Multi-player approximate Nash equilibria*

(Extended Abstract)

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ABSTRACT

In this paper we study the complexity of finding approximate Nash equilibria in multi-player normal-form games. First, for any constant number $\mathfrak{n}$, we present a polynomial-time algorithm for computing a relative $\left(1-\frac{1}{1+(\mathfrak{n}-1)^{\mathfrak{n}}}\right)$-Nash equilibrium in arbitrary $\mathfrak{n}$-player games and a relative $\left(1-\frac{1}{1+(\mathfrak{n}-1)^{\mathfrak{n}-1}}\right)$-Nash equilibrium in symmetric $\mathfrak{n}$-player games. Next, we show that there is an additive ε-well-supported Nash equilibrium, for any $\varepsilon > 0$, with support equal to $O(\ln(\mathfrak{n}\mathfrak{m}/\varepsilon)/\varepsilon^2)$, where $\mathfrak{m}$ is the number of pure strategies. Finally, we prove that finding additive approximate Nash equilibria is easy in random multi-player games.

1. INTRODUCTION

The problem of computing of a Nash equilibrium is one of the most fundamental problems in modern game theory. Given the evidence of its intractability, in recent years an extensive research has focused on the approximation of Nash equilibria. In this paper we study the fundamental problem of the complexity of approximating Nash equilibria in multi-player normal-form games.

We will consider (finite) games with $\mathfrak{n}$ players, $1,\ldots,\mathfrak{n}$, each player having a finite set of $\mathfrak{m}$ strategies, $\mathfrak{n},\mathfrak{m} \geq 2$. A *Nash equilibrium* is any strategy profile such that none of the players has an incentive to change her strategy. While we normally would like to consider Nash equilibria and study their properties, there is a strong evidence that finding them is computationally hard (see [7, 11]). Therefore we will focus on a relaxed version thereof, on the approximation of Nash equilibria. There are two different notions of approximation of Nash equilibria, one with *additive* and one with *relative* error/incentive, and then a further variant of the *ε-well-supported Nash equilibrium*. An *additive ε-Nash equilibrium* is a strategy profile such that any player has an incentive of at most ε to change her strategy. A *relative ε-Nash equilibrium* is a strategy profile in which the payoff of each player is at least $(1-\varepsilon)$ times the payoff of the best-response strategy. An *additive ε-well-supported* Nash equilibrium requires that any player has an incentive of at most ε to deviate from any of the pure strategies that she uses in her mixed strategy. Every additive ε-well-supported Nash equilibrium is also an additive ε-Nash equilibrium, but the converse does not always hold. Further, a *relative ε-well-supported Nash equilibrium* is a strategy profile in which any pure strategy that is played by the players has payoff at least $(1-\varepsilon)$ times the best-response payoff.

Given the importance and numerous applications of Nash equilibria (see, e.g., [22] and the references therein), there has been substantial amount of research devoted to the study of the complexity of computing various types of approximate Nash equilibria (see Section 1.2). However, while many applications assume multi-player games, the majority of the *algorithmic studies* have been focusing solely on two-player games (a possible exception here are games with compact representation, for example, graphical games; but our focus is on the study of *general* multi-player games). The goal of this paper is to advance our understanding of the complexity of approximating Nash equilibria games for multi-player games.

1.1 Our contributions

We present a thorough study of the complexity of finding approximate Nash equilibria in multi-player games.

We give the first approximation bound for *relative approximate Nash equilibria* in arbitrary $\mathfrak{n}$-player games, assuming $\mathfrak{n}$ is constant.

THEOREM 1. *For any $\mathfrak{n}$-player normal-form game, where $\mathfrak{n}$ is a constant, with entries in $[0,1]$, we can construct in polynomial-time a relative $(1-\frac{1}{1+(\mathfrak{n}-1)^{\mathfrak{n}}})$-Nash equilibrium.*

(Notice that, for example, for $\mathfrak{n}=3$ players this gives a relative $\frac{8}{9}$-Nash equilibrium, for $\mathfrak{n}=4$ players it gives a relative $\frac{81}{82}$-Nash equilibrium, and so on.)

Then, we extend Theorem 1 to obtain a stronger bound for *relative approximate Nash equilibria* in *symmetric* multi-player games.

THEOREM 2. *For any symmetric $\mathfrak{n}$-player normal-form game, where $\mathfrak{n}$ is a constant, with entries in $[0,1]$, we can construct in polynomial-time a relative $(1-\frac{1}{1+(\mathfrak{n}-1)^{\mathfrak{n}-1}})$-Nash equilibrium.*

(For example, for $\mathfrak{n}=3$ players this gives a relative $\frac{4}{5}$-Nash equilibrium in symmetric games, for $\mathfrak{n}=4$ players it gives a relative $\frac{27}{28}$-Nash equilibrium in symmetric games, and so on.)

Next, inspired by the works of Lipton et al. [21], Hémon et al. [17], Kontogiannis and Spirakis [19], and of Babichenko et al. [3], we prove existence of an *additive ε-well-supported Nash equilibrium* of small (poly-logarithmic) support size.

THEOREM 3. *Consider an $\mathfrak{m}$-strategies $\mathfrak{n}$-player game. Then, one can construct an empirical strategy profile that is an additive*

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ε*-well-supported Nash equilibrium with support of size* $O(\ln \mathfrak{n} + \ln \mathfrak{m} + \ln(1/\varepsilon))/\varepsilon^2$.

Finally, inspired by the work of Panagopoulou and Spirakis [23] we prove that it is easy to find *additive approximate Nash equilibria in random multi-player normal-form games*. Let a *fully uniform distribution* be a mixed strategy in which player plays every pure strategy with probability $\frac{1}{\mathfrak{m}}$.

THEOREM 4. *Consider an* $\mathfrak{m}$*-strategies* $\mathfrak{n}$*-player random normal-form game and let* $\varepsilon > 0$*. Let* x *be a fully uniform distribution. Then with high probability, the strategy profile* $\mathbf{x} = (x, \dots, x)$ *is an additive* ε*-well-supported Nash equilibrium.*

We note that in Theorem 4, parameter ε does not need to be constant and it may be as small as $\varepsilon = \sqrt{3\ln(\mathfrak{m}\mathfrak{n})/\mathfrak{m}^{\mathfrak{n}-1}}$.

As it can be seen in Section 1.2, no previous non-trivial bounds have been known earlier for any of these problems (for multi-player games).

1.2 Related works

While in classical game theory and in many applications the study of *multi-player games* (games with $\mathfrak{n} > 2$ players) plays a fundamental role, as it should be evident from the overview below, most of the earlier works on the complexity of approximation of Nash equilibria have been focusing on *bimatrix games*, that is, games with $\mathfrak{n} = 2$ players.

An *additive* ε*-Nash equilibrium* is a strategy profile such that any player has an incentive of at most ε to change her strategy. A number of polynomial-time algorithms to find additive ε-Nash equilibria for bimatrix games have been developed over the years: for $\varepsilon = \frac{3}{4}$ by Kontogiannis et al. [18], for $\varepsilon = \frac{1}{2}$ by Daskalakis et al. [13], for $\varepsilon = \frac{3-\sqrt{5}}{2} \approx 0.382$ by Daskalakis et al. [12], for $\varepsilon = \frac{1}{2} - \frac{1}{3\sqrt{6}} \approx 0.364$ by Byrka et al. [5], and finally with $\varepsilon \approx 0.3393$ by Tsaknakis and Spirakis [26]. In the special case of *symmetric* bimatrix games, a polynomial-time algorithm for additive $(\frac{1}{3} + \delta)$-Nash equilibria, for any $\delta > 0$, was given by Kontogiannis and Spirakis [20]. Bárány et al. [4] (see also [23]) have proved that additive ε-Nash equilibria are easy to find in random bimatrix games for all $\varepsilon > 0$.

While the bimatrix games have been extensively investigated, much less is known for the *games with more than two players*. For general games with $\mathfrak{n}$ players, there is a polynomial-time algorithm that outputs additive $\left(1 - \frac{1}{\mathfrak{n}}\right)$-Nash equilibria, which also can be generalized to a recursive method that gives an additive $\left(\frac{1}{2-\alpha}\right)$-Nash equilibrium for $\mathfrak{n}$ players if we can find an additive α-Nash equilibrium for $\mathfrak{n} - 1$ players, see [5, 6, 17]. Furthermore, for the special class of polymatrix games, one can improve this bound to get a polynomial-time algorithm for computing additive $(\frac{1}{2} + \delta)$-Nash equilibria for any $\delta > 0$ [14].

Lipton et al. [21] have proved existence of additive ε-Nash equilibria with support of size $O((\mathfrak{n}^2 \log \mathfrak{m})/\varepsilon^2)$, which was then improved to $O((\mathfrak{n} \log \mathfrak{m})/\varepsilon^2)$ by Hémon et al. [17], and finally to $O(\log(\mathfrak{n}\mathfrak{m}/\varepsilon)/\varepsilon^2)$ by Babichenko et al. [3].

On the other hand, it is known that if one limits the players to strategies with support of total size $o(\log \mathfrak{m})$, then it is not possible to find an additive ε-Nash equilibrium in bimatrix games, for any $\varepsilon < \frac{1}{2}$ [16]. In a similar flavor, very recently Rubinstein [25] proved that for some constant $\varepsilon > 0$, assuming the Exponential Time Hypothesis for PPAD computing an additive ε-Nash equilibrium in bimatrix game requires time $\mathfrak{m}^{\log^{1-o(1)} \mathfrak{m}}$, matching (up to the $o(1)$ term) the complexity of the algorithm of Lipton et al. [21].

The notion of an *additive* ε*-well-supported* Nash equilibrium (that forms an important class of approximate Nash equilibrium) requires that any player has an incentive of at most ε to deviate from any of the pure strategies that she uses in her mixed strategy. Since every additive ε-well-supported Nash equilibrium is also an additive ε-Nash equilibrium, but the converse does not always hold, the problem of finding an additive ε-well-supported Nash equilibrium is harder than that of finding an additive ε-Nash equilibrium. We now know how to find an additive 0.6528-well-supported Nash equilibrium in bimatrix games [8, 15, 19] in polynomial time. There are also polynomial-time algorithms for finding additive $\frac{1}{2}$-well-supported Nash equilibria in win-lose bimatrix games [19], and additive $(\frac{1}{2} + \delta)$-well-supported Nash equilibria in symmetric bimatrix games, for any $\delta > 0$ [9]. Panagopoulou and Spirakis [23] proved that additive ε-well-supported Nash equilibria are easy to find in random bimatrix games for all $\varepsilon > 0$. Furthermore, the quasi-polynomial algorithm for additive ε-Nash equilibria can be also applied to finding additive ε-well-supported Nash equilibria in bimatrix games, and so one can find additive ε-well-supported Nash equilibria in quasi-polynomial time $\mathfrak{m}^{O(\ln \mathfrak{m}/\varepsilon^2)}$ for arbitrarily small $\varepsilon > 0$ [19]. On the negative side, a poly-logarithmic number of strategies is needed to find ε-well-supported Nash equilibria in polynomial-time, for any $\varepsilon < 1$ [1, 2, 25].

While the notion of the additive approximation of the Nash equilibria has been studied extensively in the past, significantly less attention has been given to the notion of the relative ε-Nash equilibria. A *relative* ε*-Nash equilibrium* is a strategy profile in which the payoff of each player is at least $(1 - \varepsilon)$ times the payoff of the best-response strategy. Most of the relevant results we are aware of appeared in the paper of Feder et al. [16] for bimatrix games. On a positive side, Feder et al. [16, Theorem 3] give a polynomial time algorithm that finds a relative ε-Nash equilibrium for ε slightly smaller than $\frac{1}{2}$. (There exists a function $f(\mathfrak{m}) = (2 + o(1))^{\mathfrak{m}}$ such that for any α, $0 < \alpha < \frac{1}{8\mathfrak{m}f(\mathfrak{m})}$, one can find in polynomial time a relative $(\frac{1}{2} - \alpha)$-Nash equilibrium; the relative $(\frac{1}{2} - \alpha)$-Nash equilibrium found is a pure row strategy and a mixed column strategy.) On a negative side, Feder et al. [16, Theorem 1] show that for any α, $0 < \alpha \le \frac{1}{2}$, if one limits the column player to strategies with support of size less than $\frac{\log_2 \mathfrak{m}}{\log_2(1/\alpha)}$, then it is not possible to find a relative $(\frac{1}{2} - \alpha)$-Nash equilibrium, even for constant-sum zero-one games. Furthermore, Feder et al. [16, Theorem 2] present bimatrix constant-sum zero-one games for which for any ε, $0 < \varepsilon < \frac{1}{2}$, no pair of mixed strategies with supports of size smaller than $O(\varepsilon^{-2} \log \mathfrak{m})$ is a relative ε-Nash equilibrium. Further, in a related work, Daskalakis [10] considers the notion of the *relative* ε*-well-supported Nash equilibria*, strategy profiles in which any strategy that is played by the players has payoff at least $(1 - \varepsilon)$ times the best-response payoff. He shows that the problem of finding a relative ε-well-supported Nash equilibrium in two-player games, with payoff values in $[-1, 1]$, is PPAD-complete, even for constant values of the approximation. Finally, Rubinstein [24] extends this result and he proves that finding a relative ε-well-supported Nash equilibrium in a bimatrix game with positive payoffs is PPAD-complete (see also [25]).

REFERENCES

[1] Y. Anbalagan, H. Huang, S. Lovett, S. Norin, A. Vetta, and H. Wu. Large supports are required for well-supported Nash equilibria. In *Proceedings of the 18th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX'15)*, pages 78–84, 2015.

[2] Y. Anbalagan, S. Norin, R. Savani, and A. Vetta. Polylogarithmic supports are required for approximate well-supported Nash equilibria below 2/3. In *Proceedings of the 9th International Conference Web and Internet Economics (WINE'13)*, pages 15–23, 2013.

[3] Y. Babichenko, S. Barman, and R. Peretz. Simple approximate equilibria in large games. In *Proceedings of the 15th ACM Conference on Economics and Computation (EC'14)*, pages 753–770, 2014.

[4] I. Bárány, S. Vempala, and A. Vetta. Nash equilibria in random games. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05)*, pages 123–131, 2005.

[5] H. Bosse, J. Byrka, and E. Markakis. New algorithms for approximate Nash equilibria in bimatrix games. *Theoretical Computer Science*, 411(1):164–173, 2010.

[6] P. Briest, P. W. Goldberg, and H. Roeglin. Approximate equilibria in games with few players, April 2008. arXiv:0804.4524 [cs.GT].

[7] X. Chen, X. Deng, and S.-H. Teng. Settling the complexity of computing two-player Nash equilibria. *Journal of the ACM*, 56(3), 2009.

[8] A. Czumaj, A. Deligkas, M. Fasoulakis, J. Fearnley, M. Jurdziński, and R. Savani. Distributed methods for computing approximate equilibria. In *Proceedings of the 12th International Conference Web and Internet Economics (WINE'16)*, pages 15–28, 2016.

[9] A. Czumaj, M. Fasoulakis, and M. Jurdziński. Approximate well-supported Nash equilibria in symmetric bimatrix games. In *Proceedings of the 7th International Symposium on Algorithmic Game Theory (SAGT'14)*, pages 244–254, 2014.

[10] C. Daskalakis. On the complexity of approximating a Nash equilibrium. *ACM Transactions on Algorithms*, 9(3), June 2013.

[11] C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou. The complexity of computing a Nash equilibrium. *SIAM Journal on Computing*, 39(1):195–259, 2009.

[12] C. Daskalakis, A. Mehta, and C. Papadimitriou. Progress in approximate Nash equilibria. In *Proceedings of the 8th ACM Conference on Electronic Commerce (EC'07)*, pages 355–358, 2007.

[13] C. Daskalakis, A. Mehta, and C. Papadimitriou. A note on approximate Nash equilibria. *Theoretical Computer Science*, 410(17):1581–1588, 2009.

[14] A. Deligkas, J. Fearnley, R. Savani, and P. G. Spirakis. Computing approximate Nash equilibria in polymatrix games. In *Proceedings of the 10th International Conference Web and Internet Economics(WINE'14)*, pages 58–71, 2014.

[15] J. Fearnley, P. W. Goldberg, R. Savani, and T. B. Sørensen. Approximate well-supported Nash equilibria below two-thirds. *Algorithmica*, 76(2):297–319, 2016.

[16] T. Feder, H. Nazerzadeh, and A. Saberi. Approximating Nash equilibria using small-support strategies. In *Proceedings of the 8th ACM Conference on Electronic Commerce (EC'07)*, pages 352–354, 2007.

[17] S. Hémon, M. de Rougemont, and M. Santha. Approximate Nash equilibria for multi-player games. In *Proceedings of the 1st International Symposium on Algorithmic Game Theory (SAGT'08)*, pages 267–278, 2008.

[18] S. C. Kontogiannis, P. N. Panagopoulou, and P. G. Spirakis. Polynomial algorithms for approximating Nash equilibria of bimatrix games. In *Proceedings of the 2nd Workshop on Internet and Network Economics (WINE'06)*, pages 286–296, 2006.

[19] S. C. Kontogiannis and P. G. Spirakis. Well supported approximate equilibria in bimatrix games. *Algorithmica*, 57(4):653–667, 2010.

[20] S. C. Kontogiannis and P. G. Spirakis. Approximability of symmetric bimatrix games and related experiments. In *Proceedings of the 10th International Symposium on Experimental Algorithms (SEA'11)*, pages 1–20, 2011.

[21] R. Lipton, E. Markakis, and A. Mehta. Playing large games using simple strategies. In *Proceedings of the 4th ACM Conference on Electronic Commerce (EC'03)*, pages 36–41, 2003.

[22] N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, 2007.

[23] P. N. Panagopoulou and P. G. Spirakis. Random bimatrix games are asymptotically easy to solve (A simple proof). *Theory of Computing Systems*, 54(3):479–490, 2014.

[24] A. Rubinstein. Inapproximability of Nash equilibrium. In *Proceedings of the 47th Annual ACM Symposium on Theory of Computing (STOC'15)*, pages 409–418, 2015.

[25] A. Rubinstein. Setting the complexity of computing approximate two-player Nash equilibria. In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science (FOCS'16)*, pages 258–265, 2016.

[26] H. Tsaknakis and P. G. Spirakis. An optimization approach for approximate Nash equilibria. *Internet Mathematics*, 5(4):365–382, 2008.