Optimal Capacity Modification for Many-To-One Matching Problems*

Extended Abstract

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ABSTRACT

We consider many-to-one matching problems, where one side consists of students and the other side of schools with capacity constraints. We study how to optimally increase the capacities of the schools so as to obtain a stable and perfect matching (i.e., every student is matched) or a matching that is stable and Pareto-efficient for the students. We consider two common optimality criteria, one aiming to minimize the sum of capacity increases of all schools (abbrv. as MinSum) and the other aiming to minimize the maximum capacity increase of any school (abbrv. as MinMax). We obtain a complete picture in terms of computational complexity: Except for stable and perfect matchings using the MinMax criteria which is polynomial-time solvable, all three remaining problems are NP-hard. We further investigate the parameterized complexity and approximability and find that achieving stable and Pareto-efficient matchings via minimal capacity increases is much harder than achieving stable and perfect matchings.

KEYWORDS

School choice; Stable matching; Pareto-efficient matching; Parameterized complexity; Approximation algorithms

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1 INTRODUCTION

In many-to-one matching with two-sided preferences [1, 4, 11, 15, 18, 20, 21, 24–27], we are given two disjoint sets of agents, U and W, such that each agent has a strict preference list over some members of the other set, and each agent in W has a capacity constraint (aka. quota) which specifies how many agents from U can be matched to it. The goal is to find a *good* matching between U and W without violating the capacity constraints. For school choice and university admission, for example, the agents in U would be students or high-school graduates, while the agents in W would be schools and universities, respectively. To unify the terminology, we call the agents in U the students and the agents in W the schools.

As to what defines a good matching, the answer varies from application to application. The arguably most prominent and well-known concept is that of *stable* matchings [11, 12, 16], which ensures that no student u and school w will form a blocking pair, that is u is either unmatched or prefers w to his matched school, and w is either under-filled (i.e. w does not receive enough students) or prefers u to one of its matched students. Stability is a key desideratum and has been a standard criterion for many matching applications. On the other hand, the simplest concept is to ensure that every student is matched, and we call such matching a *perfect* matching. Note that having a perfect matching is particularly important in school choice or university admission since every student should at least be admitted to some school/university. A *Pareto-efficient* (abbrv. *efficient*) matching ensures that no other matching can make one student better off without making another student worse off[1] [1, 10]. Efficiency is very desirable for the students since it saves them from trying to find a mutually better solution.

Stability and efficiency, even though equally desirable, are not compatible with each other (i.e., they may not be satisfiable simultaneously). Neither is stability compatible with perfectness. But what if we can modify the capacities of the schools? Clearly, if we increase each school's capacity to $|U|$ so that every student is assigned her first choice, then we obtain a stable, efficient, and perfect matching. However, this is certainly not cost effective, so we are facing the following question:

How can we modify the capacities as little as possible to obtain a stable and efficient, or stable and perfect matching?

In this paper, we aim to answer this question computationally, and look at two common cost functions, the total and maximum capacity increases of all schools.

2 OUR CONTRIBUTIONS

We introduce the problem and thoroughly investigate the computational complexity of determining an optimal capacity increase vector for obtaining a stable and efficient (resp. stable and perfect) matching. We consider two optimality criteria: minimizing the sum k^+ of capacity increases and the maximum capacity increase $k^{\max}$. This gives rise to four problems: MinSumSP, MinMaxSP, MinSumSE, and MinMaxSE. We show that except for MinMaxSP, which is polynomial-time solvable, the other three problems are NP-hard, and remain so even when the preference lists have constant length. For MinSumSP, we prove a structural

*Full version available on arXiv [7].

[1] Not to be confused with Pareto-optimality which requires that no matching exists that can make an agent (either student or school) better off without making another agent worse off.

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Table 1: An overview of the complexity results for the three NP-complete problems. We omit the results for MinMaxSP since it is polynomial-time solvable. Here, "const-approx" means a constant-factor approximation algorithm, and "paraNP-h" means that the problem remains NP-hard even if the corresponding parameter (see the first column) has constant value.

	MinSumSP	MinSumSE	MinMaxSE
const-approx	NP-h	NP-h	NP-h
$\|U_{\mathrm{un}}\|$-approx	P	NP-h	NP-h
$\|\Delta_{\mathrm{un}}\|$-approx	P	NP-h	NP-h
$k^+/k^{\max}$	W[1]-h, XP	W[1]-h, XP	paraNP-h
$\|U_{\mathrm{un}}\|$	W[1]-h, XP	paraNP-h	paraNP-h
$(\Delta_{\mathrm{st}}, \Delta_{\mathrm{sc}})$	paraNP-h	paraNP-h	paraNP-h
$(\|U_{\mathrm{un}}\|, \Delta_{\mathrm{un}})$	FPT	paraNP-h	paraNP-h

property about agents with justified envies which may be of independent interest. We further search for parameterized and approximation algorithms. We are especially interested in three structural parameters:

– Capacity bounds k^+ and $k^{\max}$.
– Number $|U_{\mathrm{un}}|$ of initially unmatched students.
– Max. length Δ_{st} of the preference list of any student resp. the maximum length Δ_{sc} of the preference list of any school.
– Max. length Δ_{un} of the preference list of any unmatched student.

We show that both MinSumSP and MinSumSE can be solved in polynomial time if the capacity bound k^+ is a constant (i.e., in XP wrt. k^+). We strengthen this by showing that it is essentially tight since it cannot be improved to obtain an FPT algorithm, i.e., an algorithm with running time $f(k^+) \cdot (|U| + |W|)^{O(1)}$, where f is a computable function solely depending on the parameter k^+. On the other hand, MinMaxSE remains NP-hard even if $k^{\max} + \Delta_{\mathrm{st}} + \Delta_{\mathrm{sc}} + |U_{\mathrm{un}}|$ is a constant.

Since $|U_{\mathrm{un}}| \leq k^+$ in the case of MinSumSP and MinSumSE, the parameterized hardness result for k^+ also holds for $|U_{\mathrm{un}}|$. For the combined parameter $|U_{\mathrm{un}}|$ and Δ_{un}, MinSumSP admits an FPT algorithm, while MinSumSE remains NP-hard even if $|U_{\mathrm{un}}| = 0$ and either Δ_{st} or Δ_{sc} is a constant. Finally, as for approximation algorithms, while MinSumSP is in general hard to approximate to any constant factor, it admits a Δ_{un}-approximation (resp. $|U_{\mathrm{un}}|$-approximation) algorithm. However, both MinSumSE and MinMaxSE cannot be approximated within a factor of $(|U| + |W|)^{1-\varepsilon}$ (for any constant $\varepsilon > 0$) even if $|U_{\mathrm{un}}| = 0$ and Δ_{st} is a constant. Table 1 summarizes our findings.

3 RELATED WORK

Studying the trade-off and tension between stability and efficiency has a long tradition not only in Economics [1, 2, 9, 10, 13, 22], but also in Computer Science [4, 23]. For instance, Abdulkadiroğlu and Sönmez [1] examined Gale/Shapley's student-proposing deferred acceptance algorithm (which always yields a student-optimal stable matching) and the simple top trading cycles algorithm (which is efficient). Ergin [10] characterized priority structures of the schools–the so-called acyclic structure–under which a stable and efficient matching always exists. Kesten [13] proposed an efficiency-adjusted deferred acceptance algorithm to obtain an unstable matching which is efficient and Pareto superior to the student-optimal stable matching. We are not aware of any work on achieving stable and efficient matching via capacity increase.

Chen et al. [8] investigated the trade-off between stability and perfectness in one-to-one matchings which is to find a perfect matching that becomes stable after a few modification to the preference lists. Our model differs from theirs as we do not allow modifications to the agents' preferences. Limaye and Nasre [14] introduced two related matching problems, where each school has unbounded capacity and a value that measures the cost of assigning an (arbitrary) student there, and the goal is to find a stable and perfect matching with minimum sum of costs or minimum maximum cost. Their models are different from ours since they assume that any place at a given school must have the same cost, whereas in our framework each school has a capacity q so that the first q places are considered free, and only the additional places have non-zero cost. Furthermore, they allow no initial quotas, which may be interpreted in our framework as setting each initial quota to zero.

Recently, capacity variation in many-to-one matching has been studied, albeit for different objectives. Ríos et al. [19] proposed a seat-extension mechanism to increase student's welfare. Ueda et al. [23] designed a strategy-proof mechanism to address minimum and maximum quotas. Nguyen and Vohra [17] studied many-to-one matching with couples and propose algorithms to find a stable matching by perturbing the capacities. Bobbio et al. [5, 6] considered capacity variations to obtain a stable matching with minimum average rank of the matched schools (AvgRank) or maximum cardinality (CarSM). The capacity variations can be either sum of capacity increases or sum of capacity decreases. Our problem MinSumSP can be reduced to their MinSumAvgRank problem by introducing sufficiently many dummy students and garbage collector schools with very high ranks so to ensure that each original student is matched. Hence, our hardness results apply to MinSumAvgRank. They left open the complexity of MinSumCarSM, which is NP-hard by our hardness for MinSumSP. Abe et al. [3] propose some alternative method and conduct experiments for MinSumAvgRank. Yahiro and Yokoo [28] introduced a matching problem which combines school matching with resource allocation where the capacity of the schools depend on which resource is allocated to them. None of our results follows from theirs.

4 CONCLUSION

For future work, one could investigate parameterized complexity for other parameters such as the number n of students and the number m of schools. It is not very difficult to see that our problems are FPT wrt. n since one can upper-bound the number of relevant schools by a function in n. For constant number m of schools, it is easy to see that all problems are in XP wrt. m. Secondly, one could look at stable matching with maximum cardinality instead of perfectness. It would be interesting to see whether the algorithmic results for MinSumSP and MinSumSP transfer to this case. Finally, one could look at other objectives such as stable and popular matching. Preliminary results show that the problem behaves similarly to the case with stable and perfect matching.

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REFERENCES

[1] Atila Abdulkadiroğlu and Tayfun Sönmez. 2003. School Choice: A Mechanism Design Approach. *American Economic Review* 93 (2003), 729–747.
[2] Atila Abdulkadiroğlu, Yeon-Koo Che, Parag A. Pathak, Alvin E. Roth, and Olivier Tercieux. 2020. Efficiency, Justified Envy, and Incentives in Priority-Based Matching. *American Economic Review Insights* 2, 4 (2020), 425–442.
[3] Kenshi Abe, Junpei Komiyama, and Atsushi Iwasaki. 2022. Anytime Capacity Expansion in Medical Residency Match by Monte Carlo Tree Search. In *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI '22)*, Luc De Raedt (Ed.). 3–9.
[4] Mourad Baïou and Mihcel Balinski. 2004. Student admissions and faculty recruitment. *Theoretical Computer Science* 322, 2 (2004), 245–265.
[5] Federico Bobbio, Margarida Carvalho, Andrea Lodi, Ignacio Rios, and Alfredo Torrico. 2022. *Capacity Planning in Stable Matching: An Application to School Choice.* Technical Report. arXiv:2110.00734v2.
[6] Federico Bobbio, Margarida Carvalho, Andrea Lodi, and Alfredo Torrico. 2022. *Capacity Variation in the Many-to-one Stable Matching.* Technical Report. arXiv:2205.01302v1.
[7] Jiehua Chen and Gergely Csáji. 2023. *Optimal Capacity Modification for Many-To-One Matching Problems.* Technical Report. arXiv:2302.01815.
[8] Jiehua Chen, Piotr Skowron, and Manuel Sorge. 2022. Matchings under Preferences: Strength of Stability and Trade-Offs. *ACM Transactions on Economics and Computation* 6, 4 (2022), 1–55.
[9] Lars Ehlers and Aytek Erdil. 2010. Efficient assignment respecting priorities. *Journal of Economic Theory* 145, 3 (2010), 1269–1282.
[10] Haluk Ergin. 2002. Efficient Resource Allocation on the Basis of Priorities. *Econometrica* 70, 6 (2002), 2489–2497.
[11] David Gale and Lloyd S. Shapley. 1962. College Admissions and the Stability of Marriage. *The American Mathematical Monthly* 120, 5 (1962), 386–391.
[12] Dan Gusfield and Robert W. Irving. 1989. *The Stable marriage problem–Structure and algorithms.* MIT Press.
[13] Onur Kesten. 2010. School Choice with Consent. *The Quarterly Journal of Economics* 125, 3 (2010), 1297–1348.
[14] Girija Limaye and Meghana Nasre. 2022. *Stable Matchings with Flexible Quotas.* Technical Report. arXiv:2101.04425v4.
[15] D. Manlove. 2016. Hospitals/Residents Problem. In *Encyclopedia of Algorithms*, Ming-Yang Kao (Ed.).
[16] David F. Manlove. 2013. *Algorithmics of Matching Under Preferences.* Series on Theoretical Computer Science, Vol. 2. WorldScientific.
[17] Thành Nguyen and Rakesh Vohra. 2018. Near-Feasible Stable Matchings with Couples. *American Economic Review* 108, 11 (2018), 3154–3169.
[18] Parag A. Pathak, Tayfun Sönmez, M. Utku Ünver, and M. Bumin Yenmez. 2020. *Fair Allocation of Vaccines, Ventilators and Antiviral Treatments: Leaving No Ethical Value Behind in Health Care Rationing.* Technical Report. arXiv:2008.00374v2.
[19] Ignacio Ríos, Tomás Larroucau, Giorgiogiulio Parra, and Roberto Cominetti. 2014. *College Admissions Problem with Ties and Flexible Quotas.* Technical Report.
[20] Antonio Romero-Medina. 1998. Implementation of stable solutions in a restricted matching market. *Review of Economic Design* 3, 2 (1998), 137–147.
[21] Alvin Roth and Elliott Peranson. 1999. The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design. *American Economic Review* 89, 4 (1999), 748–780.
[22] Alvin E. Roth. 1982. The Economics of Matching: Stability and Incentives. *Mathematics of Operations Research* 7, 4 (1982), 617–628.
[23] Suguru Ueda, Daniel Fragiadakis, Atsushi Iwasaki, Peter Troyan, and Makoto Yokoo. 2012. Strategy-proof mechanisms for two-sided matching with minimum and maximum quotas. In *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems (AAMAS '12)*, Wiebe van der Hoek, Lin Padgham, Vincent Conitzer, and Michael Winikoff (Eds.). 1327–1328.
[24] www.carms.ca. 2023. Canadian Resident Matching Service website.
[25] www.jrmp.jp. 2023. Japan Resident Matching Program website.
[26] www.nes.scot.nhs.uk/sfas. 2023. Scottish Foundation Allocation Scheme website.
[27] www.nrmp.org. 2023. National Resident Matching Program website.
[28] Kentaro Yahiro and Makoto Yokoo. 2020. Game theoretic analysis for two-sided matching with resource allocation. In *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems.* 1548–1556.