Łukasiewicz Games

Enrico Marchioni
Institut de Recherche en
Informatique de Toulouse
Université Paul Sabatier, France
enrico.marchioni@irit.fr

Michael Wooldridge
Dept of Computer Science
University of Oxford, UK
mjw@cs.ox.ac.uk

ABSTRACT

Boolean games provide a simple, compact, and theoretically attractive abstract model for studying multi-agent interactions in settings where players will act strategically in an attempt to achieve personal goals. A standard critique of Boolean games, however, is that the binary nature of goals (satisfied or unsatisfied) inevitably trivialises the nature of such strategic interactions: a player is assumed to be indifferent between all outcomes that satisfy his goal, and indifferent between all outcomes that do not satisfy his goal. In this paper, we introduce Łukasiewicz Games, which overcome this limitation by considering goals to be specified using finitely-valued Łukasiewicz logics. The significance of this is that formulae of Łukasiewicz logic can express every continuous piecewise linear polynomial function with integer coefficients over $[0,1]^n$, thereby allowing goal formulae in Łukasiewicz Games to naturally express a much richer range of utility functions. After introducing the formal framework of Łukasiewicz Games, we present a number of detailed worked examples to illustrate the framework, and then investigate some of the properties of Łukasiewicz Games. In particular, after investigating the complexity of decision problems in Łukasiewicz Games, we give a logical characterisation of the existence of Nash equilibria in such games.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems
I.2.4 [**Knowledge representation formalisms and methods**]

General Terms

Theory

Keywords

Boolean games, Łukasiewicz Games, Łukasiewicz logic

1. INTRODUCTION

Boolean games provide a simple, compact, and theoretically attractive abstract model for studying multi-agent interactions in settings where players will act strategically in an attempt to achieve private goals [8, 2, 5, 6]. In a Boolean game, each player i exercises unique control over a set of Boolean variables V_i, and will attempt to assign values for these variables in such a way as to satisfy an individual goal φ_i, expressed as a formula of propositional logic. Strategic concerns in Boolean games arise from the fact that whether i's goal is in fact satisfied will depend in part on the choices made by other players. Players in Boolean games can be understood as representing non-deterministic computer programs, and the overall framework of Boolean games provides an elegant mathematical model through which to investigate issues of strategic interaction in multi-agent systems.

Now, a standard critique of Boolean games is that the binary nature of goals (satisfied or unsatisfied) inevitably trivialises the nature of strategic interactions. For example, players are assumed to be indifferent between all outcomes that satisfy their goal, and are indifferent between all outcomes that do not satisfy their goal. This assumption is clearly a gross simplification for many situations, a concern which led researchers to extend the original Boolean games model with costs, leading to richer and more realistic preference structures for agents [11]. While these refinements make it possible to model much richer types of interaction, the inherently dichotomous nature of preferences in Boolean games is surely one of their most debated features, and the work in the present paper is directly motivated by this limitation.

Specifically, we introduce *Łukasiewicz Games*, which overcome this limitation of Boolean games by allowing goals to be specified as formulae of finitely-valued Łukasiewicz logics [3][1]. The rationale for using Łukasiewicz logic in this way is given by the McNaughton Theorem, which says that every continuous piecewise linear polynomial function with integer coefficients over $[0,1]^n$ can be expressed as a formula of Łukasiewicz logic in n variables (see, e.g., [3]). Thus, Łukasiewicz logic provides a natural, compact, formally well-defined and expressive logical representation language for payoff functions, allowing much richer preference structures than is easily possible in conventional Boolean games.

The present paper makes three main contributions. First, we provide a formal definition of Łukasiewicz Games. Second, we present a number of detailed worked examples, which illustrate how a range of strategic scenarios can be formalized within this framework. In particular, we argue, these scenarios cannot naturally be formalized using conventional Boolean games. Third, we investigate properties of Łukasiewicz Games. We show that despite their expressive power, the key decision problems for Łukasiewicz Games are no more complex than for conventional Boolean games; in addition, we give a logical characterisation for the existence of equilibria in Łukasiewicz Games.

[1]In the remainder of the paper, when we refer to Łukasiewicz logic, it should be understood that we are referring specifically to finitely-valued Łukasiewicz logics, as distinct from the many other extensions of Łukasiewicz logic that have been investigated in the literature [3].

Appears in: *Alessio Lomuscio, Paul Scerri, Ana Bazzan, and Michael Huhns (eds.), Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2014), May 5-9, 2014, Paris, France.*

2. PRELIMINARY DEFINITIONS

Since Łukasiewicz logic is fundamental to our present work, but is not as widely known as the classical logics that underpin conventional Boolean games, we begin by introducing the concepts of Łukasiewicz logic that are used in the remainder of the paper.

The language of Łukasiewicz logic Ł [3] is built from a countable set of variables $\mathsf{V} = \{p_1, p_2, \ldots\}$, the binary connective "$\to$", and the truth constant $\overline{0}$ (for falsity). Further connectives are defined as follows:

$$\begin{array}{rcl rcl}
\neg\varphi & \text{is} & \varphi \to \bar{0}, & \varphi \wedge \psi & \text{is} & \varphi\&(\varphi \to \psi), \\
\varphi\&\psi & \text{is} & \neg(\varphi \to \neg\psi), & \varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi), \\
\varphi \oplus \psi & \text{is} & \neg(\neg\varphi\&\neg\psi), & \varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi)\&(\psi \to \varphi), \\
\varphi \ominus \psi & \text{is} & \varphi\&\neg\psi, & d(\varphi,\psi) & \text{is} & \neg(\varphi \leftrightarrow \psi).
\end{array}$$

We ofter write $n\varphi$ as an abbreviation for $\underbrace{\varphi \oplus \cdots \oplus \varphi}_{n}$, with $n \geq 1$.

Let $Form$ denote the set of formulae of Łukasiewicz logic. A *valuation*, e, is a mapping $e : \mathsf{V} \to [0, 1]$, which assigns to all propositional variables a value from the real unit interval; we require that $e(\overline{0}) = 0$. The semantics of Łukasiewicz logic are then defined, with a small abuse of notation, by extending the valuation e to complex formulae. Although strictly speaking we only need state the rule for the "$\to$" operator (as we can define the remaining operators in terms of this operator and $\overline{0}$), we present the complete ruleset in the interest of clarity:

$$\begin{array}{rcl}
e(\varphi \to \psi) & = & \min(1 - e(\varphi) + e(\psi), 1) \\
e(\neg\varphi) & = & 1 - e(\varphi) \\
e(\varphi\&\psi) & = & \max(0, e(\varphi) + e(\psi) - 1) \\
e(\varphi \oplus \psi) & = & \min(1, e(\varphi) + e(\psi)) \\
e(\varphi \ominus \psi) & = & \max(0, e(\varphi) - e(\psi)) \\
e(\varphi \wedge \psi) & = & \min(e(\varphi), e(\psi)) \\
e(\varphi \vee \psi) & = & \max(e(\varphi), e(\psi)) \\
e(\varphi \leftrightarrow \psi) & = & 1 - |e(\varphi) - e(\psi)| \\
e(d(\varphi, \psi)) & = & |e(\varphi) - e(\psi)|
\end{array}$$

A valuation e is a *model* for a formula φ if $e(\varphi) = 1$. A valuation e is a model for a theory T, if $e(\psi) = 1$, for every $\psi \in T$.

In this paper, we restrict our attention to *finite-valued Łukasiewicz logics* $Ł_k$. In such logics, it is assumed that the domain is a set of the following form: $L_k = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$. The notions of valuation and model for $Ł_k$ are defined analogously just replacing $[0, 1]$ by L_k as set of truth values for the logic, that is, valuations are functions with the signature $e : \mathsf{V} \to L_k$. It is sometimes useful to introduce constants in addition to $\overline{0}$ that will denote values in the domain L_k. Specifically, we will denote by $Ł_k^c$ the Łukasiewicz logic obtained by adding constants $\overline{c}$ for every value $c \in L_k$. We assume that valuation functions e interpret such constants in the natural way: $e(\overline{c}) = c$.

A well-known result by McNaughton established that *every continuous piecewise linear polynomial function with integer coefficients over* $[0, 1]^n$ (called a *McNaughton function*) *is definable by a formula in Łukasiewicz logic* [3].

In the case of finite-valued Łukasiewicz logics $Ł_k^c$, the functions defined by a formula are combinations of the restrictions of McNaughton functions over $(L_k)^n$ and the constant functions for each $c \in L_k$. Notice that the class of functions definable by $Ł_k^c$-formulas exactly coincides with the class of all functions $f : (L_k)^n \to L_k$, for every $n \geq 0$.[2] In this sense, we can associate to every formula $\varphi(p_1, \ldots, p_n)$ from $Ł_k^c$ a function $f_\varphi : (L_k)^n \to L_k$.

Finally, note that the satisfiability problem for finite Łukasiewicz logics is no more complex than for classical propositional logic: the problem is NP-complete [7, 3].

[2]This result can be obtained by using Lemma 3.

3. ŁUKASIEWICZ GAMES

We now introduce the framework of Łukasiewicz games. First, let $\mathsf{V} = \{p_1, \ldots, p_m\}$ be a finite set of propositional variables, as above. Our games are populated by a finite set of players $\mathsf{P} = \{P_1, \ldots, P_n\}$ (also referred to as "agents"). Note that throughout this paper, we assume that $|\mathsf{P}| = n$. Each player P_i controls a subset of propositional variables $\mathsf{V}_i \subseteq \mathsf{V}$, so that the sets V_i form a partition of V. The fact that player P_i is in control of the set V_i of propositional variables means that P_i has the unique ability within the game to choose values for the variables in V_i. It is assumed that variables take values from the set $L_k = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$.

A *strategy* for an agent P_i is a function $s_i : \mathsf{V}_i \to L_k$, which corresponds to a valuation of the variables controlled by P_i. A strategy profile is a collection of strategies $(s_1, \ldots, s_n)$, one for each player. Every strategy profile directly corresponds to a valuation function $e : \mathsf{V} \to L_k$ and *vice versa*; we find it convenient to abuse notation a little by treating strategy profiles as valuations and valuations as strategy profiles.

As in conventional Boolean games, we assume that each player is associated with a $Ł_k^c$-formula φ_i, with propositional variables from V, whose valuation is interpreted as the payoff function for player P_i. That is, the player P_i seeks a valuation e that maximises the value of the corresponding function f_{φ_i}. Of course, not all the variables in φ_i will in general be under P_i's control and, consequently, the utility P_i obtains by playing a certain strategy (i.e., choosing a certain variable assignment) also potentially depends in part on the choices made by other players.

Łukasiewicz Games can be seen as a generalization of Boolean Games [8, 2], since the latter are obviously a special case of the former. However, Łukasiewicz Games also incorporate the notion of cost/effort, given that each player's strategic choice can be seen as an assignment to each controlled variable carrying an intrinsic cost. This makes it possible to formalize situations in which agents aim at a better tradeoff between the costs of making certain choices and the resulting payoff.

EXAMPLE 1. *Consider the following example with two players P_1 and P_2, who control variables $\{x\}$ and $\{y, z\}$, respectively. The payoffs are given by the following formulae:*

$$\varphi_1 := x \wedge \neg y \wedge z \quad \textit{and} \quad \varphi_2 := x \oplus z.$$

In order for P_2 to maximize the payoff, it suffices to assign $s_2(z) = 1$, no matter what P_1 does. Player P_1, however, has no chance of maximizing the payoff except by assigning $s_1(x) = 1$ and hoping that $s_2(z) = 1$ and $s_2(y) = 0$. The latter makes sense, since, given $s_2(z) = 1$, any assignment to y does not change the value of φ_2, so P_2 has no incentive in assigning any other value to y but 0 and raising the costs. Still, P_2 knows that P_1 needs $s_1(x) = 1$ and consequently can get the maximum payoff by simply keeping the costs at minimum with $s_2(z) = 0$ and $s_2(y) = 0$.

We now formally define Łukasiewicz games.

DEFINITION 1 (ŁUKASIEWICZ GAMES). *A Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$ is given by a structure:*

$$\mathcal{G} = \langle \mathsf{P}, \mathsf{V}, \{\mathsf{V}_i\}, \{\mathsf{S}_i\}, \{\varphi_i\} \rangle$$

where:

1. $\mathsf{P} = \{P_1, \ldots, P_n\}$ *is a finite set of* players.
2. $\mathsf{V} = \{p_1, \ldots, p_m\}$ *is a finite set of propositional variables taking values from*

$$L_k = \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}.$$

3. $\mathsf{V}_i \subseteq \mathsf{V}$ *is the set of propositional variables under control of player* P_i*, so that the sets* V_i *form a partition of* V*, i.e.:*

$$\mathsf{V} = \bigcup_{i=1}^{n} \mathsf{V}_i, \text{ and for } i \neq j, \mathsf{V}_i \cap \mathsf{V}_j = \emptyset.$$

4. S_i *is the strategy set for player* i *that includes all valuations* $s_i : \mathsf{V}_i \to L_k$ *of the propositional variables in* V_i*, i.e.*

$$\mathsf{S}_i = \{s_i \mid s_i : \mathsf{V}_i \to L_k\}.$$

5. φ_i *is a* $Ł_k^c$*-formula, built from variables in* V*, whose associated function*

$$f_{\varphi_i} : (L_k)^t \to L_k$$

corresponds to the payoff function *(also called* utility function*) of* P_i*, and whose value will be determined by the valuations in* $\{\mathsf{S}_1, \dots, \mathsf{S}_n\}$.

A *strategy profile* $\vec{s}$ for $\mathcal{G}$ is a tuple $\vec{s} = (s_1, \dots, s_n)$, with each $s_i \in \mathsf{S}_i$ being the strategy selection for the corresponding player in $\mathcal{G}$. Given a strategy s_i for P_i, we denote by s_{-i} the collection of strategies $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ not including s_i, and S_{-i} is the set of all s_{-i}'s. With an abuse of notation, we use

$$f_{\varphi_i}(s_1, \dots, s_n) \qquad \text{and} \qquad f_{\varphi_i}(s_i, s_{-i})$$

to denote P_i's payoff under the strategy profile $(s_1, \dots, s_n)$: recall that φ_i defines a payoff function f_{φ_i}, and a strategy profile $(s_1, \dots, s_n)$ corresponds to a valuation $e : \mathsf{V} \to L_k$.

We now introduce the concepts of dominance, best response and Nash equilibrium adapted to our framework. In what follows, we always assume we are working with an arbitrary n-player Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$.

DEFINITION 2 (DOMINANCE). *Let* $\mathcal{G}$ *be a Łukasiewicz game. A strategy* $s_i \in \mathsf{S}_i$ *for player* P_i *is called* strictly dominated *(*weakly dominated*) if there exists a strategy* $s'_i \in \mathsf{S}_i$ *such that for all* $s_{-i} \in \mathsf{S}_{-i}$

$$f_{\varphi_i}(s_i, s_{-i}) < f_{\varphi_i}(s'_i, s_{-i}) \qquad (f_{\varphi_i}(s_i, s_{-i}) \leq f_{\varphi_i}(s'_i, s_{-i}), \textit{resp.}).$$

DEFINITION 3 (BEST RESPONSE). *Let* $\mathcal{G}$ *be Łukasiewicz game and let* $(s_1, \dots, s_n)$ *be a strategy profile for* $\mathcal{G}$*. The strategy* s_i *for* P_i *is called a* best response *whenever, fixing* s_{-i}*, there exists no strategy* s'_i *such that*

$$f_{\varphi_i}(s_i, s_{-i}) < f_{\varphi_i}(s'_i, s_{-i}).$$

DEFINITION 4 (PURE STRATEGY NASH EQUILIBRIUM). *Let* $\mathcal{G}$ *be a Łukasiewicz game. A strategy profile* $(s_1^\star, \dots, s_n^\star)$ *is called a* pure strategy Nash Equilibrium (NE) *for* $\mathcal{G}$*, whenever* $s_i^\star$ *is a best response to* $s_{-i}^\star$*, for each* $1 \leq i \leq n$.

Complexity results are not at all the main focus of the present paper, but it is natural to ask whether the enriched framework of Łukasiewicz games leads to a blow-up in computational complexity compared with conventional Boolean games. We will assume in what follows that we are working with games defined on a fixed domain L_k for $k \in \mathbb{N}$ such that $k > 0$. Let us refer to the EVALUATION PROBLEM for Łukasiewicz games as the problem of computing, for a given valuation function $e : \mathsf{V} \to L_k$ and a formula φ of Łukasiewicz logic on L_k, the value $e(\varphi)$. Since we are dealing with rationals in a fixed domain, this problem is trivially solvable in polynomial time. The MEMBERSHIP problem is the problem of determining, for a given Łukasiewicz game $\mathcal{G}$ and strategy profile $\vec{s}$ for $\mathcal{G}$, whether $\vec{s}$ forms a pure strategy Nash equilibrium for $\mathcal{G}$. The NON-EMPTINESS problem is the problem of determining for a given game $\mathcal{G}$ whether there exist any pure strategy Nash equilibria for the game. The problems are co-NP-complete and Σ_2^p-complete for conventional Boolean games. And, perhaps surprisingly, we have:

THEOREM 1. *The* MEMBERSHIP *and* NON-EMPTINESS *problems for Łukasiewicz games are co-NP-complete and* Σ_2^p*-complete, respectively.*

The proof of this can be adapted from proofs of the corresponding results for conventional Boolean games: the key point is that strategies for Łukasiewicz games are "small witnesses" in the terminology of computational complexity. A strategy for a player simply consists of a value from the domain L_k for every variable controlled by that player. Since values in L_k are rational numbers that can be assumed to be expressed in binary, they have a compact representation.

4. EXAMPLES

We now introduce a number of examples to illustrate Łukasiewicz games, and in particular, we choose examples that we believe cannot be easily expressed in the framework of conventional Boolean games (i.e., using classical logic to express goals).

4.1 Generalized Matching Pennies

The first game we present is a generalization of Matching Pennies[3], the classic example of a zero-sum game without a pure strategy equilibrium. In the original game, two players P_1 and P_2 both have a penny and must secretly choose whether to turn it to head or tails, revealing their choices simultaneously. If their choices are the same, then P_1 takes both pennies; if they are different, P_2 takes both.

Now, imagine that both players must perform an action with a certain cost and are in charge of the variables p_1 and p_2, respectively. P_1's overall strategy is to be as close as possible to P_2's choice. In contrast, P_2 wants to keep the greatest possible distance between the choices. The players' strategy spaces are given by the sets of functions

$$\mathsf{S}_1 = \{s_1 \mid s_1 : \{p_1\} \to L_k\}, \qquad \mathsf{S}_2 = \{s_2 \mid s_2 : \{p_2\} \to L_k\}.$$

Now, recall that the Łukasiewicz logic expression $d(p_1, p_2)$, defined in Section 2, gives the difference between its arguments p_1 and p_2. Using this expression, we can define the payoff function for P_1 as the formula $\neg d(p_1, p_2)$, while P_2's payoff is defined by the formula $d(p_1, p_2)$. The game is formally defined as follows:

$$\mathcal{G} = \langle \{P_1, P_1\}, \{p_1, p_2\}, \{p_1\}, \{p_2\}, \mathsf{S}_1, \mathsf{S}_2, \neg d(p_1, p_2), d(p_1, p_2) \rangle.$$

Table 1 shows the payoff matrix for this generalized version of Matching Pennies with $k = 5$.

4.2 Weak-Link Games

Weak-link games[4] are a class of coordination games, where the players benefit from mutually coordinating on the same strategy. The original version of the game (see [4]) consists of n players who simultaneously choose a number from a finite set $\{1, \dots, m\}$. Each player i's payoff is defined by the following function:

$$u_i(x_1, \dots, x_n) = a + b \cdot \min(x_1, \dots, x_n) - c \cdot (x_i - \min(x_1, \dots, x_n)),$$

where x_i is the choice made by player i, and a, b, c are positive

[3]Note that a similar generalization, based on infinite-valued Łukasiewicz logic, was first presented in [9].

[4]Nothing to do with the popular TV game show "The Weakest Link".

	P_2					
P_1	0,	$1/5$	$2/5$	$3/5$	$4/5$	1
0	1, 0	4/5, 1/5	3/5, 2/5	2/5, 3/5	1/5, 4/5	0, 1
$1/5$	4/5, 1/5	1, 0	4/5, 1/5	3/5, 2/5	2/5, 3/5	1/5, 4/5
$2/5$	3/5, 2/5	4/5, 1/5	1, 0	4/5, 1/5	3/5, 2/5	2/5, 3/5
$3/5$	2/5, 3/5	3/5, 2/5	4/5, 1/5	1, 0	4/5, 1/5	3/5, 2/5
$4/5$	1/5, 4/5	2/5, 3/5	3/5, 2/5	4/5, 1/5	1, 0	4/5, 1/5
1	0, 1	1/5, 4/5	2/5, 3/5	3/5, 2/5	4/5, 1/5	1, 0

Table 1: Generalized Matching Pennies.

parameters. Intuitively, x_i is interpreted as the effort i is willing to make in her interaction with others. The payoff u_i is heavily influenced by the choice of the agent with the lower effort level. Therefore, each player's payoff depends on the weakest link in the strategic interaction. The game has m pure strategy Nash equilibria corresponding to the strategy profiles in which the players select the same values.

Notice that in weak-link games, the payoff function's domain is a strict subset of the function's range. Therefore, we introduce a variant of such games where domain and range coincide in order to make it possible to define a representation in terms of $Ł_k^c$-formulae.

We formalize n-player weak-link games in the following game:

$$\mathcal{G} = \langle \mathsf{P}, \mathsf{V}, \{\mathsf{V}_i\}, \{\mathsf{S}_i\}, \{\varphi_i\}\rangle$$

where:

1. $\mathsf{V} = \{p_1, \ldots, p_n\}$.
2. $\mathsf{V}_i = \{p_i\}$.
3. The strategy space is defined as follows, for each i:
$$\mathsf{S}_i = \{s_i \mid s_i : \{p_i\} \to L_k\}.$$
4. Each player P_i's utility is given by the formula
$$\varphi_i(p_1, \ldots, p_n) := \bigwedge_{j=1}^{n} p_j \ominus \left(p_i \ominus \bigwedge_{j=1}^{n} p_j\right)$$
whose associated payoff function is
$$f_{\varphi_i} = \max\left(\min_{j=1}^{n} x_j - \max\left(x_i - \min_{j=1}^{n} x_j, 0\right), 0\right).$$

Table 2 shows the payoff matrix for P_1 for the two-player version of the weak-link game with $k = 7$ (P_2's payoff is obtained by replacing P_1 by P_2 and vice versa), where

$$\varphi_1(p_1, p_2) := (p_1 \wedge p_2) \ominus (p_1 \ominus (p_1 \wedge p_2))$$

with payoff function

$$f_{\varphi_1} = \max(\min(x_1, x_2) - \max(x_1 - \min(x_1, x_2), 0), 0).$$

Notice that in this game there is no strictly dominated strategy, while the strategy $s_1(p_1) = 0$ is weakly dominated.

4.3 Traveler's Dilemma

The Traveler's Dilemma was introduced by Basu [1] in order to illustrate the tension between the rational solution suggested by the existence of a Nash Equilibrium, and apparently reasonable behavior based on intuition. We introduce (a slightly modified variant of) the Dilemma and show how to formalize it as a Łukasiewicz game.

The game is as follows. Two travelers fly back home from a trip to a remote island where they bought exactly the same antiques. Unfortunately for them, their luggage gets damaged and all the items acquired are broken. An agent of the airline promises a refund for the inconvenience, but, not knowing the exact value of the objects, she puts forward the following proposal. Both travelers must privately write down on paper a number between 0 and 100, corresponding to the cost of the antiques. If they both write the same number, the agent can assume that they are both telling the truth, so they will both receive exactly that amount minus one unit (with a minimum of 0). If the travelers write different numbers, the one who wrote the lower number, say x, (assumed to be the honest one) will receive x plus a reward of two units (with a maximum of 100). The other player, who is regarded by the agent as dishonest, will receive x with a penalty of two units (with a minimum of 0).

Payoffs in the Traveler's Dilemma are defined by the following function, whose payoff matrix is shown in Table 3:

$$f(x, y) = \begin{cases} \max(x - 1, 0) & x = y \\ \min(x + 2, 100) & x < y \\ \max(y - 2, 0) & y < x \end{cases}.$$

Given that each player wants to maximize her/his payoff, what choices should they make? If both travelers choose 100, they both get 99, which is almost the exact value of their items. However, each traveler soon realizes that if she deviates from the previous choice and writes 99, while the other player sticks to the original selection, she can increase her payoff to 100. Under the assumption of common knowledge and rationality, however, the other player is drawn to make the same decision, which leads to writing 99, yielding a mutual payoff of 98. Still, deviating from this selection is unilaterally beneficial for each individual, producing again a situation of coordination between the players' choices. This reasoning only ends when both players select 0, thus gaining nothing in the process. The strategy profile $(0, 0)$ is the unique pure strategy Nash equilibrium of the game. This, however, clearly clashes with what intuition would lead us to accept as a rational choice. It seems unreasonable that two individuals would follow the above line of reasoning and rationally come to the conclusion that the best solution is ending up empty-handed.

The Traveler's Dilemma can be formalized as a Łukasiewicz game over $Ł_{100}$ as follows. Define the following game

$$\mathcal{G} = \langle\{T_1, T_2\}, \{p_1, p_2\}, \{p_1\}, \{p_2\}, \{\varphi_1(p_1, p_2), \varphi_2(p_1, p_2)\}\rangle,$$

where T_1 and T_2 are the two travelers; $\{p_1, p_2\}$ is the set of propositional variables, with p_1 being controlled by T_1 and p_2 being controlled by T_2; the payoff formulas are defined as follows[5]

[5] The connective Δ is defined as follows in each finite-valued Łukasiewicz logic $Ł_k$:

$$\Delta\varphi := \neg(k(\neg\varphi))$$

						P$_2$			
		0	$1/7$	$2/7$	$3/7$	$4/7$	$5/7$	$6/7$	1
	0	**0**	0	0	0	0	0	0	0
	$1/7$	0	**1/7**	1/7	1/7	1/7	1/7	1/7	1/7
	$2/7$	0	0	**2/7**	2/7	2/7	2/7	2/7	2/7
	$3/7$	0	0	1/7	**3/7**	3/7	3/7	3/7	3/7
P$_1$	$4/7$	0	0	0	2/7	**4/7**	4/7	4/7	4/7
	$5/7$	0	0	0	1/7	3/7	**5/7**	5/7	5/7
	$6/7$	0	0	0	0	2/7	4/7	**6/7**	6/7
	1	0	0	0	0	1/7	3/7	5/7	**1**

Table 2: Weak-Link Game.

						T2				
		0	1	2	3	···	97	98	99	100
	0	**0, 0**	2, 0	2, 0	2, 0	···	2, 0	2, 0	2, 0	2, 0
	1	0, 2	0, 0	3, 0	3, 0	···	3, 0	3, 0	3, 0	3, 0
	2	0, 2	0, 3	1, 1	4, 0	···	4, 0	4, 0	4, 0	4, 0
	3	0, 2	0, 3	0, 4	2, 2	···	5, 0	4, 0	4, 0	4, 0
T1	⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮
	97	0, 2	0, 3	0, 4	0, 5	···	96, 96	99, 95	99, 95	99, 95
	98	0, 2	0, 3	0, 4	0, 5	···	95, 99	97, 97	100, 96	100, 96
	99	0, 2	0, 3	0, 4	0, 5	···	95, 99	96, 100	98, 98	100, 97
	100	0, 2	0, 3	0, 4	0, 5	···	95, 99	96, 100	97, 100	99, 99

Table 3: Traveler's Dilemma Payoff Matrix.

$$\begin{aligned}\varphi_1(p_1,p_2) \quad := \quad & \Big(\Delta\,(p_1 \leftrightarrow p_2) \wedge \Big(p_1 \ominus \overline{\tfrac{1}{100}}\Big)\Big) \vee \\ & \Big(\neg\Delta\,(p_2 \to p_1) \wedge \Big(p_1 \oplus \overline{\tfrac{2}{100}}\Big)\Big) \vee \quad , \\ & \Big(\neg\Delta\,(p_1 \to p_2) \wedge \Big(p_2 \ominus \overline{\tfrac{2}{100}}\Big)\Big)\end{aligned}$$

$$\begin{aligned}\varphi_2(p_1,q_2) \quad := \quad & \Big(\Delta\,(p_1 \leftrightarrow p_2) \wedge \Big(p_1 \ominus \overline{\tfrac{1}{100}}\Big)\Big) \vee \\ & \Big(\neg\Delta\,(p_2 \to p_1) \wedge \Big(p_1 \ominus \overline{\tfrac{2}{100}}\Big)\Big) \vee \quad . \\ & \Big(\neg\Delta\,(p_1 \to p_2) \wedge \Big(p_2 \oplus \overline{\tfrac{2}{100}}\Big)\Big)\end{aligned}$$

These objective formulae define the following payoff functions:

$$f_{\varphi_1}(x_1,x_2) = \begin{cases} \max\left(x_1 - \frac{1}{100}, 0\right) & x_1 = x_2 \\ \min\left(x_1 + \frac{2}{100}, 1\right) & x_1 < x_2 \\ \max\left(x_2 - \frac{2}{100}, 0\right) & x_2 < x_1 \end{cases}$$

$$f_{\varphi_2}(x_1,x_2) = \begin{cases} \max\left(x_1 - \frac{1}{100}, 0\right) & x_1 = x_2 \\ \min\left(x_2 + \frac{2}{100}, 1\right) & x_2 < x_1 \\ \max\left(x_1 - \frac{2}{100}, 0\right) & x_1 < x_2. \end{cases}$$

Both f_{φ_1} and f_{φ_2} can be easily seen to be the linear transformation into L_{100} of the payoff function for the Traveler's Dilemma over the set $\{0, 1, \ldots, 99, 100\}$.

5. CHARACTERISING NASH EQUILIBRIA

The aim of this section is to provide different characterizations of the existence of pure strategy Nash equilibria in Łukasiewicz games. We first need some additional terminology and preliminary results. In particular, we need payoff functions to be defined exactly on the same domain and have as inputs not only the same set of variables but the whole set of variables occurring in a game. This requirement leads to introducing the concept of a *normalized* Łukasiewicz game.

Definition 5 (Normalized Game). *A Łukasiewicz game*

$$\mathcal{G} = \langle \mathsf{P}, \mathsf{V}, \{\mathsf{V}_i\}, \{\mathsf{S}_i\}, \{\varphi_i\} \rangle,$$

where $\mathsf{V} = \{p_1, \ldots, p_m\}$*, is called* normalized *whenever each payoff function* φ_i *is of the form* $\varphi_i(p_1, \ldots, p_m)$*, i.e., all the variables from* V *occur in each* φ_i*.*

As mentioned above, not every Łukasiewicz game is normalized, since each payoff function might contain a different subset of variables. However, we now show that each game can be transformed into a normalized one. This amounts to showing that any payoff formula can be rewritten in an equivalent form in the whole set V of variables.

Definition 6. *Let* $\varphi(p_1, \ldots, p_w)$ *be a* $Ł_k^c$*-formula. We say that* $\varphi(p_1, \ldots, p_w)$ *has an* equivalent extension

$$\varphi^\sharp(p_1, \ldots, p_w, q_1, \ldots, q_v)$$

in $\{q_1, \ldots, q_v\}$*, if, for every* $(a_1, \ldots, a_w) \in (L_k)^w$

$$f_\varphi(a_1, \ldots, a_w) = f_{\varphi^\sharp}(a_1, \ldots, a_w, b_1, \ldots, b_v)$$

for all $(b_1, \ldots, b_v) \in (L_k)^v$*.*

The next Lemma shows that for any arbitrary Łukasiewicz formula over $Ł_k^c$, we can always find an equivalent extension in any set of variables.

Lemma 1. *Let* $\varphi(p_1, \ldots, p_w)$ *be a be any* $Ł_k^c$*-formula. For any set of variables* $\{q_1, \ldots, q_v\}$*, there exists an equivalent extension of* $\varphi(p_1, \ldots, p_w)$ *in* $\{q_1, \ldots, q_v\}$*.*

and its semantic interpretation in L_k is

$$e(\Delta\varphi) = \begin{cases} 1 & e(\varphi) = 1 \\ 0 & \text{otherwise} \end{cases} \quad .$$

The above Lemma makes it possible to assume every Łukasiewicz game is normalized. To make this precise, we introduce first a notion of equivalence for games.

DEFINITION 7 (EQUIVALENT GAMES). *Let*

$$\mathcal{G} = \langle \mathsf{P}, \mathsf{V}, \{\mathsf{V}_i\}, \{\mathsf{S}_i\}, \{\varphi_i\}\rangle \text{ and } \mathcal{G}' = \langle \mathsf{P}', \mathsf{V}', \{\mathsf{V}'_i\}, \{\mathsf{S}'_i\}, \{\varphi'_i\}\rangle$$

be two Łukasiewicz games over $Ł^c_k$. *We say that* $\mathcal{G}$ *and* $\mathcal{G}'$ *are equivalent whenever:*

1. $\mathsf{P} = \mathsf{P}'$,
2. $\mathsf{V} = \mathsf{V}'$,
3. *For each* i, $\mathsf{V}_i = \mathsf{V}'_i$ *and* $\mathsf{S}_i = \mathsf{S}'_i$,
4. $(s^\star_1, \dots, s^\star_n)$ *is a* NE *for* $\mathcal{G}$ *if and only if* $(s^\star_1, \dots, s^\star_n)$ *is a* NE *for* $\mathcal{G}'$.

Given Lemma 1 it is straightforward to prove that all Łukasiewicz games have an equivalent normalized counterpart.

LEMMA 2. *Every Łukasiewicz game is equivalent to a normalized game.*

From now on, we will tacitly assume each game to be normalized. Also, notice that, so far, we have been denoting by

$$f_{\varphi_i}(s_i, s_{-i})$$

the value of the function f_{φ_i} given the strategy combination (s_i, s_{-i}). As mentioned above this actually is an abuse of notation since the strategy combination (s_i, s_{-i}) corresponds to a specific assignment to all the variables in the game, but for the valuation of f_{φ_i} only the assignments to the variables actually occurring in f_{φ_i} are taken into account. Since every game can be considered normalized, the use of this notation can now be regarded as correct.

5.1 Equilibria and Satisfiability

We now show that the existence of equilibria for an arbitrary game over $Ł^c_k$ is equivalent to the satisfiability of a special $Ł^c_k$-formula. Notice that, if we restrict to the language of $Ł^c_k$, such a formula always exists. In fact, for each variable p, we can encode a valuation $e(p) = \frac{j}{k}$ by using constants through formulas of the form

$$p \leftrightarrow \overline{\frac{j}{k}},$$

which are satisfiable if and only if $e(p)$ does equal $\frac{j}{k}$. Therefore, we can build a formula that expresses the fact that a NE actually exists by encoding all possible strategy combinations and all possible changes of strategy by each player. Still, we are going to show that even if we do not have additional constants in our logic, it is still possible to write such a formula.

In order to show how, we need some preliminary results. We begin by proving that valuations can be encoded by formulas.

LEMMA 3. *For every propositional variable* p *and every valuation* $e : \{p\} \to L_k$ *there exists a formula* $\psi(p)$ *such that*

$$e(p) = \frac{j}{k} \qquad \textit{if and only if} \qquad e(\psi(p)) = 1.^6$$

PROOF. We assume j and k to be coprime. If that is not the case then we have that

$$e\left(\psi_{\frac{j'}{k'}}(p)\right) = 1 \text{ if and only if } e(p) = \tfrac{j'}{k'},$$

where j' and k' are coprime and $\frac{j}{k} = \frac{j'}{k'}$.

Let $\mathsf{q}_{j,k}$ and $\mathsf{r}_{j,k}$ denote the quotient and the remainder, respectively, of the Euclidean division of k by j.

If $e(p) = 0$, let

$$\psi_0(p) := \neg p.$$

Then

$$e(p) = 0 \qquad \text{iff} \qquad e(\neg p) = 1.$$

If $e(p) = \frac{1}{k}$, then let

$$\psi_{\frac{1}{k}}(p) := \neg d(\neg((k-1)p), p).$$

It is easy to check that

$$e(p) = \tfrac{1}{k} \qquad \text{if and only if} \qquad e(\neg d(\neg((k-1)p), p)) = 1.$$

In fact,

$$e(\neg d(\neg((k-1)p), p)) = 1 - |(1-(k-1)x) - x|,$$

and

$$1 - |(1-(k-1)x) - x| = 1 \qquad \text{iff} \qquad x = \frac{1}{k}.$$

For $e(p) = \frac{j}{k}$, with $j \geq 2$, the proof proceeds by induction. For j and k coprime, let

$$\psi_{\frac{j}{k}}(p) = \psi_{\left(\mathsf{r}_{j,k},k\right)}\left(\neg\left(\mathsf{q}_{j,k}p\right)\right),$$

while, for j and k not coprime, take j' and k' coprime such that and $\frac{j}{k} = \frac{j'}{k'}$ and let

$$\psi_{\frac{j}{k}}(p) = \psi_{\frac{j'}{k'}}(p) = \psi_{\left(\mathsf{r}_{j',k'},k'\right)}\left(\neg\left(\mathsf{q}_{j',k'}p\right)\right),$$

Notice that $\mathsf{r}_{j,k} < j$. So, for instance, if $j = 2$, then

$$\psi_{\frac{2}{k}}(p) := \psi_{\frac{1}{k}}\left(\neg\left(\mathsf{q}_{j,k}p\right)\right) = \neg d\left(\neg\left((k-1)\left(\neg\left(\mathsf{q}_{j,k}p\right)\right)\right), \neg\left(\mathsf{q}_{j,k}p\right)\right).$$

This concludes the proof of the Lemma. □

Now we are ready to show that for each game the existence of an equilibrium is equivalent to the existence of a special satisfiable formula $\mathcal{E}_\mathcal{G}$. We prove this by giving an explicit construction of $\mathcal{E}_\mathcal{G}$. As before, we assume the game to be normalized.

Notice that as an immediate consequence of Lemma 3, we have:

COROLLARY 1. *In every Łukasiewicz game* $\mathcal{G}$ *on* $Ł^c_k$, *for every strategy combination* $(s_1, \dots, s_n)$ *there exists a* $Ł^c_k$*-formula* ψ *so that*

$$f_\psi(s'_1, \dots, s'_n) = 1 \qquad \textit{if and only if} \qquad s_i = s'_i$$

for all i.

In fact, let $\mathsf{V}_i = \{p_{1_i}, \dots, p_{m_i}\}$ be the set of variables in control of player i, and let

$$(\alpha_{1_i}, \dots, \alpha_{m_i}) \in (L_k)^{m_i}.$$

The formula

$$\psi_{\alpha_{1_i}}(p_{1_i})$$

encodes the assignment by player i of the value α_{1_i} to the variable p_{1_i}, i.e.

$$e(\psi_{\alpha_{1_i}}(p_{1_i})) = 1 \qquad \text{iff} \qquad e(p_{1_i}) = \alpha_{1_i}.$$

[6]Notice that the following proof translates into logical terms the algebraic proof of Lemma 19 in [10], whose context and content are significantly different from those of the present article.

So, the formula

$$\psi_{\alpha_{1_i}}(p_{1_i}) \wedge \cdots \wedge \psi_{\alpha_{m_i}}(p_{m_i})$$

encodes player i's strategy

$$\{\alpha_{1_i}, \ldots, \alpha_{m_i}\},$$

and the formula

$$\bigwedge_{i=1}^{n} \Big(\psi_{\alpha_{1_i}}(p_{1_i}) \wedge \cdots \wedge \psi_{\alpha_{m_i}}(p_{m_i}) \Big) \tag{1}$$

encodes the strategy combination

$$\{\alpha_{1_1}, \ldots, \alpha_{m_1}, \ldots, \alpha_{1_i}, \ldots, \alpha_{m_i}, \ldots, \alpha_{1_n}, \ldots, \alpha_{m_n}\}.$$

To avoid any possible confusion, notice that for $j \neq i$, we might have that $m_i \neq m_j$, since i and j might be in control of a different number of variables.

Now, take, for each player i the set of all strategies

$$\mathsf{S}_i = \{s_i \mid s_i = (\beta_{1_i}, \ldots, \beta_{m_i}) \in (L_k)^{m_i}\}.$$

Assign to each player i a new set of variables

$$\mathsf{V}^{s_i} = \left\{ q_{1_i}^{\beta_{1_i}}, \ldots, q_{m_i}^{\beta_{m_i}} \right\},$$

for each $s_i \in (L_k)^{m_i}$. This means that if a player controls m_i variables, she has $(k+1)^{m_i}$ different strategy combinations and is therefore assigned $m_i \cdot (k+1)^{m_i}$ new variables.

Proceeding as above, take the formula

$$\psi_{\beta_{1_i}}\left(q_{1_i}^{\beta_{1_i}}\right)$$

that encodes the assignment by player i of the value β_{1_i} to the variable $q_{1_i}^{\beta_{1_i}}$, so that the formula

$$\psi_{\beta_{1_i}}\left(q_{1_i}^{\beta_{1_i}}\right) \wedge \cdots \wedge \psi_{\beta_{m_i}}\left(q_{m_i}^{\beta_{m_i}}\right) \tag{2}$$

encodes player i's strategy

$$\{\beta_{1_i}, \ldots, \beta_{m_i}\}.$$

Let

$$\begin{aligned}\varphi_i(p_{1_1}, \ldots, p_{m_1}, \ldots, p_{1_{i-1}}, \ldots, p_{m_{i-1}}, \ldots, p_{1_i}, \ldots, p_{m_i}, \\ p_{1_{i+1}}, \ldots, p_{m_{i+1}}, \ldots p_{1_n}, \ldots, p_{m_n})\end{aligned} \tag{3}$$

be player i's payoff formula, and let

$$\begin{aligned}\varphi_i(p_{1_1}, \ldots, p_{m_1}, \ldots, p_{1_{i-1}}, \ldots, p_{m_{i-1}}, \ldots, q_{1_i}^{\beta_{1_i}}, \ldots, q_{m_i}^{\beta_{m_i}}, \\ p_{1_{i+1}}, \ldots, p_{m_{i+1}}, \ldots p_{1_n}, \ldots, p_{m_n})\end{aligned} \tag{4}$$

be the formula obtained from (3) by replacing the variables

$$\{p_{1_i}, \ldots, p_{m_i}\}$$

with the variables

$$\left\{ q_{1_i}^{\beta_{1_i}}, \ldots, q_{m_i}^{\beta_{m_i}} \right\}.$$

So, using (3) and (4), the formula

$$\begin{aligned}\varphi_i(p_{1_1}, \ldots, p_{m_1}, \ldots, p_{1_{i-1}}, \ldots, p_{m_{i-1}}, \ldots, q_{1_i}^{\beta_{1_i}}, \ldots, q_{m_i}^{\beta_{m_i}}, \\ p_{1_{i+1}}, \ldots, p_{m_{i+1}}, \ldots p_{1_n}, \ldots, p_{m_n}) \to \\ \varphi_i(p_{1_1}, \ldots, p_{m_1}, \ldots, p_{1_{i-1}}, \ldots, p_{m_{i-1}}, \ldots, p_{1_i}, \ldots, p_{m_i}, \\ p_{1_{i+1}}, \ldots, p_{m_{i+1}}, \ldots p_{1_n}, \ldots, x_{m_n})\end{aligned} \tag{5}$$

encodes the fact that player i's payoff does not increase. To simplify the notation, we denote the formula (5) by χ.

Now, define the formula $\mathcal{E}_{\mathcal{G}}$, where each

$$\vec{s} \in (L_k)^{\sum_{i=1}^{n} m_i}$$

is a strategy combination:

$$\begin{aligned}\mathcal{E}_{\mathcal{G}} := \bigvee_{\vec{s} \in (L_k)^{\sum_{i=1}^{n} m_i}} \Bigg[\bigwedge_{i=1}^{n} \Big(\psi_{\alpha_{1_i}}(p_{1_i}) \wedge \cdots \wedge \psi_{\alpha_{m_i}}(p_{m_i}) \Big) \wedge \\ \bigwedge_{i=1}^{n} \Bigg[\bigwedge_{s_i \in (L_k)^{m_i}} \Big[\psi_{\beta_{1_i}}\left(q_{1_i}^{\beta_{1_i}}\right) \wedge \cdots \wedge \psi_{\beta_{m_i}}\left(q_{m_i}^{\beta_{m_i}}\right) \wedge \chi \Big] \Bigg] \Bigg]\end{aligned} \tag{6}$$

From the above construction, it is easy to check that $\mathcal{E}_{\mathcal{G}}$ actually encodes the existence of equilibria. In fact, $\mathcal{E}_{\mathcal{G}}$ is a disjunction indexed by all possible strategy combinations. The existence of an equilibrium requires at least one of the disjuncts to be satisfiable. Each disjunct is a conjunction of formulas encoding the requirement that for a given strategy combination and for every player, every change of strategy does not result in any payoff increase. So, if any such a disjunct is satisfiable, the related strategy combination actually corresponds to a NE.

THEOREM 2. *A Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$ admits a Nash Equilibrium if and only if $\mathcal{E}_{\mathcal{G}}$ is satisfiable.*

5.2 Equilibria and Satisfiable Games

Our purpose now is to show that whenever a game $\mathcal{G}$ has a Nash equilibrium, there exists a special kind of game, called a satisfiable game, that is equivalent to $\mathcal{G}$.

DEFINITION 8 (SATISFIABLE GAMES). *A Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$ is called* satisfiable *if there exists a strategy combination $(s_1, \ldots, s_n)$ such that $f_{\varphi_i}(s_1, \ldots, s_n) = 1$ for all i.*

The following lemma is an immediate consequence:

LEMMA 4. *Every satisfiable Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$ admits a Nash Equilibrium.*

As said above, we want to show that every game $\mathcal{G}$ on $Ł_k^c$ admits a NE if and only if it is equivalent to a satisfiable game. In order to do so, we need to use some concepts related to the algebraic semantics of Łukasiewicz logics and their model theory. Space limitations prevent us from defining these concepts; we refer the reader to [10].

Take any linearly ordered finite MV-algebra MV_k.[7] The first-order theory $\mathsf{Th}(\mathrm{MV}_k)$ of MV_k is the set of first order sentences in the language $\langle \oplus, \neg, 0 \rangle$ that hold over MV_k. As shown in [10], each theory $\mathsf{Th}(\mathrm{MV}_k)$ admits quantifier elimination in $\langle \oplus, \neg, 0 \rangle$. This means that every formula of $\mathsf{Th}(\mathrm{MV}_k)$ is equivalent to a quantifier-free formula.

Now, take any Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$, and let $\vec{x}_i, \vec{y}_i$ denote tuples of variables assigned to player i, and let $E_{\mathcal{G}}$ be the following sentence in $\mathsf{Th}(\mathrm{MV}_k)$ (where $\sqcap$ denotes the Boolean metalanguage conjunction and $\leq$ the order relation):

$$\begin{aligned}&\exists \vec{x}_1, \ldots, \vec{x}_n \ \forall \vec{y}_1, \ldots, \vec{y}_n \\ &\prod_{i=1}^{n} \left(\begin{array}{l} \varphi_i(\vec{x}_1, \ldots, \vec{x}_{i-1}, \vec{y}_i, \vec{x}_{i+1}, \ldots, \vec{x}_n) \\ \leq \\ \varphi_i(\vec{x}_1, \ldots, \vec{x}_{i-1}, \vec{x}_i, \vec{x}_{i+1}, \ldots, \vec{x}_n) \end{array} \right).\end{aligned}$$

It is then easy to check that:

[7] MV-algebras generalize Boolean algebras and form the algebraic semantics of Łukasiewicz logics [3].

LEMMA 5. *A Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$ admits a Nash Equilibrium if and only if $E_\mathcal{G}$ holds in* $\mathsf{Th}(\mathrm{MV}_k)$.

From Lemma 5, we obtain a characterization of the existence of NE in terms of the validity of a first-order formula $E_\mathcal{G}$. The fact that $E_\mathcal{G}$ is valid in $\mathsf{Th}(\mathrm{MV}_k)$ is equivalent to the definability of a non-empty set that corresponds to the set of equilibria of $\mathcal{G}$. By quantifier-elimination, such a set is in turn definable by a quantifier-free formula $\Phi(\vec{x}_1, \dots, \vec{x}_n)$. In order to show the existence of an equivalent satisfiable game, we exploit the fact that this quantifier free-formula can be translated into a formula of the logic $Ł_k^c$. This step will be a consequence of the following lemma.

LEMMA 6. *For every quantifier-free formula Φ (with parameters) of $\mathsf{Th}(\mathrm{MV}_k)$ in the language $\langle \oplus, \neg, 0\rangle$ there exists a $Ł_k^c$-formula φ such that Φ is satisfiable over MV_k if and only if there exists a valuation e such that $e(\varphi) = 1$.*

PROOF. Let $Terms_{\mathrm{MV}}$ be the set of terms t of $\mathsf{Th}(\mathrm{MV}_k)$ in the language of MV-algebras and let $Form_{Ł_k^c}$ be the set of $Ł_k^c$-formulas. Define a translation $\tau : Terms_{\mathrm{MV}} \to Form_{Ł_k^c}$ such that

1. If t is x, then $\tau(t) = p$.
2. If t is $\frac{j}{k}$, then $\tau(t) = \overline{\frac{j}{k}}$.
3. If t is $t' \oplus t''$, then $\tau(t) = \tau(t') \oplus \tau(t'')$.
4. If t is $\neg t'$, then $\tau(t) = \neg\tau(t')$.

Now, every quantifier-free formula Φ of $\mathsf{Th}(\mathrm{MV}_k)$ is a Boolean combination of equalities and (strict) inequalities between terms. So, define a new mapping $\lambda : Form_{\mathrm{MV}} \to Form_{Ł_k^c}$, where $Form_{\mathrm{MV}}$ is the set if quantifier-free formulas Φ of $\mathsf{Th}(\mathrm{MV}_k)$, as follows (where $\sim$ denotes the Boolean metalanguage negation):

1. If Φ is $(t = t')$, then $\lambda(\Phi) = \Delta(\tau(t) \leftrightarrow \tau(t'))$.
2. If Φ is $(t < t')$, then $\lambda(\Phi) = \neg\Delta(\tau(t') \to \tau(t))$.
3. If Φ is $\Phi' \sqcap \Phi''$, then $\lambda(\Phi) = \lambda(\Phi') \wedge \lambda(\Phi'')$.
4. If Φ is $\sim\Phi'$, then $\lambda(\Phi) = \neg\lambda(\Phi')$.

It is easy to check that every formula Φ is satisfiable in MV_k if and only if $\lambda(\Phi)$ is satisfiable. In fact, on the one hand, every function symbol in $\langle \oplus, \neg, 0\rangle$ (and every parameter as well) has an interpretation in MV_k corresponding to the interpretation of the related connective (and constants) in $Ł_k^c$. On the other hand, the use of the operator Δ forces each formula of the form $\Delta(\tau(t) \leftrightarrow \tau(t'))$ and $\neg\Delta(\tau(t') \to \tau(t))$ to behave like a Boolean formula, making compositions of such formulas into Boolean combinations. $\square$

Given the previous lemma, and following the above reasoning, it is possible to prove that, by replacing each payoff formula φ_i in $\mathcal{G}$ with the formula

$$\tau(\Phi(\vec{x}_1, \dots, \vec{x}_n)) \vee \varphi_i,$$

we obtain a satisfiable game $\mathcal{G}'$ equivalent to $\mathcal{G}$.

THEOREM 3. *A Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$ admits a Nash Equilibrium if and only if it is equivalent to a satisfiable game.*

6. CONCLUSIONS

Boolean games are a rich and natural model for understanding strategic behaviour in logic-based goal-oriented multi-agent systems. Although they have many desirable features and have attracted much interest, the dichotomous nature of the preferences in conventional Boolean games severely limits the types of scenarios that can easily be expressed using Boolean games based on classical (two valued) logic. By expressing goal formulae with Łukasiewicz logic, we are able to naturally and compactly express much richer objectives for agents, as we have demonstrated here.

Several questions suggest themselves for future work. For example, it is natural to consider whether we can adapt techniques for theorem proving with Łukasiewicz logic to solving Łukasiewicz games. In addition, further consideration of the computational complexity of Łukasiewicz games would be of interest.

Acknowledgments. Marchioni acknowledges support of the Marie Curie Intra-European Fellowship NAAMSI (FP7-PEOPLE-2011-IEF). Wooldridge was supported by the ERC Advanced Grant RACE (291528).

7. REFERENCES

[1] K. Basu. The traveler's dilemma: Paradoxes of rationality in game theory. *American Economic Review*, 84(2): 391–395, 1994.

[2] E. Bonzon, M.-C. Lagasquie-Schiex, J. Lang, and B. Zanuttini. Boolean games revisited. In *Proceedings of the Seventeenth European Conference on Artificial Intelligence (ECAI-2006)*, pages 265–269, Riva del Garda, Italy, 2006.

[3] R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning (Trends in Logic Volume 7)*. Kluwer Academic Publishers: Dordrecht, The Netherlands, 2000.

[4] G. Devetag, H. Hosni, and G. Sillari. You better play 7: Mutual versus common knowledge of advice in a weak-link experiment. *Synthese*, 190(8): 1351–1381, 2013.

[5] P. E. Dunne, S. Kraus, W. van der Hoek, and M. Wooldridge. Cooperative Boolean games. In *Proceedings of the Seventh International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS-2008)*, pages 1015–1022, Estoril, Portugal, 2008.

[6] J. Grant, S. Kraus, M. Wooldridge, and I. Zuckerman. Manipulating Boolean games through communication. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence (IJCAI-11)*, pages 210–215, Barcelona, Catalonia, Spain, 2011.

[7] R. Hähnle. Many-valued logic and mixed integer programming. *Annals of Mathematics and Artificial Intelligence*, 12: 231–263, 1994.

[8] P. Harrenstein, W. van der Hoek, J.-J.Ch. Meyer, and C. Witteveen. Boolean games. In *Proceedings of the Eighth Conference on Theoretical Aspects of Rationality and Knowledge (TARK VIII)*, pages 287–298, Siena, Italy, 2001.

[9] T. Kroupa and O. Majer. Nash Equilibria in a class of constant-sum games represented by McNaughton functions. In *Logic, Algebra and Truth Degrees 2012, Book of Abstracts*, pages 94–97, Kanazawa, Ishikawa, Japan, 2012.

[10] G. Lenzi and E. Marchioni. An algebraic characterization of o-minimal and weakly o-minimal MV-chains. *Journal of Pure and Applied Algebra*, 218(1): 90–100, 2014.

[11] M. Wooldridge, U. Endriss, S. Kraus, and J. Lang. Incentive engineering for Boolean games. *Artificial Intelligence*, 195: 418–439, 2013.