Fair Cake-Cutting Algorithms with Real Land-Value Data

Extended Abstract

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ABSTRACT
Fair division of land is an important practical problem that is commonly handled either by hiring assessors or by selling and dividing the proceeds. A third way to divide land fairly is via algorithms for fair cake-cutting. However, the current theory of fair cake-cutting is not yet ready to optimally share a plot of land and such algorithms are seldom used in practical land-division.

We attempt to narrow the gap between theory and practice by performing extensive simulations of a classic cake-cutting algorithm on real land-value data. We improve the practical performance of this algorithm using heuristics we developed, and show their effectiveness on real land-value maps compared to actual assessment and sale data on various performance metrics. The cake-cutting algorithms perform better in most metrics.

We further examined the cake cutting algorithm with respect to strategic gain of an agent relative to a truthful agent. The strategic gain was found to be insignificant effect in cake-cutting algorithms.

KEYWORDS
fair allocation; cake-cutting; land division; simulation

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1 INTRODUCTION
Fair division occurs in inheritance cases, partnership resolutions, and public land allocations. If the partners do not trust an assessor, they often sell the land in the market and split the revenues.

In the last 70 years, economists and computer scientists have developed various algorithms for fair cake-cutting - fair division of a continuous heterogeneous resource among agents with different preferences. Using such automatic methods is not only cheaper but also more practical. We measured the strategic gain of an agent relative to a truthful agent. The strategic gain was found to be insignificant effect in cake-cutting algorithms.

A land-estate $C$ (‘cake’) has to be divided among $n$ agents. Each agent $i$ has a value-density function $v_i$, mapping each point of $C$ to its monetary value for $i$. $C$ should be partitioned into $n$ disjoint contiguous pieces, $X_1, \ldots, X_n$, one piece per agent. $C$ is assumed to be a rectangle, and each piece should be a rectangle too.

There are infinitely many land partitions, and there are various metrics by which a partition can be evaluated. We compared the methods using several metrics:

- **Utilitarian Value** (UV): the sum of the agents’ relative values.
- **Egalitarian Value** (EV): the smallest relative value of an agent.
- **Largest Envy** (LE): the largest amount by which an agent considers another agent’s share as better than his/her own share.

- **Average/Smallest Face-Ratio** (AFR/SFR): the average/smallest of the calculated face-ratio of all allocated land-plots.

There are many algorithms for finding a proportional cake partition. In this experiment we focus on the Even-Paz algorithm [3], since we believe it is the algorithm most likely to be used in a cake-cutting problem with a large number of agents. This is due both to its simplicity and to its optimal run-time complexity: it runs in time $O(n \log n)$, which is provably the best possible [7]. We compare the results of Even-Paz to another classic proportional cake-cutting algorithm we examined - Last-Diminisher [5].

Both Even-Paz and Last-Diminisher are well-defined for a one-dimensional cake. However, when $C$ is two dimensional, in each iteration, the agents can make their query-marks in many different directions. We applied 10 different heuristic for choosing the cut direction (horizontal or vertical) in each iteration and compared those heuristics empirically. We tested two sets of heuristics: pre-defined heuristics (all $n – 1$ cuts are defined in advanced) and greedy heuristics (at each iteration level the cut direction is selected in order to maximize a certain quality function).

Assessor division. We compare our algorithms’ results to the performance of an assessor. The division is done using the following method. For some integers $k_1, k_2$ with $k_1 \cdot k_2 = n$, the assessor first partitions the land using vertical cuts into $k_1$ parallel strips of value 1/$k_1$, and then partitions each such strip using horizontal cuts into $k_2$ plots of value $(1/k_1)/k_2 = 1/n$. We take $k_1 = 2^{\lceil \log_2 n \rceil / 2}$ and $k_2 = 2^{\lceil \log_2 n \rceil / 2}$, so that the pieces have a balanced aspect ratio.

3 EXPERIMENTS AND RESULTS

We constructed three land-value maps. The first map was of New Zealand — based on the Forest Profit Expectations Dataset (U10073). The second map was of Israel — based on a commercial website for classified real-estate ads (http://madlan.co.il). For comparison, the third map was generated uniformly at random.

From each land-value map we created datasets, each dataset containing $n$ variants, one variant per agent. The subjective preference of each agent was captured by a random noise that is added to the land-value map. For each map and $n$ in 4, 8, 16, 32, 64, 128, we ran 50 experiments with different randomly-generated noise. In most experiments the noise-ratio was $r = 0.6$.

The results for the New Zealand map and Israel map were very similar, but the results for the random map were substantially different. For all tested metrics we found that Even-Paz is at least as good, and often better, than Last-Diminisher.

Below, we discuss the Even-Paz results for New Zealand map averaged over the 50 runs and a 95% confidence interval.

The best-performing heuristic is MostValuableMargin (MVM), a greedy heuristic that chooses a cut direction resulting in a more valuable margin between the actual cut proposed by the algorithm and the two closest cuts proposed by agents. Intuitively, the reason is that this heuristic takes the most advantage of the differences between the agents’ valuations.

The UV of MVM is significantly better than the market sale ($p < 0.001$) and the advantage grows with $n$. For $n = 128$, the advantage is about 2.9%. In contrast, the UV of an assessor division is 1 – the same as market sale.

The EV of Even-Paz is always higher than the EV of selling the land and with MVM it is significantly better ($p < 0.001$). With $n = 128$, the advantage is about +1%. The EV of an assessor division is always lower than of selling the land. This means that, in an assessor division, there is always at least one person who receives less than his/her fair share.

Besides market sale which trivially attains the minimum LE, the best heuristic is MVM: it scores better than the other heuristics we tested, but the advantage is not statistically significant ($p = 0.2$). The LE of Even-Paz is better (smaller) than of the assessor-division. For $n = 128$, MVM is significantly smaller than assessor division in about 3% ($p < 0.001$).

The best heuristic for both AFR and SFR, is SquarePiece (SP) ($p < 0.001$), a greedy heuristic that chooses a cut direction resulting in a higher face-ratio. In contrast to the previous metrics, we did not find a significant difference between SP and an assessor division. The AFR of SP is always above 1/3, and sometimes above 1/2. However, the SFR (for $n = 128$) is very small — around 0.02. This means that at least one agent might get a very thin plot.

We ran experiments to check how much an agent may gain by being unsubmissive (assuming all other agents are truthful). The largest strategic gain attained by an agent was 6.54%. However, the average strategic gain in all experiments was less than 1.5%.

4 CONCLUSION AND DISCUSSION

Cake-cutting algorithms may be a viable alternative to the common methods for land division, namely assessor division and market sale. In particular, Even-Paz attains higher social welfare than both these methods, and lower envy than assessor division. The effect is larger when there are more agents and when there is more variation in the agents’ valuations. Even-Paz performs better than the other cake-cutting algorithm we checked (Last Diminisher).

When adapting cake-cutting algorithms to two dimensions, in terms of social welfare and envy MVM is superior, while SP is superior in terms of face-ratio. This shows a trilateral trade-off between run-time, social welfare and geometric shape as well as illustrating the importance of metric driven practical decision making.

Strategic manipulation may improve the welfare of an agent with complete information, but the improvement is relatively small.
REFERENCES


