Fractional Hedonic Games

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ABSTRACT

An important issue in multi-agent systems is the exploitation of synergies via coalition formation. We initiate the formal study of *fractional hedonic games*. In fractional hedonic games, the utility of a player in a coalition structure is the average value he ascribes to the members of his coalition. Among other settings, this covers situations in which there are several types of agents and each agent desires to be in a coalition in which the fraction of agents of his own type is minimal. Fractional hedonic games not only constitute a natural class of succinctly representable coalition formation games, but also provide an interesting framework for network clustering. We propose a number of conditions under which the core of fractional hedonic games is non-empty and provide algorithms for computing a core stable outcome.

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; J.4 [**Computer Applications**]: Social and Behavioral Sciences - Economics

General Terms

Economics, Theory and Algorithms

Keywords

Game theory (cooperative and non-cooperative), teamwork, coalition formation, and coordination

1. INTRODUCTION

Hedonic games—as introduced by Drèze and Greenberg [15] and further explored by, e.g., Banerjee et al. [5], Bogomolnaia and Jackson [7], Elkind and Wooldridge [16], Gairing and Savani [19], Branzei and Larson [10], Aziz et al. [3, 4]—present a natural versatile formal framework to study the formal aspects of coalition formation. In hedonic games, coalition formation is approached from a game-theoretic angle. The outcomes are coalition structures—partitions of the players—over which the players have preferences. Moreover, the players have different individual or joint strategies at their disposal to affect the coalition structure to be formed. Various solution concepts—such as the *core*, the *strict core*, and various kinds of *individual stability*—have been proposed to analyze these games.

The characteristic feature of hedonic games is a non-externalities condition, which incorporates the useful but arguably simplifying assumption that each player's preferences over the coalition structures are fully determined by his preferences over the coalition he belongs to and do not depend on how the remaining players are grouped. Nevertheless, the number of coalitions a player can be a member of is exponential in the total number of players and the development and analysis of concise representations as well as interesting subclasses of hedonic game are an ongoing concern in computer science and game theory. Particularly prominent in this respect are representations in which the players are assumed to entertain preferences over the other players, which are then systematically lifted to preferences over coalitions [see e.g., 12, 3].

The work presented in this paper pertains to what we will call *fractional hedonic games*, a subclass of hedonic games in which each player is assumed to have cardinal utilities or values over the other players. These induce preferences over coalitions by considering the *average value* of the members in each coalition. The higher this value, the more preferred the respective coalition is. Fractional hedonic games belong to the succinct classes of hedonic games in which preferences over other agents are extended to preferences over coalitions in a natural way. Previously the min, max, and sum operators have been used respectively for *hedonic games based on worst players* [12], *hedonic games based on best players* [11], and *additively separable hedonic games* [see e.g., 4]. In this paper we use the average operator. Despite the natural appeal of fractional hedonic games, they have, to our knowledge, enjoyed surprisingly little attention.[1] Fractional hedonic games can be represented by a weighted directed graph where the weight of edge (i, j) denotes the value agent i has for agent j. Since the games have a natural graphical representation, we will show how desirable outcomes for fractional hedonic games also provide interesting and desirable ways to perform network clustering.

Natural economic scenarios can be modeled as fractional hedonic games. A particular economic setting that we will consider is what we refer to as *Bakers and Millers*. We assume there to be two types of players, for expositional rea-

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[1]Hajduková [20] first mentioned the possibility of using the average value of coalition members in hedonic games but did not further analyze this concept.

sons called bakers and millers, where individuals of the same type are competitors and those of different types can be seen as suppliers and purveyors in a situation of free movement of labor. Both types of players can freely choose the neighborhood, where to set up their enterprises. Millers want to be situated in a neighborhood with as many purchasing bakers relative to competing millers as possible, so as to keep prices high for the wheat they produce. On the other hand, bakers seek as high a ratio of the number of millers to the number of bakers as possible, so as to keep the price of wheat down and that of bread up. We find that these settings (which belong to the class of fractional hedonic games) always admit a core stable partition. This result is generalized to situations in which there are more than two types and agents want to keep the fraction of agents of their own type as low as possible. Our study of the Baker and Millers setting was inspired by Schelling's famous dynamic model of segregation [30, 31].

The contributions of the paper are as follows. First we formally define fractional hedonic games and show that fractional hedonic games bear negative existence and computational results. To be precise, the core of fractional hedonic games can be empty and even if the core is not empty, computing and verifying a core stable partition is NP-hard and coNP-complete, respectively. In light of such negative results, we then turn our attention to natural sub-classes of games that are based on unweighted and undirected graphs. We identify a number of classes of graphs for which the core of the corresponding hedonic game is non-empty. These include graphs with degree at most two, forests, multipartite complete graphs, bipartite graphs which admit perfect matchings, regular bipartite graphs, and graphs with girth at least five. For each of these classes we present algorithms to compute core stable partitions. We argue that the Bakers and Millers setting can be appropriately modeled as a fractional hedonic game based on complete bi-partite graphs. Generalizing this setting to complete multi-partite graphs, we find that the partition consisting of the grand coalition is generally core stable. Finally, we formulate a number of open problems and directions for future research and elaborate on how computing desirable outcomes in fractional hedonic games provides an interesting game-theoretic perspective to community detection [see e.g., 26, 18] and network clustering.[2]

2. PRELIMINARIES

Let N be a set $\{1, \dots, n\}$ of agents or players. For each $i \in N$, we let $\mathcal{N}_i$ denote the set $\{S \subseteq N : i \in S\}$ of coalitions i is a member of. With a slight abuse of terminology we refer to both the set N of all players and the coalition partition $\{N\}$ as the *grand coalition*. A *hedonic game* is a pair $(N, \succsim)$, where $\succsim = (\succsim_1, \dots, \succsim_n)$ is a profile of complete and transitive relations $\succsim_i$, modeling the preferences of the players. We use $\succ_i$ and $\sim_i$ to refer to the strict and indifferent parts of $\succsim_i$, respectively. If $\succsim_i$ is also anti-symmetric we say that i's preferences are *strict*. A coalition $S \in \mathcal{N}_i$ is *acceptable* to a player i if i prefers S to being alone, that is, $S \succsim_i \{i\}$ and *unacceptable*, otherwise.

By a *value function* for a player i we understand a function $v_i : N \to \mathbb{R}$ assigning a real value to each of the players.

[2] Clauset et al. [13] discuss how social network analysis can be used to identify clusters of like-minded buyers and sellers in `amazon`'s purchasing network.

Unless stated otherwise we will assume $v_i(i) = 0$. A value function v_i can be extended to a value function over coalitions such that for all $S \in \mathcal{N}_i$,

$$v_i(S) = \frac{\sum_{j \in S} v_i(j)}{|S|}.$$

A hedonic game $(N, \succsim)$ is now said to be a *fractional hedonic game* if for each player i in N there is a value function v_i such that for all coalitions $S, T \subseteq N$,

$$S \succsim_i T \text{ if and only if } v_i(S) \geq v_i(T).$$

A fractional hedonic game is said to be *symmetric* if $v_i(j) = v_j(i)$ and *simple* if $v_i(j) \in \{0, 1\}$. Simple fractional hedonic games have a natural appeal. Politicians may want to be in a party that maximizes the fraction of like-minded members and, for whatever reasons, people may want to be with as large a fraction of people of the same ethnic or social group. These situations can be fruitfully modeled as simple fractional hedonic games by having the agents assign value 1 to like-minded or otherwise similar people, and 0 to others.

A simple fractional hedonic game $(N, \succsim_i)$ can be represented by a *directed* graph (V, E) in which $V = N$ and $(i, j) \in E$ if and only if $v_i(j) = 1$. In a much similar fashion, if $(N, \succsim_i)$ is both symmetric and simple, it can be represented by an *undirected* graph (V, E) such that $V = N$ and $\{i, j\} \in E$ if and only if $v_i(j) = v_j(i) = 1$. The complete undirected graph on n vertices is denoted by K_n whereas an undirected cycle on n vertices is denoted by C_n. A graph (V, E) is said to be *k-partite* if V can be partitioned into k independent sets $V_1, \dots, V_k$, that is, $v, w \in V_i$ implies $\{v, w\} \notin E$. A k-partite graph is *complete* if for all $v \in V_i$ and $w \in V_j$ we have $\{v, w\} \in E$ if and only if $i \neq j$.

The outcomes of hedonic game are partitions of the players, or coalition structures. Given a partition $\pi = \{\pi_1, \dots, \pi_m\}$ of the players, $\pi(i)$ denotes the coalition in π of which player i is a member. We also write $v_i(\pi)$ for $v_i(\pi(i))$, which reflects the hedonic nature of the games we consider.

Hedonic games are analyzed using *solution concepts*, which formalize desirable or optimal ways in which the players can be partitioned (as based on the players' preferences over the coalitions). Among the most prominent solution concepts rank the *core* and the *strict core*, both of which formalize a concept of stability of partitions. We say that a coalition $S \subseteq N$ *(strongly) blocks* a partition π, if each player $i \in S$ strictly prefers S to his current coalition $\pi(i)$ in the partition π, that is, if $S \succ_i \pi(i)$ for all $i \in S$. A partition that does not admit a strongly blocking coalition is said to be in the *core*. In a similar vein, we say that a coalition $S \subseteq N$ *weakly blocks* a partition π, if each player $i \in S$ weakly prefers S to $\pi(i)$ and there exists at least one player $j \in S$ who strictly prefers S to his current coalition $\pi(j)$, that is, $S \succsim_i \pi(i)$ for all $i \in N$ and $S \succ_i \pi(j)$ for some $j \in S$. A partition does not admit a weakly blocking coalition is in the *strict core*.

EXAMPLE 1. *Consider the simple and symmetric fractional hedonic game based on the graph depicted in Figure 1. In the grand coalition, the utility of each player is $\frac{1}{2}$. There is only one core stable partition: $\{\{1, 2, 3\}, \{4, 5, 6\}\}$, which yields utility $\frac{2}{3}$ for each player. Observe that, when interpreted as an additively separable hedonic game, this is not a stable partition, as the grand coalition would yield a higher utility—namely, 3 instead of 2—to all and thus be a strongly blocking coalition.*

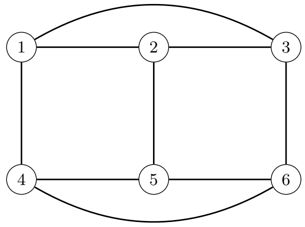

Figure 1: Example of a simple and symmetric fractional hedonic game. The only core stable partition is $\{\{1,2,3\},\{4,5,6\}\}$. In contrast, if the graph represents an additively separable hedonic game, then the partition consisting of the grand coalition is the only core stable partition.

3. RELATED WORK

Fractional hedonic games are related to *additively separable* hedonic games [see e.g., 4]. In both fractional hedonic games and additively separable hedonic games, each agent has a cardinal value for every other player. In additively separable hedonic games, utility in a coalition is derived by adding the values for the other players. By contrast, in fractional hedonic games, utility in a coalition is derived by adding the values for the other players and then dividing the sum by the total number of players in the coalition. Although conceptually additively separable and fractional hedonic games are similar, their formal properties are quite different. As neither of the two models is obviously superior, this shows how slight modeling decisions may affect the formal analysis. For example, in unweighted and undirected graphs, the grand coalition is trivially core stable for additively separable hedonic games. On the other hand, this is not the case for fractional hedonic games.

A fractional hedonic game approach to social networks with only non-negative weights may help detect like-minded and densely connected communities. In comparison, when the network only has non-negative weights for the edges, any reasonable solution for the corresponding additively separable hedonic game returns the grand coalition, which is not very discerning.

Recently, Olsen [27] has examined a variant of symmetric simple fractional hedonic games and investigated the computation and existence of Nash stable outcomes. In the games he considered, however, every maximal matching is core stable and every perfect matching is a best possible outcome even if there are large cliques present in the graph. By contrast, in our setting agents have an incentive to form large cliques.

Fractional hedonic games are different from but related to another class of hedonic games called *social distance games* introduced by Branzei and Larson [10]. In social distance games, an agent not only derives utility from agents he likes directly but also some utility from agents which are at smaller distances from him. In many situations, one does not derive an additional benefit from friends of friends and may in fact prefer to minimize the fraction of people one does not agree with or have direct connections with. In such scenarios, fractional hedonic games are more suitable than social distance games.

Fractional hedonic games also exhibit some similarity with the segregation and status-seeking models considered by Milchtaich and Winter [25] and Lazarova and Dimitrov [23]. Group formation models based on types were first considered by Schelling [30]. In Section 6, we argue why fractional hedonic games are helpful for network clustering.

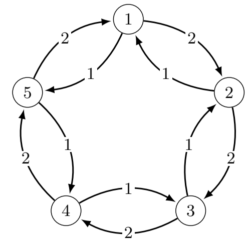

Figure 2: Fractional hedonic game used in the proof of Theorem 1. For all missing edges $\{i,j\}$ we have $v_i(j) = -10$.

Identifying sufficient and necessary conditions for the existence of stability in coalition formation has also been an active area of research [see e.g., 5, 7, 1, 2]. In this paper, we identify a number of conditions under which a core stable partition is guaranteed to exist in a particularly natural model of coalition formation.

Recently, and independently from our work, Feldman et al. [17] have also considered the hedonic games framework as an approach to graph clustering. However, their research does not relate to core and strict core stability. Moreover, they study different classes of hedonic games.

4. NEGATIVE RESULTS

In this section, we outline some negative results concerning the existence and computation of stable partitions. Firstly, we note that fractional hedonic games based on graphs that are both directed and weighted may have an empty core.

THEOREM 1. *For fractional hedonic games, the core can be empty.*

PROOF. There exists a five-player fractional hedonic game for which the core can be empty. Let $N = \{0,\dots,4\}$ and, assuming arithmetic modulo 5, the preferences over N of each player i be given by: $v_i(i+1) = 2$, $v_i(i-1) = 1$ and $v_i(j) = -10$ for $j \in \{i+2, i+3\}$. The game is illustrated in Figure 2. Then, each individually rational coalition consists of at most two players. It can be shown that no individually rational partition is core stable. $\square$

Computing a core stable partition turns out to be NP-hard, even if the core is guaranteed to be non-empty. The key idea in the proof below is that a subclass of fractional hedonic game is essentially equivalent to a subclass of additively separable hedonic games called *aversion-to-enemies games*.

THEOREM 2. *Computing a core stable partition for fractional hedonic games with symmetric preferences is NP-hard. Moreover, even checking whether a partition is core stable is coNP-complete.*

PROOF. Consider an undirected graph (V, E). This graph can represent an *additively separable hedonic game* or a *fractional hedonic game* in which the set of players N is equal to V and the valuations of players are defined as follows:

$v_i(j) = v_j(i) = 1$ if $\{i,j\} \in E$ and $v_i(j) = v_j(i) = -n$ if $\{i,j\} \notin E$. Let the resulting games be called G and G^*, respectively. We make two claims. Firstly, a coalition $S \in \mathcal{N}_i$ is unacceptable to i in G if and only if it is unacceptable to i in G^* and this is the case if S contains a player j such that $v_i(j) = -n$. Secondly, for any two acceptable coalitions $S, T \in \mathcal{N}_i$, $S \succsim_i T$ in G if and only if $S \succsim_i T$ in G^*. We already know that S does not contain a player j such that $v_i(j) = -n$. Therefore $S \setminus \{i\}$ consists of players $j \in N$ such that $v_i(j) = 1$. Therefore $S \succsim_i T$ in G iff $|S| \geq |T|$ iff $(|S|-1)/|S| \geq (|T|-1)/|T|$ iff $S \succsim_i T$ in G^*.

Based on these two claims, it follows that a partition is core stable in G^* if and only if it is core stable in G. Therefore any computational hardness results we have concerning core stability for symmetric *aversion-to-enemies games* also carry over to symmetric fractional hedonic games. It has previously been shown that the core of aversion-to-enemies games is non-empty, but computing a core stable partition is NP-hard even if $v_i(j) = v_j(i)$ for all $i, j \in N$ [14]. Furthermore, checking whether a partition is core stable is coNP-complete [32] for this class of games. □

It is worth observing that the grand coalition is not necessarily core stable, even in simple symmetric fractional hedonic games (see Example 1). Next, we point out that strict core can be empty.

THEOREM 3. *In simple symmetric fractional hedonic games, the strict core can be empty.*

PROOF. Consider the fractional hedonic game represented by a cycle of size five (C_5). For C_5, any coalition of size three or more admits a blocking coalition of size two. Even the partition consisting of one singleton and two coalitions of size two admits a weakly blocking coalition. □

5. POSITIVE RESULTS

In this section, we present a number of subclasses of simple symmetric fractional hedonic games for which the core is non-empty. Since these games can be represented by unweighted and undirected graphs, we will focus on different graph classes. In particular we show existence results for the following classes of graphs: graphs with degree at most two, forests, multi-partite complete graphs, bipartite graphs which admit perfect matchings, regular bipartite graphs, and graphs with girth at least five. All our proofs are constructive in the sense that we show that a core stable partition exists by outlining a way to construct a core stable partition.

5.1 Graphs with bounded degree

THEOREM 4. *For simple symmetric fractional hedonic games represented by graphs of degree at most 2, the core is non-empty.*

PROOF. We present a polynomial-time algorithm to compute a partition which is core stable. The partition is computed as follows. First keep finding K_3s until no more can be found. This takes time $\binom{n}{3}$. Let us call the set of vertices matched into K_3s as V_1. We remove V_1 from the graph along with E_1—the edges between vertices in V_1. We then repeat the procedure by deleting K_2s instead of K_3s. Let us call the set of vertices matched into pairs by V_2. In that case, $V \setminus (V_1 \cup V_2)$ are the unmatched vertices. The partition obtained is π.

In order to prove that π is core stable, consider the potential blocking coalitions. We know that vertices in V_1 cannot be in a blocking coalition because each vertex in V_1 is in one of its most favored coalitions. Also there does not exist a blocking coalition consisting solely of vertices from $V \setminus (V_1 \cup V_2)$. If this were the case, then we had not computed a maximal matching of $(V \setminus V_1, E \setminus E_1)$. Now let us assume that there exists a $v_2 \in V_2$ which is in a blocking coalition. Then the coalition is of the form $\{v_2, v_2', v_3\}$ where $v_3 \in V \setminus (V_1 \cup V_2)$ and $v_2, v_2' \in V_2$. If the utility of v_2 is greater than $1/2$, then the utility of v_2' is less than half. Since v_2' obtained utility $1/2$ in π, $\{v_2, v_2', v_3\}$ is not a blocking coalition. □

5.2 Forests

THEOREM 5. *For simple symmetric fractional hedonic games represented by undirected forests, the core is non-empty.*

PROOF. We present an algorithm to compute a core stable partition for an undirected tree. We can assume that the graph is connected—and therefore a tree—because if it were not, then the same algorithm for a tree could be applied to each connected component separately. Denote the graph representing the game by $G = (V, E)$. Pick an arbitrary vertex $v_0 \in V$ and run breadth first search on it. Let L_0 consist of v_0, L_1 of all the vertices at a distance of 1 from v_0, and L_k of all vertices at a distance of k from v_0. Let L_ℓ be the last layer of the tree. We construct a partition π, which we will later claim is core stable. Initialize π to the empty set. For each vertex v in the second last layer $L_{\ell-1}$ which has a child in the last layer L_ℓ, add the set $\{v\} \cup \{w : w \in L_\ell \text{ and } \{v,w\} \in E\}$ to π. Remove the sets of this form $\{v\} \cup \{w : w \in L_\ell \text{ and } \{v,w\} \in E\}$ from the tree and repeat the process until no more layers are left. The partition returned is π. The procedure terminates properly. In each iteration, the last layer of the tree is removed along with some or all the vertices of the second last layer. If a vertex is left alone, send it to a smallest coalition that one of its neighbors is a member of.

We now prove that π is core stable. For the base case, we show that no vertex from a coalition π consisting only the lowermost two layers, that is, L_ℓ and $L_{\ell-1}$, can be in a blocking coalition. If the vertex u in question is from the second last layer, then it will only be in a blocking coalition S if S contains u, all the children of u as well as the parent of u. But then S is not a blocking coalition for the children of u. For a leaf node v to be in a blocking coalition, it will need to be with its parent u but in a smaller coalition. But this means that u is not in a blocking coalition. We remove all vertices from coalitions only containing vertices from the last and second last layer and repeat the argument inductively. □

5.3 Complete k-partite graphs

In the introduction, we mentioned the *Bakers and Millers* setting, in which the players are of two different types. Each player likes the fraction of players of the other type as high as possible. The underlying intuition is that players of the same type are competitors whereas those of the other type purveyors. This idea is easily be extended to multiple types.

Let $\Theta = \{\theta_1, \dots, \theta_t\}$ be a set of types that partitions the set N of players, where $t = |\Theta|$. Let, furthermore, $\theta(i)$ denote the type player i belongs to. A hedonic game $(N, \succsim)$ is called a *Bakers and Millers game* if the preferences of each player i are such that for all coalitions $S, T \in \mathcal{N}_i$,

$$S \succsim_i T \quad \text{if and only if} \quad \frac{|S \cap \theta(i)|}{|S|} \leq \frac{|T \cap \theta(i)|}{|T|}.$$

Thus, each generalized Bakers and Millers setting with k types can be seen as a symmetric fractional hedonic games in which the players' preferences are given by a *complete k-partite graph* with the independent sets representing the types. More formally, it can easily be appreciated that a Bakers and Millers game can be modeled as a fractional hedonic game based on a graph (V, E), with $V = N$ and

$$E = \{\{i, j\} : \theta(i) \neq \theta(j)\}.$$

Within this setting it can easily be appreciated that the grand coalition is always strict core stable. Observe that for every coalition S we have

$$\sum_{1 \leq t' \leq t} \frac{|S \cap \theta_{t'}|}{|S|} = 1.$$

Now assume for contradiction that the grand coalition N is not core stable. Then, there is a coalition S such that $\frac{|S \cap \theta(i)|}{|S|} < \frac{|N \cap \theta(i)|}{|N|}$ for some $i \in S$ and $\frac{|S \cap \theta(j)|}{|S|} \leq \frac{|N \cap \theta(j)|}{|N|}$ for all $j \in S$. But then

$$\sum_{1 \leq t' \leq t} \frac{|S \cap \theta_{t'}|}{|S|} < \sum_{1 \leq t' \leq t} \frac{|N \cap \theta_{t'}|}{|N|},$$

that is, $1 < 1$, a contradiction. Generalizing this idea we obtain the following theorem.

Theorem 6. *Let $(N, \succsim)$ be a Bakers and Millers game with type space $\Theta = \{\theta_1, \dots, \theta_t\}$ and $\pi = \{S_1, \dots, S_m\}$ a partition. Then, π is strict core stable if and only if for all types $\theta \in \Theta$ and all coalitions $S, S' \in \pi$,*

$$\frac{|S \cap \theta|}{|S|} = \frac{|S' \cap \theta|}{|S'|}.$$

Proof. First assume that for all types $\theta \in \Theta$ and all coalitions S and S' in π we have $\frac{|S \cap \theta|}{|S|} = \frac{|S' \cap \theta|}{|S'|}$, but that a weakly blocking coalition T for π exists. Then, $\frac{|T \cap \theta(j)|}{|T|} \leq \frac{|\pi(j) \cap \theta(j)|}{|\pi(j)|}$ for all $j \in T$, while there is some $i \in T$ with $\frac{|T \cap \theta(i)|}{|T|} < \frac{|\pi(i) \cap \theta(i)|}{|\pi(i)|}$. Consider this i. Without loss of generality assume that $\theta_1, \dots, \theta_k$ are the types represented in T, that is, those types θ with $j \in \theta$ for some $j \in T$. By assumption $\frac{|\pi(j) \cap \theta(j)|}{|\pi(j)|} = \frac{|\pi(i) \cap \theta(j)|}{|\pi(i)|}$ for all $j \in T$. Hence,

$$\sum_{1 \leq k' \leq k} \frac{|T \cap \theta_{k'}|}{|T|} < \sum_{1 \leq k' \leq k} \frac{|\pi(i) \cap \theta_{k'}|}{|\pi(i)|}.$$

A contradiction follows as both

$$\sum_{1 \leq k' \leq k} \frac{|T \cap \theta_{k'}|}{|T|} = 1 \quad \text{and} \quad \sum_{1 \leq k' \leq k} \frac{|\pi(i) \cap \theta_{k'}|}{|\pi(i)|} \leq 1.$$

For the other direction, consider arbitrary $S, T \in \pi$ and assume for some type $\theta \in \Theta$ that $\frac{|S \cap \theta|}{|S|} > \frac{|T \cap \theta|}{|T|}$. Then, $S \cap \theta \neq \emptyset$. Let $i \in S \cap \theta$. As

$$\sum_{1 \leq t' \leq t} \frac{|S \cap \theta_{t'}|}{|S|} = \sum_{1 \leq t' \leq t} \frac{|T \cap \theta_{t'}|}{|T|},$$

there is some $\theta' \in \Theta$ with $\frac{|S \cap \theta'|}{|S|} < \frac{|T \cap \theta'|}{|T|}$. Thus, $T \cap \theta' \neq \emptyset$.

First consider the case in which both $S \cap \theta' = \emptyset$ and $T \cap \theta = \emptyset$. Without loss of generality, we may assume that $|S| \leq |T|$. Observe that $|S| < |T \cup \{i\}|$. The coalition $T \cup \{i\}$ is weakly blocking, as

$$\frac{|(T \cup \{i\}) \cap \theta|}{|T \cup \{i\}|} = \frac{|\{i\}|}{|T \cup \{i\}|} < \frac{|\{i\}|}{|S|} \leq \frac{|S \cap \theta|}{|S|}$$

and, for every type θ'' distinct from θ,

$$\frac{|(T \cup \{i\}) \cap \theta''|}{|T \cup \{i\}|} = \frac{|T \cap \theta''|}{|T \cup \{i\}|} \leq \frac{|T \cap \theta''|}{|T|}.$$

(The latter inequality is not strict, as $T \cap \theta''$ may be empty.)

Finally, assume without loss of generality, that $T \cap \theta \neq \emptyset$ and let $j \in T \cap \theta$. Since S and T are distinct and both in π, also $i \neq j$. We show that the coalition $T' = (T \setminus \{j\}) \cup \{i\}$ is weakly blocking. Consider an arbitrary type $\theta'' \in \Theta$. Observe that $|T| = |T'|$ and $|T \cap \theta''| = |T' \cap \theta''|$, whether $\theta'' = \theta$ or not. Therefore, $\frac{|T \cap \theta''|}{|T|} = \frac{|T' \cap \theta''|}{|T'|}$. Accordingly, every player $k \in T \setminus \{i, j\}$ is indifferent between T and T'. To conclude the proof, observe that $\frac{|T' \cap \theta|}{|T'|} = \frac{|\pi(j) \cap \theta|}{|T|}$. Hence,

$$\frac{|\pi(i) \cap \theta(i)|}{|S|} = \frac{|S \cap \theta|}{|S|} > \frac{|T \cap \theta|}{|T|} = \frac{|\pi(j) \cap \theta|}{|T|} = \frac{|T' \cap \theta|}{|T'|},$$

that is, $T' \succ_i S$, as desired. $\square$

Let d denote the greatest common divisor of $|\theta_1|, \dots, |\theta_t|$, which we know can be computed in time linear in t (cf. [8]). Theorem 6 can now be rephrased as follows: a partition π for a Bakers and Millers game is strict core stable if and only if for all coalitions S in π there is a positive integer k_S such that for all types θ_i we have $|S \cap \theta_i| = k_S|\theta_i|/d$. Thus, for the grand coalition N we have $k_N = d$. There is also a partition π such that $k_S = 1$ for all coalitions S in π. It can readily be appreciated that this π is strict core stable and that there is no finer one with the same property. We say that two partitions π and π' are *identical up to renaming players of the same type* if there is a bijection $f \colon N \to N$ such that for all players i we have $\theta(i) = \theta(f(i))$ and $\pi' = \{\{f(i) : i \in S\} : S \in \pi\}$. Hence, we have the following corollary.

Corollary 1. *For every Bakers and Millers game, there is a* unique *finest strict core stable partition (up to renaming players of the same type), which, moreover, can be computed in linear time.*

As strict core stability implies core stability, the "if"-direction of Theorem 6 also holds for the core. That is, partition π is core stable whenever $\frac{|S \cap \theta|}{|S|} = \frac{|S' \cap \theta|}{|S'|}$ for all coalitions S, S' in π. The inverse of this statement does not generally hold. Consider three players 1, 2, and 3, with 1 belonging to type θ_1, while 2 and 3 belong to type θ_2. Then, the partition $\{\{1, 2\}, \{3\}\}$ is core stable but not strict core stable: coalition $\{1, 3\}$ is weakly blocking.

5.4 Bipartite graphs

Bipartite graphs constitute one of the most natural classes for which it is unknown whether the core is generally non-empty. Still, for certain subclasses of bipartite graphs, Hall's Theorem [21] can be leveraged to obtain positive results regarding this issue. First, we have the following lemma.

Lemma 1. *For every fractional hedonic game that is represented by an undirected bipartite graph admitting a perfect matching the core is non-empty.*

Proof. Let $\{N', N''\}$ be the respective bipartition of N. For every coalition $S \subseteq N$, either $\frac{|N' \cap S|}{|S|} \leq \frac{1}{2}$ or $\frac{|N'' \cap S|}{|S|} \leq \frac{1}{2}$. Hence, every coalition S contains at least one player i with $v_i(S) \leq \frac{1}{2}$. In a perfect matching, considered as a partition, every player has value $\frac{1}{2}$. Hence, every perfect matching is core stable and the claim follows. $\square$

The following two results are corollaries of Hall's Theorem.

Corollary 2. *For every graph with bipartition $\{N', N''\}$ with $|N(S)| \geq |S|$ for all coalitions S in N', the core of the corresponding fractional hedonic game is non-empty.*

Corollary 3. *For all bipartite k-regular graphs the core of the corresponding fractional hedonic game is non-empty.*

5.5 Graphs with large girth

The *girth* of a graph is the length of the shortest cycle in the graph. Graphs with a girth of at least five do not admit triangles or cycles of length four. We say that two vertices v and w have a neighbor in common in (V, E) if either $\{v, w\} \in E$ or there is some $u \in V$ such that $\{u, v\}, \{u, w\} \in E$. We have the following lemma.

Lemma 2. *Let (V, E) be a graph with $|V| \geq 3$. Then, (V, E) has girth of at least five if and only if all $v, w \in V$ have at most one neighbor in common.*

Proof. For the if direction, assume that (V, E) contains a cycle of length three or four. In either case, it is easy to find vertices that have at least two neighbors in common. For the only-if direction, assume that there are $v, w \in V$ that have more than one neighbor in common. That is, either $\{v, w\} \in V$ and there is some $u \in V$ such that $\{u, v\}, \{u, w\} \in V$ or there are $u, u' \in V$ such that $\{u, v\}, \{u, w\}, \{u', v\}, \{u', w\} \in V$. If the former, the graph has girth of at most three. If the latter, the graph's girth is at most four. $\square$

The key idea behind the following result is to pack the vertices of the graph representing a fractional game into stars while maximizing a particular objective function.

Theorem 7. *For simple symmetric fractional hedonic games represented by graphs with girth at least five, the core is non-empty.*

Proof. We first introduce the more general notion of graph packing. Let $\mathcal{F}$ be a set of undirected graphs. An *$\mathcal{F}$-packing* of a graph G is a subgraph H of G such that each component of H is isomorphic to a member of $\mathcal{F}$. The components of an $\mathcal{F}$-packing H can be seen as coalitions, and thus $\mathcal{F}$-packings naturally induce a coalition partition, with each vertex not contained in a connected component forming a singleton coalition. We will consider *star-packings* of graphs, that is, $\mathcal{F}$-packings with $\mathcal{F} = \{S_2, S_3, S_4, \dots\}$ such that each S_i is a star with i vertices. Each star S_i with $i > 2$ has one center c and $i-1$ leaves $\ell_1, \dots, \ell_{i-1}$. We assume S_2 to have two centers and no leaves.

With each star packing, denoted by π, we associate an *objective vector* $\vec{x}(\pi) = (x_1, \dots, x_{|V|})$ such that $x_i \leq x_j$ if $1 \leq i \leq j \leq |V|$, and there is a bijection $f\colon V \to \{1, \dots, |V|\}$ with $u_v(\pi) = x_{f(v)}$. Thus, in $\vec{x}(\pi)$ the vertices/players are ordered according to their value for π in ascending order. We assume these objective vectors to be ordered lexicographically by $\geq$, e.g., $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \geq \left(0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$ but not vice versa. The goal is to compute a star packing that maximizes its objective vector. Intuitively, this balances the sizes of the stars in the star packing without leaving vertices needlessly on their own. Also see Figure 3 for an illustration.

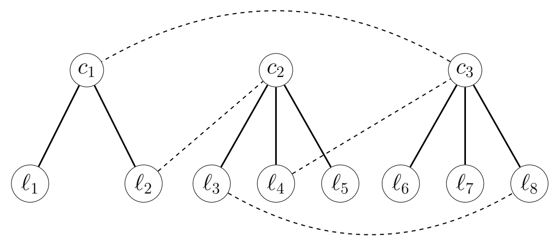

Figure 3: A graph with girth 5 and a star packing indicated by the solid edges. This star packing does not have an optimal objective vector: a better one would result if ℓ_3 and ℓ_8 were to form a star. Observe that $\{\ell_3, \ell_8\}$ is here also a strongly blocking coalition.

Obviously, star packings minimizing the objective are guaranteed to exist and in the remainder of the proof we argue that such star packings are core stable.

To this end, let π be a star packing of a graph (V, E) that maximizes the objective vector. Observe that a vertex v has utility 0 under π if and only if v has no neighbors in (V, E). In that case v has utility 0 in every partition and in every coalition. For contradiction assume that there is a coalition S strongly blocking π. Obviously, S contains no isolated vertices, as these would obtain utility 0 and thus not be strictly better off joining S. Therefore, S consists entirely of vertices that are either centers or leaves of π.

Also observe that, for any two leaves ℓ, ℓ' in π we have $\{\ell, \ell'\} \notin E$. For a contradiction assume the opposite. Then, ℓ and ℓ' come from different centers, otherwise (V, E) would contain a triangle. Moreover, $\pi' = \{\{\ell, \ell'\}, \pi'_1 \dots, \pi'_k\}$, where $\pi'_i = \pi_i \setminus \{\ell, \ell'\}$, is a star packing for which the objective vector is larger than the one for π.

Now three cases can be distinguished: (i) S contains no centers of π, (ii) S contains exactly one center of π and (iii) S contains more than one center of π.

If (i), S only contains leaves of π, between which we know there are no edges. Hence, every member of S has utility 0 and S cannot be blocking.

If (ii), we show that $\vec{x}(\pi)$ is not optimal. Let S consists of one center c and m leaves $\ell_1, \dots, \ell_m$ of π. Since no leaves in π are neighbors, some reflection reveals that S is a star with c as center and $\ell_1, \dots, \ell_m$ as leaves. Let ℓ denote one of the leaves and c' the center of π such that $\ell \in \pi(c')$. Consider the partition π' such that

$$\pi'(k) = \begin{cases} \pi(c) \cup \{\ell\} & \text{if } k \in \pi(c) \cup \{\ell\}, \text{ and} \\ \pi(k) \setminus \{\ell\} & \text{otherwise.} \end{cases}$$

We claim that $\vec{x}(\pi') > \vec{x}(\pi)$, contradicting our initial assumption. Observe that it suffices to prove that (a) $u_\ell(\pi') > u_\ell(\pi)$ and (b) $u_k(\pi') \geq u_\ell(\pi')$ for all k with $u_k(\pi') < u_k(\pi)$.

For (a), observe that if $u_c(\pi) < u_c(S)$ and c is a center

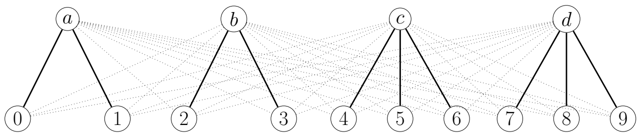

Figure 4: A complete bipartite graph in which no star packing yields a stable partition. For instance, the partition indicated by the solid edges is not stable as $\{a, b, 4, 5, 6, 7, 8\}$ would deviate.

in both π and S, then $\frac{|\pi(c)|-1}{|\pi(c)|} < \frac{|S|-1}{|S|}$. Moreover, $u_\ell(\pi) < u_\ell(S)$, that is, $\frac{1}{|\pi(\ell)|} < \frac{1}{|S|}$. Accordingly, $|\pi(c)| < |S| < |\pi(\ell)|$. It follows that $|\pi'(\ell)| = |\pi(c) \cup \{\ell\}| \leq S < |\pi(\ell)|$ and thus $u_\ell(\pi') > u_\ell(\pi)$.

For (b), let k be such that $u_k(\pi') < u_k(\pi)$. Then either $k = c'$ or $k \in \pi(c) \setminus \{c\}$. As c' is a center and ℓ a leaf in π, c' still is a center in π'. Hence, $u_{c'}(\pi') \geq \frac{1}{2}$. Moreover, ℓ is also a leaf in π and thus $u_\ell(\pi') < \frac{1}{2}$, proving the case. Now assume that $k \in \pi(c) \setminus \{c\}$. Then, with k and ℓ being both leaves in $\pi'(c)$, $u_k(\pi') = u_\ell(\pi')$.

If (iii), assume that S contains at least two centers c and c' in π. Then, $u_c(\pi) \geq \frac{1}{2}$ and $u_{c'}(\pi) \geq \frac{1}{2}$. Either $|S| = 2k + 2$ or $|S| = 2k + 3$ for some $k \geq 1$. As both $u_c(S) > \frac{1}{2}$ and $u_{c'}(S) > \frac{1}{2}$, also $|\{i \in S : \{c, i\} \in E\}| \geq k + 2$ and $|\{i \in S : \{c', i\} \in E\}| \geq k + 2$. It follows that c and c' must have at least two neighbors in common, contradicting Lemma 2. This completes the proof. $\square$

It is worth observing that, even if a core-stable partition by stars exists, there may still be other stable partitions. In a 3-star, for instance, letting the center and two leaves form a coalition is core stable. Furthermore, it is an interesting question whether our constructive method to compute a core stable partition can be implemented in polynomial time using known algorithms for packing graphs into stars.

It also be noted that for fractional hedonic games on *general* bipartite graphs it is not always the case that there are star-packings that also yield core stable partitions. Consider, e.g., the complete bipartite graph with 14 vertices with bipartition $\{\{a, b, c, d\}, \{i : 0 \leq i \leq 9\}\}$, as depicted in Figure 4. Still, by virtue of Theorem 6 the corresponding fractional hedonic game allows for stable partitions, e.g., the grand coalition. Hence, any method of finding core stable partitions in games on bipartite graphs based on finding star-packings is bound to fail.

6. NETWORK CLUSTERING

The analysis of simple fractional hedonic games is also related to social network analysis, which aim is to identify viable clustering of vertices [9]. Particularly relevant within this line of research is the work on identifying and detecting communities in networks [see e.g., 26, 29]. In network clustering, the main approaches for identifying "good" clusters is to minimize intra-cluster density measures or to maximize inter-cluster density measures. Such global parameters, however, do not take into account the incentives members of the network may have to from a (blocking) coalition and deviate from their respective clusters. Papadimitriou [28, Section 7] identified this as an interesting line of research to pursue. Hedonic games offer a promising formal framework to capture the dynamic nature of cluster formation as based on the members' incentives.

Apart from core stability, there are also other solution concepts for hedonic games that may be suitable for network clustering/community detection, e.g., finding the partitions that maximize the sum of agent utilities (utilitarian welfare) or maximize the utility of the worst off agent (egalitarian welfare). This perspective is not only more fine-grained than local-density measures such as *'maximum cliques', 'plexes and their duals', 'dense subgraphs'* etc. [Chapter 6, 9], it also captures the rationale behind bi-criteria optimization criteria such as maximizing intra-cluster density and minimizing inter-cluster density [29]. It can achieve both objectives simultaneously via a single criterion such as maximizing social welfare. In a similar way as the Shapley value of a suitably defined game based on a graph has been used to propose new centrality indices [see e.g., 24], computing the maximum welfare partition of the fractional hedonic game represented by the graph appears to be an attractive and fundamental way of performing network clustering.

The field of *data clustering* has some relation with network clustering. In contrast to network clustering, however, the edge weight/distance between two points that plays such an important role in data clustering represents a level of difference rather than one of similarity or affinity. In a seminal paper on data clustering, Kleinberg [22] emphasized the need for an axiomatic theory of clustering that is independent of specific algorithms, objective functions or data models. He defined three natural axioms—namely, *scale invariance*, *richness*, and *consistency*—and proved that no clustering method can simultaneously satisfy all three of them. Later Ben-David and Ackerman [6] showed that consistency can be slightly weakened so as to obtain a consistent set of axioms. One can define analogous axioms that are appropriate for *network clustering*. It turns out that, when the network is modeled as a fractional hedonic game, the analogues of the axioms of scale invariance, richness, and consistency as in the paper by Ben-David and Ackerman [6] are satisfied by both the core and the maximum welfare measure. Note that, if the graph is unweighted, additively separable hedonic games are not useful for network clustering, because the partition consisting of the grand coalition is always optimal.

7. CONCLUSIONS

We initiated the study of core stable partitions in fractional hedonic games. The model promises to be of interest in both network clustering and coalition formation. We focussed on core stable partitions and identified a number of classes of graphs for which the core is non-empty. Interestingly, it remains open whether simple fractional hedonic games always admit core stable partitions. The problem appears to be quite challenging as we have not even been able to prove non-emptiness of the core for bipartite graphs.

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