committee monotonicity. Our results also show that EJR and PJR are compatible with strong SMWPI and weak SMWOPI (in particular, PAV satisfies all these axioms). Our incompatibility results are mostly of theoretical interest because PR rules are NP-hard to compute [29] and therefore of little practical use. However, we believe that these results are interesting because they illustrate the existence of a certain conflict between representation and monotonicity. With respect to the compatibility between EJR and PJR with monotonicity axioms, several interesting open questions remain open. First of all, the only rule that we have found that satisfies EJR (or PJR) and strong SMWPI is PAV, which is known to be NP-hard to compute. Therefore, it would be very interesting to find a rule that satisfies EJR (or PJR) and strong SMWPI but can be computed in polynomial time. Very recently, Aziz et al. [3] have identified a set of voting rules that satisfy EJR and can be computed in polynomial time. The study of these rules could be interesting in this regard. In the second place, it is also open whether EJR and PJR are compatible with strong SMWOPI. Other open issues would be to find a rule that satisfies both EJR and committee monotonicity and to find a rule that satisfies strong SMWOPI and PR. The similarity between the strategyproofness axiom proposed by Peters [23] and SMWOPI makes us think that the use of SAT solvers could be a possible approach to address these research questions.

We have also studied the relevance of our axioms to several types of scenarios. We have found that our support monotonicity axioms fit well with all the cases studied except in the case of the second type of proportional representation discussed by Sánchez-Fernández et al. [29], where the goal is to satisfy large cohesive groups according to their size. Therefore, it would be interesting to define an additional support monotonicity axiom that fits well in this scenario and is compatible with EJR (which is oriented to this type of proportional representation). The development of an adapted version of the notion of non-crossing monotonicity proposed by Elkind et al. [12] for ranked ballots could also be a line of continuation of this work.

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