Optimal False-name-proof Single-Item Redistribution Mechanisms

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ABSTRACT

Although the celebrated Vickrey auction is strategy-proof and guaranteed to achieve an efficient allocation in a single-object auction, if there exists no outside party (i.e., a seller or an auctioneer) with the right to collect the payment, the collected payment will be wasted. Redistribution mechanisms try to redistribute the payment to participating agents as much as possible without violating strategy-proofness. However, when a losing agent can obtain part of the payment, she may have an incentive to participate under multiple identities and receive a greater share of the redistribution.

Our goal is to develop *false-name-proof* redistribution mechanisms that are robust against such manipulations. First, we prove that no mechanism simultaneously satisfies false-name-proofness and allocative efficiency, except for the Vickrey auction. Next, we propose a class of false-name-proof redistribution mechanisms, which are characterized by several parameters. We show that each mechanism in the class is not dominated by any other false-name-proof mechanism in terms of social welfare. Precisely, by choosing these parameters appropriately, all instances of this class are guaranteed to achieve at least the same amount of social welfare obtained by any false-name-proof mechanism. Furthermore, we formalize an optimization problem that determines appropriate parameter values based on prior information about participating agents.

Categories and Subject Descriptors

I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multiagent systems*; J.4 [**Social and Behavioral Sciences**]: Economics

Keywords

Mechanism design, Redistribution, False-name-proofness

1. INTRODUCTION

Mechanism design is a subfield of microeconomic theory and game theory, which concerns designing collective decision-making rules for multiple agents. Such rules are expected to achieve several desirable properties, such as maximizing the social welfare, while each agent pursues her own utility. Due to the growing needs for agent technology and the Internet's popularity. vigorous research on mechanism design has been conducted in the AI and MAS research communities.

In this paper, we consider *redistribution mechanisms* for allocating a single object to participating agents. Although the celebrated Vickrey auction is strategy-proof and guaranteed to achieve an efficient allocation, if there exists no outside party who has a right to collect the payment, the collected payment will be wasted. Redistribution mechanisms try to distribute the payment to participating agents as much as possible without violating strategy-proofness. As a result, the welfare loss caused by wasting the payment can be reduced to a certain extent. However, there exists no strategy-proof mechanism that always achieves all of the following properties: allocative efficiency, individual rationality, and strong budget-balance (which means we run neither a surplus nor a deficit) [6, 11, 13].

Several researchers have proposed budget-balanced redistribution mechanisms by abandoning efficient allocation [3, 5, 7]. An alternative approach is to relax budget balance, so that the mechanism achieves an efficient allocation but it wastes part of the payment [1, 2, 8, 9, 10, 12]. In this case, the mechanism is usually required to be non-deficit, i.e., it never runs a loss. If it suffers a deficit, it cannot be sustained without an outside party (e.g., the government) that is willing to subsidize the deficit.

One possible application domain of redistribution mechanisms consists of resource allocation problems in which a resource is shared among the members of a community, e.g., a college's tennis court, a car, or an electricity charger for electric cars, which is co-owned by a community. It appears unlikely that a redistribution mechanism would be applied to a domain where anybody can participate. If receiving redistribution is possible just by participating/registering for a mechanism, it will attract many participants who are not really interested in the shared resource. For a redistribution mechanism to be successful, there must presumably be a natural restriction on the possible participants, e.g., only students/professors of a college or members of an established community.

Of course, even if participation is restricted to community members, a community member might have an incentive to join the mechanism, even though she is not interested in the shared resource at all, assuming that the redistribution consists of actual money. For example, a person who is not genuinely interested in playing tennis might create a fake

Appears in: *Alessio Lomuscio, Paul Scerri, Ana Bazzan, and Michael Huhns (eds.), Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2014), May 5-9, 2014, Paris, France.*

team and participate just to obtain the redistribution. Such fake participation can perhaps be avoided if the redistribution consists of something that is useless to agents who are not interested in the shared resource.

However, this is not the end of the story. A tennis team could split itself and act as two different teams. Also, a person could ask her friend, who does not own an electric car, to participate in the mechanism and to hand over priority tickets. All these manipulations can be considered instances of false-name manipulation, a type of manipulation that has been considered in various application domains including Internet auctions [4, 15, 16]. A mechanism that is robust against such false-name manipulation is called *false-name-proof*.

As far as the authors are aware, no existing redistribution mechanisms (other than simply running the Vickrey auction and redistributing nothing) are false-name-proof. For example, suppose that the Bailey-Cavallo mechanism [1, 2] is applied to the assignment of a tennis court to a college's intramural teams. A delegate from each team registers and bids. Suppose there are three teams, the highest bid is \$15 by team A, the second highest bid is \$12 by team B, and the third highest bid is \$9 by team C. Team A obtains the right to use the tennis court. The original payment is \$12, i.e., the second highest bid. The redistribution amount is 1/3 of the third highest bid for teams A and B and 1/3 of the second highest bid for team C. Thus, team C obtains \$4 (or tennis equipment of an equivalent value). But now, if team C creates fictitious team D and bids \$1 for this team, then teams C and D each receive 1/4 of the second highest bid, i.e., 3. As a result, team C can increase its redistribution from 4 to 6 by this false-name manipulation.

We first show an impossibility result for false-name-proof redistribution: under some natural assumptions, there exists no mechanism satisfying false-name-proofness and allocative efficiency, except for the Vickrey auction without any redistribution. We propose a class of redistribution mechanisms, each of which is false-name-proof. Furthermore, we show that our proposed class is *stable* in a sense: for any other given false-name-proof redistribution mechanism, we can find a mechanism from the proposed class that dominates it, and no dominance relation exists between any pair of mechanisms from the class. We further analyze our mechanisms with and without prior information.When there is no prior information on the distributions of the bidder valuations and the number of bidders, we can find a further stable set based on *prior-free dominance* relations. Under some assumptions, there is a class of mechanisms that prior-free-dominate any false-name-proof mechanism.

2. PRELIMINARIES

We introduce our model of the redistribution problem where false-name manipulations are possible. Let $\mathcal{N}$ be the set of all potential agents/identities and $N \subseteq \mathcal{N}$ be a set of attending agents/identities. Note that in our model, set N is a variable, since due to false-name manipulations the number of attending identities can change. For a given set of agents/identities $N \subseteq \mathcal{N}$, let $k(N) := |N|$ denote the number of elements in the set N. We use k instead of $k(N)$.

There exists an object that is going to be sold to the attending agents N. Attending agent $i \in N$ has a *valuation* $v_i \in V$ where $V = [0, \bar{V}]$ for the object, where V denotes the set (or *domain*) of all the possible valuations of an agent. Let $v = (v_i)_{i \in N} \in V^k$ denote the profile of the valuations of attending agents N, and let $v_{-i} \in V^{k-1}$ denote the profile of the valuations of all agents except for i, when the set of agents N is attending.

For each $N \subseteq \mathcal{N}$, let $A_k \subseteq \{0,1\}^k$ be the set of all possible *allocations* $a = (a_i)_{i \in N} \in \{0,1\}^k$ that satisfy $\sum_{i \in N} a_i \leq 1$. Here $a_i = 1$ indicates that agent i receives the object, and $a_i = 0$ indicates that she does not receive it. *Mechanism* $M = (f, p)$ consists of allocation rule f and payment rule p. Allocation rule f is a set of functions $f^l : V^l \to A_l$, where, for each $l \in \{1, \dots, |\mathcal{N}|\}$, f^l is a function that maps any valuation profile $v \in V^l$ to an allocation of the object $a \in A_l$. For given $v \in V^k$ and $i \in N$, we let $f_i(v) := f_i^k(v)$ denote the allocation to agent i. Similarly, payment rule p is a set of functions p^l, where $p^l : V^l \to \mathbb{R}^l$ gives the payment for each agent. More precisely, for given $v \in V^k$ and $i \in N$, let $p_i(v) := p_i^k(v)$ denote the amount of money that agent i *pays* (so that a negative amount means that i receives money).

In this paper, we restrict our attention to mechanisms that satisfy all of the following six assumptions.

Assumption 1 (Determinism). *Mechanism $M = (f, p)$ is* deterministic *if for any $N \subseteq \mathcal{N}$ and for any $v \in V^k$, there exists a (deterministic) allocation $a \in A$ such that $f(v) = a$.*

That is, for any set of attending agents N and any valuation profile v that they might have, the mechanism returns a unique allocation, without any randomization.

Assumption 2 (Non-Deficit). *Mechanism $M = (f, p)$ is said to satisfy* non-deficit (ND) *if for any $N \subseteq \mathcal{N}$ and any $v \in V^k$, $\sum_{i \in N} p_i(v) \geq 0$.*

That is, the mechanism never runs a deficit.This assumption seems natural, since if a mechanism does not satisfy it, we need an outside party that is willing to subsidize the deficit to execute the mechanism. Note that this assumption does not require that no agent ever gets any money.

Assumption 3 (Anonymity). *Mechanism $M = (f, p)$ is said to be almost* anonymous *if for any $N, N' \subseteq \mathcal{N}$ such that $|N| = |N'| = k$ and for any $v, v' \in V^k$, the existence of bijection $\sigma : N \to N'$ satisfying $v_i = v'_{\sigma(i)}$ for any $i \in N$ implies that*

$$v_i \cdot f_i(v) - p_i(v) = v'_{\sigma(i)} \cdot f_{\sigma(i)}(v') - p_{\sigma(i)}(v')$$

holds for any $i \in N$.

That is, agents' names do not matter for their utility. This is a natural extension of a traditional definition of anonymity, which requires that if the valuations of the agents are permuted, then their utilities must be correspondingly permuted. Indeed, this corresponds to our requirement when we set $N = N'$.

Assumption 4 (Individual Rationality). *Mechanism $M = (f, p)$ is said to be* individually rational (IR) *if for any $N \subseteq \mathcal{N}$, any $i \in N$, any $v_{-i} \in V^{k-1}$, and any $v_i \in V$,*

$$v_i \cdot f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq 0.$$

That is, the mechanism must not give negative utility to any agent who reports her true valuation to the mechanism. If a mechanism violates this assumption, an agent might have an incentive not to participate.

Assumption 5 (Sale Monotonicity). *Mechanism* $M = (f, p)$ *is said to be* sale monotone *if for any* $N \subseteq \mathcal{N}$, *any* $i, j (i \neq j) \in N$, *any* $v_{-i} \in V^{k-1}$, *any* $v_i \in V$, *and any* $\epsilon > 0$,

$$\sum_{i \in N} f_i(v) = 1 \Rightarrow \sum_{i \in N} f_i(v_j + \epsilon, v_{-j}) = 1.$$

That is, when the good is sold to some agent under bids v, it must also be sold to some agent (possibly a different one) whenever another agent bids higher.

Assumption 6 (Cross-Monotonicity on Competition). *Mechanism* $M = (f, p)$ *is said to be* cross-monotone on competition *if for any* $N \subseteq \mathcal{N}$, *any* $i, j (i \neq j) \in N$, *any* $v_{-i} \in V^{k-1}$, *any* $v_i \in V$, *and any* $\epsilon > 0$,

$$f_i(v) = 1 \Rightarrow f_i(v_j - \epsilon, v_{-j}) = 1.$$

That is, if an agent wins, then she must still win whenever another agent decreases her bid. According to Assumptions 5 and 6, if we consider mechanisms with a reserve price, the reserve price is determined independently of any valuation profile.

We also introduce the following properties of mechanisms:

Definition 1 (Strategy-Proofness). *Mechanism* $M = (f, p)$ *is said to be weakly* strategy-proof (SP) *if for any* $N \subseteq \mathcal{N}$, *any* $i \in N$, *any* $v_{-i} \in V^{k-1}$, *any* $v_i \in V$, *and any* $v'_i \in V$,

$$v_i \cdot f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i \cdot f_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i}).$$

That is, a mechanism is strategy-proof if reporting the true valuation is a dominant strategy for every agent when creating additional identifiers is not considered as a possibility.

Definition 2 (False-Name-Proofness). *Mechanism* $M = (f, p)$ *is said to be wealky* false-name-proof (FNP) *if for any* $N \subseteq \mathcal{N}$, *any* $i \in N$, *any* $v_{-i} \in V^{k-1}$, *any* $v_i \in V$, *any* $v'_i \in V$, *any* $S \subseteq \mathcal{N} \setminus N$, *and any* $v_S \in V^{|S|}$,

$$v_i \cdot f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i \cdot \sum_{l \in S \cup \{i\}} f_l(v'_i, v_{-i}, v_S) - \sum_{l \in S \cup \{i\}} p_l(v'_i, v_{-i}, v_S),$$

where $v_S \in V^{|S|}$ *is a valuation profile reported by set of identities* S.

This condition is stronger than strategy-proofness as defined above. Under a false-name-proof mechanism, for every agent, *using only one identity* and reporting her true valuation is a weakly dominant strategy, even though she can use multiple identities. Indeed, if we set $S = \emptyset$, we obtain strategy-proofness as a special case.

Definition 3 (Allocative Efficiency). *Mechanism* $M = (f, p)$ *is said to satisfy* allocative efficiency (AE) *if for any* $N \subseteq \mathcal{N}$, *and any* $v \in V^k$,

$$f(v) \in \arg\max_{a \in A_k} \sum_{i \in N} v_i \cdot a_i.$$

This condition requires that for any valuation profile, an agent with the highest valuation must win.

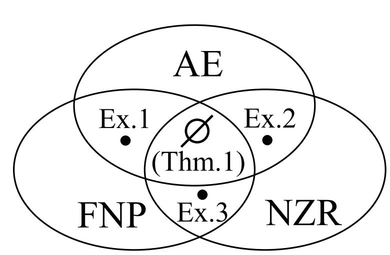

Figure 1: Illustration of Theorem 1.

Definition 4 (Non-Zero Redistribution). *Mechanism* $M = (f, p)$ *is said to satisfy* non-zero redistribution (NZR) *if for some* $N \subseteq \mathcal{N}$, *and some* $v \in V^k$,

$$\exists i \in N \text{ s.t., } f_i(v) = 0 \wedge p_i(v) < 0.$$

This condition is intended as a minimal requirement for a mechanism to be considered a redistribution mechanism. It requires that for at least one bid profile, the mechanism returns some money to at least one loser.

3. IMPOSSIBILITY

Although the three conditions FNP, AE, and NZR defined in the previous section seem quite natural for our redistribution problem, we now show that there exists no mechanism that simultaneously satisfies all of them. The result is visually represented in Fig. 1.

Theorem 1. *There exists no mechanism satisfying false-name-proofness, allocative efficiency, and non-zero redistribution.*

We introduce two lemmas to establish this result.

Lemma 1. $\forall N \subseteq \mathcal{N}$ *such that* $|N| = 2$ *and* $\forall v \in V^2$, *any mechanism* $M = (f, p)$ *satisfying strategy-proofness and allocative efficiency cannot return any amount of money to the loser. That is, when* $|N| = 2$,

$$\forall j \in N, [f_j(v) = 0 \Rightarrow p_j(v) = 0].$$

Proof. Consider an arbitrary N such that $|N| = 2$ and an arbitrary $v \in V^2$. Without loss of generality, let $N = \{i, j\}$ and $v_i \geq v_j$, so that (by AE) agent i is the winner. It's clear that when both agents have valuation 0, IR and ND imply this lemma.

Assume for the sake of contradiction that for some $v \in V^2$, loser j receives $\pi > 0$. Then, the winner must pay at least π; otherwise, the mechanism violates ND.

Then, consider a modified valuation profile $v' = (v_i, v'_j)$ such that $v'_j < \pi$. Under it, loser j must receive the same amount of money π; otherwise, the mechanism violates SP, because j would have an incentive either to report v'_j when her valuation is v_j, or to report v_j when her valuation is v'_j. Therefore, winner i still has to pay at least π.

Let us next consider another modification, $v'' = (v'_i, v'_j)$, such that $\pi > v'_i > v'_j$. By AE, agent i is still winning. By IR, i pays at most v'_i, which is strictly smaller than π.

Therefore, agent i with true valuation v_i has an incentive to misreport her valuation as v'_i when the other agent j has valuation v'_j, to reduce her payment. This violates the assumption that M is strategy-proof. □

LEMMA 2. *When $|N| = 2$, any mechanism $M = (f, p)$ satisfying false-name-proofness, allocative efficiency, and non-zero redistribution must sometimes return some money to the loser, i.e., $\exists v \in V^2$ such that for some agent j,*

$$f_j(v) = 0 \wedge p_j(v) < 0.$$

PROOF. For a given mechanism M satisfying FNP and NZR, let $\hat{N}$, $\hat{v}$, and $\hat{j}$ be the parameters that imply satisfaction of the NZR condition: $f_{\hat{j}}(\hat{v}) = 0$ and $p_{\hat{j}}(\hat{v}) < 0$. Then, by ND and IR for the losing agents, there is an agent $i \in \hat{N}$ that wins the object and pays at least $p_{\hat{j}}(\hat{v})$. Furthermore, from AE, $\hat{v}_i \geq \hat{v}_{\hat{j}}$ holds.

Now, consider removing all losers except for $\hat{j}$, resulting in a situation with only two agents, $\{i, \hat{j}\}$. By AE, agent $\hat{j}$ still loses. But we can now conclude that $\hat{j}$ must receive at least an amount of $|p_{\hat{j}}(\hat{v})| > 0$; otherwise, $\hat{j}$ has an incentive to make the situation identical to the above case by adding a set of accounts $N \setminus \{i, \hat{j}\}$.The fake identities would not win, and therefore, by IR, they would not be made to pay anything. Thus, $f_{\hat{j}}(\hat{v}_i, \hat{v}_{\hat{j}}) = 0 \wedge p_{\hat{j}}(\hat{v}_i, \hat{v}_{\hat{j}}) < 0$ holds for the set of two agents $\{i, \hat{j}\}$. □

Lemma 2 states that if a mechanism never redistributes when $|N| = 2$, then it violates at least one of AE, FNP, and NZR. It should be noted that an NZR mechanism does not necessarily redistribute anything when $|N| = 2$. For example, the Bailey-Cavallo redistribution mechanism satisfies NZR, but does not redistribute anything when $|N| = 2$.

PROOF OF THEOREM 1. For the sake of contradiction, we assume there exists a mechanism M satisfying all the three properties simultaneously. Since false-name-proofness implies strategy-proofness, for the case of $k = 2$, M cannot return any money to the loser for any valuation profile $v \in V^2$ (Lemma 1). However, Lemma 2 states that mechanism M must return a non-zero amount of money to the loser for at least one valuation profile $v \in V^2$. This is a contradiction. □

Thus, if we require FNP, then we cannot have both NZR and AE. Our mechanisms satisfying both NZR and FNP involve reserve prices, so their worst-case allocative efficiency and social welfare (counting payments) are both zero. Mechanisms satisfying both AE and FNP are efficient, but they cannot redistribute anything. Thus, their worst-case social welfare is also zero. On the contrary, if we only require SP (rather than FNP), then there are many redistribution mechanisms that achieve perfect allocative efficiency and almost perfect social welfare for large numbers of agents.

As a conclusion to this section, we show the "tightness" of the impossibility result by presenting three mechanisms, each of which satisfies two of the three properties.

EXAMPLE 1 (FNP, AE, BUT NOT NZR). *Consider the Vickrey auction without any redistribution.*

EXAMPLE 2 (AE, NZR, BUT NOT FNP). *Consider (e.g.) the Bailey-Cavallo mechanism proposed in [1, 2].*

EXAMPLE 3 (FNP, NZR, BUT NOT AE). *Consider a modified Vickrey auction with a reserve price, so that only when there are two agents, some money is given to the loser, and no redistribution is made otherwise.*

Since our main objective in this paper is to find false-name-proof redistribution mechanisms, we must abandon satisfying allocative efficiency. However, although it satisfies both FNP and NZR, the third example seems poor in terms of social welfare. In the next section, we seek a class of false-name-proof redistribution mechanisms that outperforms Example 3 above.

4. FALSE-NAME-PROOF REDISTRIBUTION MECHANISMS

We now introduce a class of false-name-proof redistribution mechanisms that we call Exponentially Decreasing Redistribution (EDR) mechanisms.

DEFINITION 5 (EDR). *Mechanism $M = (f, p)$ is an* exponentially decreasing redistribution (EDR) *mechanism if we can find a pair of sequences $(c_k)_{1 \leq k \leq |\mathcal{N}|}$ and $(r_k)_{1 \leq k \leq |\mathcal{N}|}$ satisfying (i) $c_1 = r_1 = 0$, (ii) $c_2 \geq 0$, (iii) $\forall k \geq 3, 0 \leq c_k \leq \frac{1}{2}c_{k-1}$, and (iv) $\forall k \geq 2, r_k = r_{k-1} + 2c_k$ such that for any $N \subseteq \mathcal{N}$, and $v \in V^k$, and any $i, j \in N$,*

$$f_i(v) = \begin{cases} 1 & \text{if } v_i \geq \max\{\max_{j \neq i} v_j, r_k\} \\ 0 & \text{otherwise,} \end{cases}$$

$$p_i(v) = \begin{cases} r_k & \text{if } v_i \geq r_k > \max_{j \neq i} v_j \\ \max_{j \neq i} v_j - c_k & \text{if } v_i \geq \max_{j \neq i} v_j \geq r_k \\ -c_k & \text{if } \max_{j \neq i} v_j \geq \max\{v_i, r_k\} \\ 0 & \text{otherwise.} \end{cases}$$

Each parameter r_k indicates a *reserve price*, a well-known concept in auction theory. When all agents bid under the reserve price, the object is not sold and all agents get zero utility. The mechanism starts by running the Vickrey auction, using r_k as the reserve price when the number of attending agents is k. Each parameter c_k indicates the amount of money redistributed to each agent when there are k bidders and the item is sold. This includes the winner when more than one agents bid over r_k.

Note that when there exist more than one bidders whose valuations are the highest, the mechanism uses a deterministic tie-breaking rule — the agent with the smallest index gets the object. This tie-breaking rule still keeps our proposed mechanism anonymous, because the winner and all the losers with the same valuation always get the same utility.

We first show that every EDR mechanism is strategy-proof. That is, no agent can improve her utility by misreporting her valuation.

PROPOSITION 1. *Any EDR mechanism is strategy-proof.*

PROOF. Suppose there are k agents and the others' bids are such that agent i with valuation v_i does not win the object when bidding truthfully. If she changes her bid in such a way that she still loses, then she receives the same amount c_k (or 0 in the case where no agent wins) of redistribution. On the other hand, if she overbids so that she wins, then her payment is at least $v_i - c_k$ (or $r_k \geq v_i$ in the case where no agent wins originally), by the definition of the mechanism. Hence, her utility cannot exceed c_k (or 0 in the case where no agent wins originally), which is the utility she receives from telling the truth.

Next consider the case where i does win when reporting truthfully. If she is the only agent whose valuation exceeds reserve price r_k, then she has no incentive to misreport, because the only way in which she can change her utility is by bidding below r_k, which will result in zero utility. On the other hand, if there exists some other agent whose bid v_j also exceeds r_k, then agent i pays $\max_{j \neq i} v_j - c_k$. Since $v_i \geq \max_{j \neq i} v_j$, her utility is at least c_k, which is the utility she would get from underbidding in a way that makes her lose. □

Using this proposition, we can show that any EDR mechanism satisfies FNP. In the proof, we also use two lemmas that are given in the appendix.

Theorem 2. *Any EDR mechanism is false-name-proof.*

Proof. The proof is obtained by combining Proposition 1 and Lemmas 3 and 4 in the appendix.

First, we observe that for any fixed valuation profile v_{-i} and any false-name manipulation (v'_i, v_S) (by agent i with true valuation v_i, using a set of identities $S \cup \{i\}$ and misreport v'_i of her own valuation), we can always find misreport v''_i (without using false names) that gives the same allocation to manipulator i,

$$\sum_{l \in S \cup \{i\}} f_l(v'_i, v_{-i}, v_S) = f_i(v''_i, v_{-i}).$$

Therefore, by Lemmas 3 and 4, it always holds that

$$p_i(v''_i, v_{-i}) \leq \sum_{l \in S \cup \{i\}} p_l(v'_i, v_{-i}, v_S),$$

regardless of whether i obtains the item or not. Hence, the simple misreport of v''_i is at least as beneficial to i as the false-name manipulation (v'_i, v_{-i}), because they result in the same allocation for i but the former has a (weakly) lower payment.

Furthermore, by Proposition 1, for agent i, misreporting her valuation as v''_i is not beneficial:

$$v_i \cdot f_i(v) - p_i(v) \geq v_i \cdot f_i(v''_i, v_{-i}) - p_i(v''_i, v_{-i}).$$

Combining these three equations/inequalities, we have

$$\begin{aligned} v_i \cdot f_i(v) - p_i(v) &\geq v_i \cdot f_i(v''_i, v_{-i}) - p_i(v''_i, v_{-i}) \\ &\geq v_i \cdot \sum_{l \in S \cup \{i\}} f_l(v'_i, v_{-i}, v_S) - \sum_{l \in S \cup \{i\}} p_l(v'_i, v_{-i}, v_S), \end{aligned}$$

which coincides with the definition of FNP. □

We demonstrate the behavior of an EDR mechanism on the tennis court example and illustrate its robustness to false-name manipulations.

Example 4. *Consider the same problem setting described in Section 1, and apply an EDR mechanism with $c_2 = 4$, $c_3 = 2$, $c_4 = 1$, $r_2 = 8$, $r_3 = 12$, and $r_4 = 14$.*

When only three teams, A, B and C, participate in the EDR mechanism, A gets the right to use the tennis court at price \$10 = \$12 − \$2, and the other teams receive \$2 each. Next, consider the situation where team C uses two identifiers, team C and team D.

Since there are 4 participants from the mechanism's perspective, *it uses parameters r_4 and c_4. Thus, the sum of the redistributions to teams C and D becomes \$2, which is the same as the original redistribution to team C.*

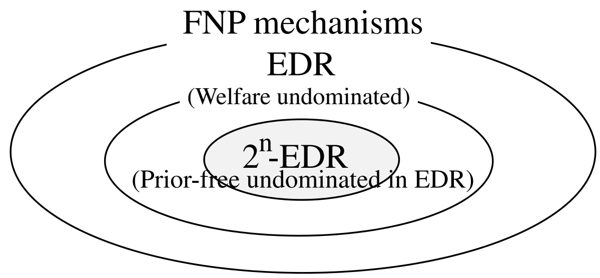

Figure 2: Welfare/prior-free dominance relations

Algorithm 1 Obtaining an EDR Mechanism $((c_k)_{1 \leq k \leq |\mathcal{N}|}, (r_k)_{1 \leq k \leq |\mathcal{N}|})$ which Welfare Dominates a Given FNP Mechanism $M' = (f', p')$.

Init: $c_1 = r_1 = 0$.
for $k = 2, \ldots, |\mathcal{N}|$ **do**
  $c_k \leftarrow \frac{1}{2} \max_{\{i,j\} \in N, v \in V^k} \sum_{l \in \{i,j\}} (-p'_l(v) + f'_l(v) \cdot \max_{l' \in N \setminus l} \{r'_k, v_{l'}\})$
  $r_k \leftarrow r_{k-1} + 2c_k$
end for
return $((c_k)_{1 \leq k \leq |\mathcal{N}|}, (r_k)_{1 \leq k \leq |\mathcal{N}|})$

The class of EDR mechanisms includes the Vickrey auction as a special case, which can be obtained by setting $c_k = r_k = 0$ for all k. However, the Vickrey auction obviously violates NZR. The next result shows a necessary and sufficient condition for an EDR mechanism to satisfy NZR.

Theorem 3. *An EDR mechanism satisfies non-zero redistribution if and only if it satisfies $c_2 > 0$.*

Proof. First we show the "if" part. When $c_2 > 0$, we can find at least one valuation profile $v \in V^2$ such that the loser gets a non-zero amount of redistribution. For instance, consider $N = \{1, 2\}$ and $v = (v_1, v_2) = (2c_2, 0) \in V^2$. Then agent 1 wins the object and pays $2c_2$ (because $r_2 = r_1 + 2c_2 = 2c_2$), and agent 2 receives c_2. Hence, the EDR mechanism satisfies the NZR condition.

To prove the "only if" part: if $c_2 = 0$, then the corresponding EDR mechanism coincides with the Vickrey auction, which violates NZR. □

As a corollary, we can obtain the following result.

Corollary 1. *An EDR mechanism allocates efficiently if and only if $c_2 = 0$.*

5. OPTIMALITY OF PROPOSED MECHANISMS

In this section, we show the "optimality" of the proposed mechanisms among all false-name-proof mechanisms. By introducing a binary relation called *welfare dominance* over the set of all possible mechanisms, we characterize EDR mechanisms by the property of being *welfare undominated by other FNP mechanisms.*

To define the relation, we first introduce the notion of social welfare with respect to a given valuation profile.

Definition 6 (Social Welfare). *For a given mechanism M, a set of agents $N \subseteq \mathcal{N}$, and a valuation profile*

$v \in V^k$, *let*

$$SW(M, v) := \sum_{i \in N} \left[v_i \cdot f_i(v) - p_i(v) \right]$$

be the social welfare *of mechanism M with respect to valuation profile v.*

We then formally define the welfare dominance relation.

Definition 7 (Welfare Dominance). *Mechanism $\tilde{M}$ is said to* welfare dominate *mechanism M (or $\tilde{M} \xrightarrow{WD} M$) if for any $N \subseteq \mathcal{N}$ and any $v \in V^k$,*

$$SW(\tilde{M}, v) \geq SW(M, v).$$

That is, mechanism $\tilde{M}$ welfare dominates mechanism M if $\tilde{M}$ *always* has a (weakly) higher social welfare than M. We can easily observe from the definition that the relation over the set of all possible mechanisms is transitive and antisymmetric: for any three mechanisms M, M', M'', $M \xrightarrow{\text{WD}} M'$ and $M' \xrightarrow{\text{WD}} M''$ implies $M \xrightarrow{\text{WD}} M''$ (transitivity), and for any two mechanisms M, M', having both $M \xrightarrow{\text{WD}} M'$ and $M' \xrightarrow{\text{WD}} M$ implies $M' = M$ (antisymmetry). Here, we consider two mechanisms "equal" if they provide the same social welfare on every profile.

For given FNP mechanism $M' = (f', p')$, Algorithm 1 returns a pair of sequences $(c_k)_{1 \leq k \leq |\mathcal{N}|}$ and $(r_k)_{1 \leq k \leq |\mathcal{N}|}$ corresponding to an EDR mechanism M that welfare dominates M'. (Intuitively, the right-hand side of Step 3 in Algorithm 1 corresponds to the maximal average redistribution that an agent could obtain using two identities. When one of these two identities is winning the item, we need to subtract from their total payment the Vickrey-with-reserve payment, which is not counted as redistribution.) By utilizing this algorithm, we show that the EDR mechanisms are the only false-name-proof mechanisms that are not welfare dominated by any other mechanism. In other words, the EDR mechanisms form a kind of "stable set," in the sense that there is no welfare dominance relation between any two EDRs, and any FNP mechanism is welfare dominated by an EDR. Fig. 2 illustrates dominance relations among FNP mechanisms.

Theorem 4. *The class of EDR mechanisms consists exactly of all the FNP mechanisms that are not welfare dominated by any other FNP mechanism.*

The theorem is proven in Propositions 2 and 3, which correspond to "external stability" and "internal stability," respectively.

Proposition 2. *For any given FNP mechanism M', there exists an EDR mechanism M that welfare dominates M'.*

Proof. For any given FNP mechanism M', Algorithm 1 returns EDR mechanism M. We proceed by mathematical induction. Here, c_k^{sum} denotes the sum of the redistributions when the number of attending agents is k. (1) If $k = 1$, $r_1 = c_1^{\text{sum}} = 0$ holds in mechanism M. Since mechanism M' satisfies $r'_1 \geq r_1 = 0$ and $c'^{\text{sum}}_1 = 0$ by SP, M gets (weakly) higher social welfare than M'.

(2) If $k = 2$, because M' is FNP, an agent cannot increase her utility by increasing the number of attending agents from 1 to 2 using two identifiers. Thus, we obtain $r'_2 \geq r'_1 + 2c_2 \geq 2c_2 = r_2$, or, $r'_2 \geq r_2$. We next consider the difference between the sum of the redistributions in M and in M'. For mechanism M with $v \in V^2$, we have $c_2^{\text{sum}} = 2c_2$. For mechanism M' with $v \in V^2$, we have $c'^{\text{sum}}_2 \leq 2c_2$. Thus, we obtain $c_2^{\text{sum}} - c'^{\text{sum}}_2 \geq 0$. As a result, we have $r'_2 \geq r_2$ and $c_2^{\text{sum}} \geq c'^{\text{sum}}_2$ and thus we see that M obtains higher social welfare than M'.

(3) We assume by induction that for $k = k'-1$ ($k' \geq 3$), we have $SW(M, v) \geq SW(M', v)$, implying that $r'_{k'-1} \geq r_{k'-1}$. When we consider the case of $k = k'$ under this assumption, we obtain $r'_k \geq r_k$ and $c_k^{\text{sum}} \geq c'^{\text{sum}}_k$ in a similar manner as the case of $k = 2$. Therefore, we can obtain $M' \xrightarrow{\text{WD}} M$. □

Proposition 3. *No EDR mechanism is welfare dominated by any other FNP mechanism.*

Proof. We first show that no EDR mechanism is welfare dominated by another EDR mechanism. For a given pair of EDRs, M and M', let (c_k) and (c'_k) be the corresponding parameters, respectively. Then, we can find the minimum k such that $c_k \neq c'_k$; otherwise, these two mechanism coincide, i.e, $M = M'$. Let k^* be this minimum number and assume, without loss of generality, that $c_k > c'_k$. From the definition of EDR mechanisms, it holds that $r_k > r'_k$. Now, for a valuation profile $v \in V^{k^*}$ such that $v_i > r_{k^*}$ for some i, $\text{SW}(M, v) > \text{SW}(M', v)$ holds, because the former mechanism M redistributes more money to the agents than the latter mechanism M'. In contrast, for a valuation profile $v' \in V^{k^*}$ such that $r_{k^*} > v_i > r'_{k^*}$ for all i, $\text{SW}(M, v') < \text{SW}(M', v')$ holds, because the former mechanism cannot sell the object to any agent, but the latter can. These two inequalities prove that there is no welfare dominance relation between any two different EDRs.

To complete the proof, assume for the sake of contradiction that given EDR M, there exists FNP mechanism M' that welfare dominates M, i.e, $M' \xrightarrow{\text{WD}} M$. From the discussion above, M' is not described in Definition 5. Then, by Proposition 2, we construct an EDR M'' that dominates M', i.e., $M'' \xrightarrow{\text{WD}} M' \xrightarrow{\text{WD}} M$. If we assume that $M'' = M$ holds, antisymmetry implies that M' is an EDR mechanism, which is against the assumption. Therefore, considering only the case of $M'' \neq M$ is sufficient to complete the proof. By transitivity, we have $M'' \xrightarrow{\text{WD}} M$ for the two EDR mechanisms M'' and M, contradicting the above. □

An implication of the result is that, when we are only interested in welfare-maximizing FNP mechanisms, it suffices to focus only on EDR mechanisms, even if a mechanism designer has prior knowledge about the environments, such as the number of attending agents and their valuations. However, the class is still quite broad, and selecting one out of all EDRs is a difficult task for mechanism designers. In the next section we discuss some guidelines for doing so, without prior knowledge.

6. FURTHER ANALYSES

As discussed in the previous section, focusing on EDR mechanisms is without loss of generality if we are interested in welfare-maximization. Therefore, we introduce another binary relation *over EDR mechanisms* to find a smaller "stable set" in the class.

When a mechanism designer does not have any knowledge, it seems natural to treat all possible situations equally. We

Algorithm 2 Obtaining an 2^n-EDR which Prior-Free Dominates a Given EDR Mechanism $M' = (f', p')$.

Init: $c_1^* = r_1^* = 0$.
$c_2^* \leftarrow r'_{|\mathcal{N}|}/4$
$r_2^* \leftarrow r'_{|\mathcal{N}|}/2$
for $k = 3, \ldots, |\mathcal{N}|$ **do**
    $c_k^* \leftarrow \frac{1}{2} c_{k-1}^*$
    $r_k^* \leftarrow r_{k-1}^* + 2c_k^*$
end for
return $((c_k^*)_{1 \le k \le |\mathcal{N}|}, (r_k^*)_{1 \le k \le |\mathcal{N}|})$

introduce a new relation called *prior-free dominance*, which successfully structures a stable set. Furthermore, mechanisms among the stable set can be characterized to have (c_k) proportional to 2^{-k}.

DEFINITION 8 (PRIOR-FREE DOMINANCE). *A mechanism $\tilde{M}$ is said to* prior-free dominate *another mechanism M (or shortly $\tilde{M} \xrightarrow{PFD} M$) if $\forall N \subseteq \mathcal{N}$,*

$$[\exists v \in V^k, SW(M, v) > SW(\tilde{M}, v)] \Rightarrow [\exists v' \in V^k, SW(\tilde{M}, v') > SW(M, v')]$$

and $\exists N' \subseteq \mathcal{N}$,

$$\forall v \in V^{k(N')}, SW(\tilde{M}, v) > SW(M, v) \quad (1)$$

Intuitively, mechanism $\tilde{M}$ prior-free dominates another mechanism M if (i) for any set of agents, $\tilde{M}$ does not always "lose" to M, and (ii) for some set of agents, $\tilde{M}$ always "defeats" M. Mechanism M is *prior-free dominated* by mechanism $\tilde{M}$ if $\tilde{M}$ prior-free dominates M. The following is this section's main result, which gives the subclass of EDR mechanisms that consists exactly of the EDRs that are not prior-free dominated by any EDR mechanism.

THEOREM 5. *The class of EDR mechanisms whose corresponding pair of sequences $(c_k)_{1 \le k \le |\mathcal{N}|}$ and $(r_k)_{1 \le k \le |\mathcal{N}|}$ satisfies $\forall k \ge 3, c_k = \frac{1}{2} c_{k-1}$ are the only EDR mechanisms that are not prior-free dominated by any EDR mechanism.*

We refer to this class of mechanisms as 2^n-EDR mechanisms. Algorithm 2 returns a 2^n-EDR mechanism when an EDR mechanism M' is given. If a 2^n-EDR mechanism is given to Algorithm 2, it the same 2^n-EDR mechanism. Fig. 2 illustrates the relations among FNP, EDR, and 2^n-EDR mechanisms. This theorem is proven separately in Propositions 4 and 5.

PROPOSITION 4. *For a given non-2^n-EDR mechanism M', there exists a 2^n-EDR mechanism M^{*g} that prior-free dominates M'.*

PROOF. Here, we assume that EDR mechanism M' is of the form given in Definition 5 and that an 2^n-EDR mechanism M^* is generated by Algorithm 2. Since $r'_k = r'_{k-1} + 2c'_k$ and $c'_k \le \frac{1}{2} c'_{k-1}$ for any $k \ge 2$, it must holds for any EDR mechanism M' that $r'_2 \ge \frac{1}{2} r'_{|\mathcal{N}|}$, and thus for M^* with parameters (c_k^*) and (r_k^*), we have that for $k \ge 2$, $r'_k \ge r_k^*$. Furthermore, since sequence (c'_k) is decreasing (weakly) faster than (c_k^*), there exists a number $k^* \ge 2$ such that $c'_k > c_k^*$ for all $k < k^*$ and $c'_k \le c_k^*$ for all $k \ge k^*$. Importantly, when $k^* > 2$, we have that for any $k < k^*$, $r'_k > r_k^*$ holds, because both are EDR mechanisms.

As a result, it holds that for any $k < k^*$, $r'_k > r_k^*$ and $c'_k > c_k^*$, and for any $k \ge k^*$, $r'_k \ge r_k^*$ and $c'_k \le c_k^*$. Therefore, a valuation profile v' that gives a higher social welfare under M' than under M^* exists (i.e., the precondition of Eq. (1) holds) only when $k < k^*$. But in that case, we can always find a different valuation profile $v^* \in V^k$ satisfying $r_k^* < v_{(1)}^* < r'_k$, which gives a strictly higher social welfare under M^* than under M'. Thus Eq. (1) holds. Also, for any $k \ge k^*$, obviously there exists no valuation profile that gives a higher social welfare to M', because Eq. (1) also holds. As a result, $M^* \xrightarrow{\text{PFD}} M'$. □

PROPOSITION 5. *No 2^n-EDR mechanism is prior-free dominated by any EDR mechanism.*

PROOF. We denote by $\text{SW}(M', v)$ the social welfare of EDR mechanism M' and by $\text{SW}(M^*, v)$ the social welfare of 2^n-EDR mechanism M^*. We prove that it holds that $\forall N \subseteq \mathcal{N}$, $\exists v \in V^k$ s.t.: $\text{SW}(M', v) \le \text{SW}(M^*, v)$. It will suffice to show that any one of (a), (b), or (c) is satisfied: (a) $c_k^* > c'_k$, (b) $r_k^* \le r'_k$ and $c_k^* = c'_k$, or (c) $r_k^* < r'_k$ and $c_k^* < c'_k$. Thus, we prove this theorem by showing that at least one condition is satisfied for any k. We give a proof by mathematical induction.

(1) If $k = 1$, (b) is satisfied, since c'_k, c_k^*, r'_k, and r_k^* are 0.

(2) If $k = 2$, we obtain that $r_2^* = 2c_2^*$ and $r'_2 = 2c'_2$. Therefore, one of (a), (b), or (c) must be satisfied.

(3) By induction, we assume that at least one condition is satisfied for $k = k' - 1$ ($k' \ge 3$). If (a) is satisfied for $k = k' - 1$, we directly see that (a) is satisfied for $k = k'$, by Definition 5.

If (b) is satisfied for $k = k' - 1$, we obtain $c_{k'}^* \ge c'_{k'}$ from Definition 5 and the definition of 2^n-EDR mechanisms. If (a) is not satisfied for $k = k'$, $c_{k'}^* \le c'_{k'}$ holds. These inequalities imply $c_{k'}^* = c'_{k'}$. For the reserve price, because $c_{k'}^* = c'_{k'}$, we obtain $r_{k'}^* \le r'_{k'}$. Thus, (b) holds for $k = k'$.

Finally, if (c) is satisfied for $k = k' - 1$, then $r_{k'-1}^* < r'_{k'-1}$ holds. We assume that (c) is not satisfied for $k = k'$; we may assume this is so because $r_{k'}^* \ge r'_{k'}$ (otherwise, one of the other conditions holds). From Definition 5 and the definition of 2^n-EDR mechanisms, we obtain that $c_{k'}^* = 1/2 \cdot (r_{k'}^* - r_{k'-1}^*)$ and $c'_{k'} = 1/2 \cdot (r'_{k'} - r'_{k'-1})$. From this, we see that $c_{k'}^* > c'_{k'}$ (using that $r_{k'-1}^* < r'_{k'-1}$ and $r_{k'}^* \ge r'_{k'}$). Thus, we conclude that (a) holds for $k = k'$.

As a result, we prove that it holds that $\forall N \subseteq \mathcal{N}$, $\exists v \in V^k$ s.t. $\text{SW}(M', v) \le \text{SW}(M^*, v)$. □

7. CONCLUSION AND FUTURE WORK

In this paper, we first proved an impossibility result on FNP redistribution: under some natural assumptions, there exists no mechanism satisfying FNP and Pareto efficiency, except for the Vickrey auction without any redistribution. We then proposed a class of redistribution mechanisms, called EDR, each of which is FNP and welfare-undominated. After that, we refined the class of undominated mechanisms in a prior-free sense and we proposed a class of mechanisms, each of which is prior-free undominated. Future work will extend our results to more complicated environments. For one, it would be interesting to consider redistribution mechanisms with more than one object, such as multi-unit and combinatorial redistribution auctions. Another direction is to consider an online model of redistribution [14].

8. ACKNOWLEDGMENTS

Sakurai thanks JST PRESTO program. Yokoo thanks JSPS KAKENHI Grant Number 24220003.

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APPENDIX

LEMMA 3. *Consider an arbitrary EDR mechanism M and an arbitrary set of agents N. Suppose agent $i \in N$ wins the object when the set of agents N reports valuation profile $v \in V^k$. Further, suppose that agent i still wins when a set of agents $S \subseteq \mathcal{N} \setminus N$ also joins the market and reports $v_S \in V^{|S|}$. Then, it must be the case that for any $j \in N \setminus \{i\}$, $p_j(v) \leq \sum_{l \in S \cup \{j\}} p_l(v, v_S)$.*

PROOF. Let $(c_k)_{1 \leq k \leq |\mathcal{N}|}$ be the sequence of parameters of EDR mechanism M. From the definitions (i),(ii), and (iii) of an EDR mechanism, agent j receives $c_{k(N)}$ when the set of attending agents is N. Since all agents $l \in S \cup \{j\}$ lose when S joins the market, each agent $l \in S \cup \{j\}$ receives $c_{k(N \cup S)}$, so that the sum of their redistributions is $(|S|+1)c_{k(N \cup S)}$. By the definition of EDR mechanisms, it holds that $c_{k(N \cup S)} \leq c_{k(N)}/2^{|S|}\}$. Therefore, we have that $c_k \geq (|S|+1)c_{k(N \cup S)}$, implying that $p_j(v) \leq \sum_{l \in S \cup \{j\}} p_l(v, v_S)$. □

LEMMA 4. *Consider an arbitrary EDR mechanism M and an arbitrary set of attending agents N. Suppose agent $i \in N$ wins the object when the set of agents N reports valuation profile $v \in V^k$. Further, suppose that if set of agents $S \subseteq \mathcal{N} \setminus N$ also joins the market and reports $v_S \in V^{|S|}$, then agent $i' \in S \cup \{i\}$ wins the object. Then, it must be the case that $p_i(v) \leq \sum_{l \in S \cup \{i\}} p_l(v, v_S)$.*

PROOF. Let $(c_k)_{1 \leq k \leq |\mathcal{N}|}$ and $(r_k)_{1 \leq k \leq |\mathcal{N}|}$ be the sequences of parameters of the EDR mechanism. For any N and any $S \subseteq \mathcal{N} \setminus N$, by the definition of EDR mechanisms we have that $r_{k(N \cup S)} = r_k + 2\sum_{m=1}^{|S|} c_{k+m}$.

Let $i' \in S \cup \{i\}$ be the winning agent when S also joins the market, and let $v_{(2)}$ and $v_{(2)}^{+S}$ be the second-highest valuation when N is the set of agents and when $N \cup S$ is the set of agents, respectively. That is, $v_{(2)} = \max_{j \neq i} v_j$ and $v_{(2)}^{+S} = \max_{j' \in N \cup S \setminus \{i'\}} v_{j'}$. Then, there are four cases: (I) $r_k > v_{(2)}$ and $r_{k(N \cup S)} > v_{(2)}^{+S}$, (II) $r_k > v_{(2)}$ and $r_{k(N \cup S)} \leq v_{(2)}^{+S}$, (III) $r_k \leq v_{(2)}$ and $r_{k(N \cup S)} > v_{(2)}^{+S}$, and (IV) $r_k \leq v_{(2)}$ and $r_{k(N \cup S)} \leq v_{(2)}^{+S}$.

In (I), the LHS of the inequality that we seek to prove is r_k, and the RHS is $r_{k(N \cup S)} - |S|c_{k(N \cup S)}$. Because $r_{k(N \cup S)} = r_k + 2\sum_{m=1}^{|S|} c_{k+m}$ and the sequence (c_k) is nonincreasing with respect to k (for $k \geq 2$), so that $c_{k(N \cup S)} \leq c_{k+m}$ for any $S \subseteq \mathcal{N} \setminus N$ and any $m = \{1, \ldots, |S|\}$, we have $p_i(v) = r_k = r_{k(N \cup S)} - 2\sum_{1 \leq m \leq |S|} c_{k+m} \leq r_{k(N \cup S)} - |S|c_{k(N \cup S)} = \sum_{l \in S \cup \{i\}} p_l(v, v_S)$.

In (II), the LHS is the same as in (I), while the RHS is $v_{(2)}^{+S} - (|S|+1)c_{k(N \cup S)}$. Because $v_{(2)}^{+S} \geq r_{k(N \cup S)}$, we have $r_k = r_{k(N \cup S)} - 2\sum_{1 \leq m \leq |S|} c_{k+m} \leq r_{k(N \cup S)} - c_{k+1} - \sum_{1 \leq m \leq |S|} c_{k+m} \leq v_{(2)}^{+S} - (|S|+1)c_{k(N \cup S)}$. Thus, it holds that $p_i(v) = r_k \leq v_{(2)}^{+S} - (|S|+1)c_{k(N \cup S)} = \sum_{l \in S \cup \{i\}} p_l(v, v_S)$.

In (III), the LHS is $v_{(2)} - c_k$, and the RHS is $r_{k(N \cup S)} - |S|c_{k(N \cup S)}$. Since, when S also joins, the second-highest bid $v_{(2)}$ is still in the market and there exists no bid greater than $r_{k(N \cup S)}$, it must hold that $v_{(2)} \leq r_{k(N \cup S)}$. Moreover, as we stated in the proof of Lemma 3, $c_{k(N \cup S)} \geq \frac{1}{2^{|S|}} c_k$ for any $N \subseteq \mathcal{N}$ and $S \subseteq \mathcal{N} \setminus N$ ($k > 0$). Therefore, $p_i(v) = v_{(2)} - c_k \leq r_{k(N \cup S)} - c_k \leq r_{k(N \cup S)} - |S|c_{k(N \cup S)} = \sum_{l \in S \cup \{i\}} p_l(v, v_S)$.

Finally, in (IV), the LHS is $v_{(2)} - c_k$, and the RHS is $v_{(2)}^{+S} - (|S|+1)c_{k(N \cup S)}$. Clearly, $v_{(2)} \leq v_{(2)}^{+S}$. Furthermore, since $c_k \geq 2^{|S|} c_{k(N \cup S)}$ holds for any $N \subseteq \mathcal{N}$ and $S \subseteq \mathcal{N} \setminus N$, we obtain: $p_i(v) = v_{(2)} - c_k \leq v_{(2)}^{+S} - 2^{|S|} c_{k(N \cup S)} \leq v_{(2)}^{+S} - (|S|+1)c_{k(N \cup S)} = \sum_{l \in S \cup \{i\}} p_l(v, v_S)$. □