Stable Matching with Uncertain Pairwise Preferences

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ABSTRACT

We study a two-sided matching problem where the agents have independent pairwise preferences on their possible partners and these preferences may be uncertain. In this case, the certainly preferred part of an agent's preferences may admit a cycle and there may not even exist a matching that is stable with non-zero probability. We focus on the computational problems of checking the existence of possibly and certainly stable matchings, i.e., matchings whose probability of being stable is positive or one, respectively. We show that finding a possibly stable matching is NP-hard, even if only one side can have cyclic preferences. On the other hand we show that the problem of finding a certainly stable matching is polynomial-time solvable if only one side can have cyclic preferences and the other side has transitive preferences, but that this problem becomes NP-hard when both sides can have cyclic preferences. The latter complexity result also implies the hardness of finding a kernel in a special class of directed graphs.

CCS Concepts

•Theory of computation → Design and analysis of algorithms; •Computing methodologies → Multi-agent systems; •Applied computing → Economics;

Keywords

Matching under preferences; stable matchings; pairwise comparisons; uncertain preferences

1. INTRODUCTION

We consider a *Stable Marriage problem (SM)* in which there are two disjoint sets of agents, a set of men and a set of women, and each agent is able to compare any two given agents from the other side of the market. The goal is to compute a *stable matching* μ, i.e., a matching where there is no pair of a man and a woman that prefer to be matched to each other rather than matched to their partners in μ. The standard literature on stable matching problems [10, 12, 15] usually assumes that the preferences are linear orders and that agents are fully aware of their preferences. We relax both of these assumptions. We assume that men and women may have uncertainty in their preferences, and that the underlying preferences need not be linear orders. Uncertainty in preferences could arise for a number of reasons both practical or epistemological. For example, an agent may have not invested enough time or effort to come up with a linear order or differentiate any two potential partners.

In this paper, we focus on the *pairwise probability model* where each agent only expresses a probability of preferring one agent over another for all possible pairs. More formally, each agent a is endowed with independent pairwise probabilities for any given two partners b and c. If a prefers b to c with probability p then s/he prefers c to b with probability $1-p$. We suppose that each agent has complete pairwise comparisons, and therefore we only consider complete matchings as solutions (where *complete* means that all agents are matched). Uncertain pairwise preferences are a general model of preferences that have been well-studied in the context of voting. Pairwise uncertain preferences are also referred to as fuzzy preferences (see e.g., [3]). Since, the preferences are probabilistic, we will especially be interested in matching that are stable with probability that is one or at least nonzero. We will refer to *stability probability* as the probability that a matching is stable.

Many of our results depend on the structure of the *certainly preferred relation* that can be derived from the pairwise preferences. We define the *certainly preferred* relation $\succ_a^{\text{cert}}$ for each agent a and write $b \succ_a^{\text{cert}} c$ if and only if agent a prefers b over c with probability 1. We assume that the certainly preferred relation is not necessarily transitive. That is, it is possible that a prefers b to c and c to d, both with probability 1, but prefers d to b with some nonzero probability. Moreover,

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the certainly preferred relation is also not necessarily acyclic. In the previous example, if a prefers d to b with probability 1, then the certainly preferred relation is *cyclic*. In such a case it may happen that any matching will be certainly blocked.

Example 1. *We have three men m_1, m_2 and m_3 and three women w_1, w_2 and w_3. For every two agents on one side of the market, each agent of the other side assigns a probability to preferring one over another, as depicted in Table 1.*

This setting admits six complete matchings listed in Table 2 along with their stability probabilities. We can extract the certainly preferred relation from the preferences as follows:

m_1 : $\emptyset$;
m_2 : $\{w_2 \succ^{\text{cert}}_{m_2} w_1, w_1 \succ^{\text{cert}}_{m_2} w_3\}$;
m_3 : $\{w_2 \succ^{\text{cert}}_{m_3} w_1, w_1 \succ^{\text{cert}}_{m_3} w_3\}$;
w_1 : $\{m_2 \succ^{\text{cert}}_{w_1} m_1, m_1 \succ^{\text{cert}}_{w_1} m_3, m_2 \succ^{\text{cert}}_{w_1} m_3\}$;
w_2 : $\{m_1 \succ^{\text{cert}}_{w_2} m_3, m_2 \succ^{\text{cert}}_{w_2} m_3\}$; *and*
w_3 : $\{m_1 \succ^{\text{cert}}_{w_3} m_3, m_2 \succ^{\text{cert}}_{w_3} m_3\}$.

Understanding this relation simplifies the computation as we know certain pairs will not have positive probability of blocking. For the men, m_2 and m_3 both certainly prefer w_2 to w_1, and w_1 to w_3, and both have positive probability on $w_3 \succ w_2$, which could lead to a cyclic preference. For the women, w_1 has a strict linear order over the agents, while w_2 and w_3 both only have uncertainty over the relationship between m_1 and m_2, but are sure m_3 is the worst and hence there is no chance for a cycle.

In order to compute the probability that μ_1 is stable we have to compute the probability that none of the potential blocking pairs occurs. Specifically we compute the probability of each of the potential blocking pairs occurring:

$$p(m_1,w_2) = p(w_2 \succ_{m_1} w_1)\cdot p(m_1 \succ_{w_2} m_2) = 0.6\cdot 0.8 = 0.48,$$
$$p(m_1,w_3) = p(w_3 \succ_{m_1} w_1)\cdot p(m_1 \succ_{w_3} m_3) = 0.8\cdot 1.0 = 0.8,$$
$$p(m_2,w_1) = p(w_1 \succ_{m_2} w_2)\cdot p(m_2 \succ_{w_1} m_1) = 0.0\cdot 1.0 = 0.0,$$
$$p(m_2,w_3) = p(w_3 \succ_{m_2} w_2)\cdot p(m_2 \succ_{w_3} m_3) = 0.5\cdot 1.0 = 0.5,$$
$$p(m_3,w_1) = p(w_1 \succ_{m_3} w_3)\cdot p(m_3 \succ_{w_1} m_1) = 1.0\cdot 0.0 = 0.0,$$
$$p(m_3,w_2) = p(w_2 \succ_{m_3} w_3)\cdot p(m_3 \succ_{w_2} m_2) = 0.5\cdot 0.0 = 0.0.$$

The probability that μ_1 is stable is the probability that no blocking pair exists. Since each blocking pair occurs independently we can compute this as the probability that no blocking pair occurs independently: $(1-p(m_1,w_2))\cdot(1-p(m_1,w_3))\cdot \ldots \cdot (1-p(m_3,w_2)) = (1-0.48)\cdot(1-0.8)\cdot(1-0.5) = 0.052$. It can be easily verified that matchings μ_2, μ_4, and μ_5 have stability probability 0 since (m_2,w_1), (m_2,w_1), and (m_2,w_2), form blocking pairs with probability 1 for these matchings, respectively.

	Men		Women
m_1	$p(w_1 \succ_{m_1} w_2) = 0.4$	w_1	$p(m_1 \succ_{w_1} m_2) = 0.0$
	$p(w_2 \succ_{m_1} w_3) = 0.6$		$p(m_2 \succ_{w_1} m_3) = 1.0$
	$p(w_1 \succ_{m_1} w_3) = 0.2$		$p(m_1 \succ_{w_1} m_3) = 1.0$
m_2	$p(w_1 \succ_{m_2} w_2) = 0.0$	w_2	$p(m_1 \succ_{w_2} m_2) = 0.8$
	$p(w_2 \succ_{m_2} w_3) = 0.5$		$p(m_2 \succ_{w_2} m_3) = 1.0$
	$p(w_1 \succ_{m_2} w_3) = 1.0$		$p(m_1 \succ_{w_2} m_3) = 1.0$
m_3	$p(w_1 \succ_{m_3} w_2) = 0.0$	w_3	$p(m_1 \succ_{w_3} m_2) = 0.2$
	$p(w_2 \succ_{m_3} w_3) = 0.5$		$p(m_2 \succ_{w_3} m_3) = 1.0$
	$p(w_1 \succ_{m_3} w_3) = 1.0$		$p(m_1 \succ_{w_3} m_3) = 1.0$

Table 1: Pairwise probabilities for the agents in Example 1.

	Matching	Stability Probability
μ_1	$\{(m_1,w_1),(m_2,w_2),(m_3,w_3)\}$	0.052
μ_2	$\{(m_1,w_1),(m_2,w_3),(m_3,w_2)\}$	0.0
μ_3	$\{(m_1,w_2),(m_2,w_1),(m_3,w_3)\}$	0.48
μ_4	$\{(m_1,w_2),(m_2,w_3),(m_3,w_1)\}$	0.0
μ_5	$\{(m_1,w_3),(m_2,w_1),(m_3,w_2)\}$	0.0
μ_6	$\{(m_1,w_3),(m_2,w_2),(m_3,w_1)\}$	0.2496

Table 2: Stability probability for each matching in Example 1.

Computational Problems.

Given a stable marriage setting where agents have uncertain preferences, various natural computational problems arise. Beside computing the stability probability of a matching, the other important problem is finding a matching that is "most stable." We can think of two (not necessarily equivalent) criteria. One is to find a matching that has the highest probability of being stable, and the other is to find a matching that has the minimum expected number of blocking pairs. In this paper we mainly focus on the former and leave the latter for future work. We consider the following problems for the pairwise probability model.

- StabilityProbability: Given a matching and uncertain preferences of the agents, what is the stability probability of the matching?
- MatchingWithHighestStabilityProbability: Given uncertain preferences of the agents, compute a matching with highest stability probability.

We additionally consider problems connected to, and more restricted than, MatchingWithHighestStabilityProbability: (1) ExistsCertainlyStableMatching — Does there exist a matching that has stability probability one? (2) ExistsPossiblyStableMatching — Does there exist a matching that has non-zero stability probability?

Results.

We summarize the central complexity results of this paper in Table 3. In particular, in Theorem 2 we show that ExistsPossiblyStableMatching is NP-complete, even if one side has cyclic certainly preferred relations. With a slight extension of this theorem, in Theorem 3 we also obtain the NP-hardness of MatchingWithHighestStabilityProb even if no agent has cycle in her/his certainly preferred relations. ExistsCertainlyStableMatching is polynomial-time solvable as long as the certainly preferred relation is transitive for at least one side of the market (Theorem 5). Otherwise, if the certainly preferred relation can be cyclic on both sides of the market, then the problem is NP-hard (Theorem 4). Finally, we show how the latter result implies the NP-hardness of the kernel checking problem for a special class of directed graphs (Theorem 8).

Related work.

Aziz et al. [1, 2] studied matching problems under uncertain preferences, but only considered linear models. In a linear model the agents have linear preferences over their potential partners which are realized with some probabilities. Three cases were studied. In the *lottery model* each agent has an independent probability distribution over her possible

Computational problems	transitive|transitive	transitive|cyclic	cyclic|cyclic
ExistsPossiblyStableMatching	in P [trivial]	NP-complete [Thm 2]	NP-complete
ExistsCertainlyStableMatching	in P	in P [Thm 5]	NP-complete [Thm 4]
MatchingWithHighestStabilityProb	NP-hard [Thm 3]	NP-hard	NP-hard

Table 3: Summary of results under different assumptions over the certainly preferred relations on the two sides, where 'transitive' means that the certainly preferred relations are transitive and cyclic means that preference cycle may occur in the certainly preferred relation of an agent.

preferences. In the *compact indifference model* these probability distributions come from weak preferences where the ties are broken by uniform random lotteries. Finally, the so-called *joint probability* model allows a probability distribution over complete preference profiles. The model of uncertain pairwise preferences, that we study in this paper, is conceptually different. Here, we assume that the agents have uncertain pairwise comparisons over their potential partners, and these comparisons are independent of each other. To illustrate the significant difference between the linear and pairwise models, let us describe a simple example. Suppose that we have a man m who is completely indifferent among three women w_1, w_2, and w_3. In our linear models this situation would be described with uniform probabilities over the six potential linear orders on the three women. In particular, the probability that matching m with w_1 is stable for this small instance would be $\frac{1}{3}$, as w_1 is the best partner of m in two of the six equal probability orders. However, in our pairwise probability model the man would have 0.5 probability of preferring any woman to another one. But note that these probabilities are independent from each other. Therefore the probability of the matching $\{(m, w_1)\}$ being stable is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. This conceptual difference makes it necessary to use different approaches when studying the pairwise model.

A general model based on (certain) pairwise comparisons was studied in a recent paper by Farczadi, Georgiou and Könemann [6]. They suppose that the agents on one side of the market have complete linear preferences and the agents on the other side can have arbitrary binary relations for any pair of partners. They interpret this relation as one potential partner being at least as good as the other potential partner for the agent concerned. They define stability in the following way: if two agents are not matched then either of them should be matched to a partner that is at least as good as this potential blocking partner. Their model differs from ours in many aspects, as they do not use probabilities and they allow antisymmetric relations and no relation for a pair of potential partners, and they only consider the case when one side has pairwise comparisons. Nevertheless, in our proof for the NP-completeness of the problem of deciding the existence of a possibly stable matching for the case where one side may have cyclic preferences (Theorem 2), we use a proof that is similar to the one given by Farczadi et al. [6] for the problem of finding a stable matching in their context.

2. PRELIMINARIES

The classical setting of a Stable Marriage problem is defined as follows. There are two sets of agents: a set M of n men and a set W of n women. We use the term *agents* when making statements that apply to both men and women, and the term *candidates* to refer to the agents on the opposite side of the market to that of an agent under consideration. Each agent a has a preference ordering over the candidates, where $c_1 \succ_a c_2$ denotes that a prefers c_1 over c_2.

Contrary to the classical definition, we assume that agents' preferences are described as independent pairwise comparisons on their possible partners, and that these preferences may be uncertain. Let L denote the *uncertain pairwise preference profile* for all agents. We denote by $I = (M, W, L)$ an instance of a stable marriage problem with uncertain pairwise preferences.

A *matching* μ is a pairing of men and women such that each man is paired with at most one woman and vice versa; defining a list of (man, woman) pairs (m, w). We use $\mu(m)$ to denote the woman w that is matched to m and $\mu(w)$ to denote the match for w. If an agent a is unmatched, we let $\mu(a) = \varnothing$. The probability that a matching is *stable* is the probability that there exists no pair $(m, w) \notin \mu$ where m prefers w to his current partner in μ, i.e., $w \succ_m \mu(m)$, and vice versa. If such a pair exists, it constitutes a *blocking pair*; as the pair would prefer to defect and match with each other rather than stay with their partner in μ. A matching is *certainly stable* if it is stable with probability 1.

The *Stable Marriage problem with Ties and Incomplete lists (SMTI)* is an extension of SM, where agents are allowed to declare some candidates unacceptable—i.e. they strictly prefer remaining unmatched to pairing with an unacceptable partner, and have weak orders over acceptable candidates. A matching is *individually rational* if no agent is matched to an unacceptable partner. Every agent a strictly prefers being matched to an acceptable candidate to remaining unmatched. A matching is called *weakly stable* if it is individually rational and there is no strongly blocking pair, consisting of two agents who strictly prefer each other to their partners in the matching. A matching is *super-stable* if it is individually rational and there is no weakly blocking pair who both weakly prefer each other to their current partners. A weakly stable matching always exists, but Manlove et al. [13] proved that deciding whether an SMTI instance has a complete weakly stable matching is NP-hard. Super-stable matching may not exist, but deciding the existence can be done efficiently.

The *Stable Marriage problem with Partially ordered lists (SMP)* is another extension of SM in which agents' preferences are partial orders over the candidates. In *SMP with Incomplete lists (SMPI)*, each agent's partially ordered preferences contains only his/her acceptable candidates. A matching is *super-stable* in an instance of SMPI if it is stable w.r.t. all linear extensions of the partially ordered preferences.

The main observation in this section is that stability probability of a matching can be checked easily.

Theorem 1. *For the pairwise probabilities model,* StabilityProbability *is polynomial-time solvable.*

PROOF. We can compute the probability that a given matching μ is stable as follows. The probability $p_b(m,w)$ that a given blocking pair (m,w) exists is equal to the probability that m prefers w over his current match in μ multiplied by the probability that w prefers m over her current match in μ. Both these probabilities are given in the pairwise probabilities model. Then the probability that a given blocking pair (m,w) does not exist is $1-p_b(m,w)$. The probability that no blocking pair exists is $\prod_{(m,w)\in M\times W}(1-p_b(m,w))$. Note that the probability that no blocking pair exists is equal to the probability that μ is stable. □

3. POSSIBLY STABLE MATCHINGS

When the certainly preferred relation is acyclic, the problem of checking whether there exists a matching with non-zero probability of being stable is trivial: for each agent, a linear extension consistent with the certainly preferred relation can be taken and for such a preference profile, a stable matching exists due to the classic result of Gale and Shapley [9]. If the certainly preferred relation is not acyclic, a possibly stable matching does not necessarily exist. We first show that the problem of deciding whether a possibly stable matching exists is NP-complete.

THEOREM 2. *For the pairwise probabilities model,* EXISTSPOSSIBLYSTABLEMATCHING *is NP-complete, even if one side of the market has deterministic linear preferences.*

PROOF. Since computing STABILITYPROBABILITY can be solved in polynomial time, we can check efficiently whether a given matching has positive probability, so the problem is in NP. To prove NP-hardness, we reduce from the problem of deciding whether an instance of SMTI admits a complete stable matching. Our reduction is similar to a reduction used by Farczadi et al. [6, Theorem 1] for a different problem. Because some details are adjusted, we describe it here completely. In [13], Manlove et al. proved that determining if an instance of SMTI admits a complete stable matching is NP-complete, even if the ties appear only on the women's side, and each woman's preference list is either strictly ordered or consists entirely of a tie of size two. Let $M=\{m_1,m_2,\dots,m_n\}$ be the set of men and $W=\{w_1,w_2,\dots,w_n\}$ be the set of women in an SMTI instance I, where we suppose that the men's preferences are strict. We create an instance I' of pairwise probability model as follows. We extend the set of men with three new men, m_{n+1}, m_{n+2} and m_{n+3}, and likewise extend the set of women with three new women w_{n+1}, w_{n+2} and w_{n+3}. We refer to the acceptable partners of any given agent a in I as his/her *proper partners* in I'. For each man m_i ($i\in[n]$) in the original instance, we extend his strict preference ordering on his proper partners arbitrarily, by appending the three new women and his unacceptable candidates in I in some arbitrary order. For every woman w_i ($i\in[n]$) in the original instance, we create the pairwise preferences as follows. Firstly, w_i prefers every proper partner of hers to the rest, i.e., to every new man and every unacceptable candidate in I. Secondly, w_i prefers each of the three new men to those men who were unacceptable to her in I. The pairwise preferences of w_i over her proper partners are created in the following natural way: w_i certainly prefers m_k to m_l if w_i strictly prefers m_k to m_l in I, and if w_i is indifferent between m_k and m_l in I then we set the corresponding pairwise probability to be 0.5 in I'. We then let the pairwise preferences of w_i over the three new men to be cyclic, i.e., w_i prefers m_{n+1} to m_{n+2}, m_{n+2} to m_{n+3}, and m_{n+3} to m_{n+1}. The preferences of w_i over the unacceptable original candidates are arbitrary. What remains is to define the pairwise preferences of the new agents. Let each of the three new men have all the original women at the top of his preference list ordered according to their indices, followed with w_{n+1}, w_{n+2} and w_{n+3} (in this order). As for the three new women, let them have m_{n+1}, m_{n+2} and m_{n+3} in the top of their strict preference lists, followed by the original men in an arbitrary order. (Note that every complete linear order implies pairwise probability preferences). We claim that there exists a complete weakly stable matching in I if and only if there is a matching with positive stability probability in I'.

To see the first direction, let μ be a complete weakly stable matching in I. It is easy to see that if we extend μ with pairs (m_{n+1},w_{n+1}), (m_{n+2},w_{n+2}) and (m_{n+3},w_{n+3}) then the resulting matching μ' has positive probability of being stable in I'. This is because, as it is easy to verify, there is no pair which would be certainly blocking for μ'. In the other direction, suppose that μ' is a complete matching in I' that has positive probability of being stable (i.e., it has no certainly blocking pair). First we show that every original woman has to be matched with a proper partner. Suppose for a contradiction that w_i is the woman with the smallest index who is not matched to a proper partner. If w_i is matched to an original man who was unacceptable to her in I then w_i would form a certainly blocking pair with any of the three new men. To see this, note that w_i certainly prefers either of the three new men to her partner, and as none of the new men are matched to a original woman with index smaller than i, hence they all certainly prefer w_i to their partners. Suppose now that w_i is matched to one of the three new men. Then w_i would form a certainly blocking pair with the subsequent new man according to her cyclical preference. (For instance, if w_i is matched to m_{n+1} in μ' then she forms a certainly blocking pair with m_{n+3}.) This is because the subsequent new man cannot have any better partner, since all the women with smaller indices than i are matched to a proper partner. So we arrive at the conclusion that every original woman is matched with a proper partner. As μ' does not admit a certainly blocking pair, this conclusion implies that all the original women are matched to a proper partner in such a way that there is no certainly blocking pair involving original agents. Thus by restricting μ' to the original agents we arrive at a matching μ that must be weakly stable and complete for I. □

The above result gives us the following corollary.

THEOREM 3. MATCHINGWITHHIGHESTSTABILITYPROBABILITY *is NP-hard, even if the certainly preferred relation is transitive for one side of the market and the other side has deterministic linear orders.*

PROOF. We adjust the proof of Theorem 2, as follows. Suppose that I was the starting instance of SMTI, I' was the corresponding instance of pairwise model with certain cyclic preferred relations, and let the adjusted instance of the pairwise model be denoted by I''. Whenever some women had cyclic certainly preferred relations in I', we now modify the probabilities in these pairwise comparisons by a sufficiently small ϵ value. That is, whenever a woman w_i certainly preferred a man m_k to another man m_l belonging to a certain preference cycle in I', we set $p(m_k \succ_{w_i} m_l)=1-\epsilon$ in I''.

Thus, we have no certainly preferred relations in any cycle in I''. However, when considering the matching with the highest stability probability in I'', we can still separate between the two cases with regard to the original NP-complete problem for I. To see this, suppose first that we have a complete stable matching for I. In this case this matching extended with the three new pairs in I' will have a probability of being stable at least $\frac{1}{2^n}$ in both I' and I''. This is because every woman who is indifferent between some men has at most one tie of length two in her preference list in I, and so if this woman is matched to one of the men in her tie then only the other man in this tie may block, which happens with 0.5 probability (if that man prefers this woman to his matching partner). On the other hand, if there exists no complete stable matching for I then we have shown in the proof of Theorem 2 that there always existed a certain blocking pair in I'. This certain blocking pair now will have a probability of $1-\epsilon$ to be blocking, implying that any matching in this case has less than ϵ probability of being stable. Therefore, if we choose ϵ to be $0 < \epsilon < \frac{1}{2^n}$ then an algorithm which solves MatchingWithHighestStabilityProbability could also be used to decide the existence of a complete stable matching for SMTI efficiently. $\square$

4. CERTAINLY STABLE MATCHINGS

In this section we focus on matchings that are stable with probability one and study the ExistsCertainlyStableMatching problem. As we will see, its complexity depends on whether one or both sides can have cyclic preferences.

4.1 Hardness result

We first show that ExistsCertainlyStableMatching is NP-complete in the general model where both sides may have cyclic preferences, even if all agents are certain.

Theorem 4. *ExistsCertainlyStableMatching is NP-complete, even if all agents are certain.*

Proof. Checking whether a matching is stable can be done in linear time, so the problem is in NP. To prove NP-hardness, we reduce from the 3-SAT problem. We are given an instance B of 3-SAT with a set of variables $V = \{v_1, \ldots, v_n\}$ and clauses $C = \{c_1, \ldots, c_m\}$. We create an instance I of the pairwise probability model with certain and intransitive preferences, as follows. (Since all the preferences will be certain, for simplicity, instead of $b \succ_a^{\text{cert}} c$, we write $b \succ_a c$ if agent a prefers b over c with probability 1.) For each variable, v_i we create a simple gadget G_i involving six agents a_i, $\overline{a_i}$, t_i, f_i, t'_i and f'_i and the following set of so-called *proper pairs* $E_i = \{a_it_i, t_i\overline{a_i}, \overline{a_i}f_i, f_ia_i, t_it'_i, f_if'_i\}$. For each clause C_j, we create another gadget G'_j with six agents $m_{j,1}$, $m_{j,2}$, $m_{j,3}$, $w_{j,1}$, $w_{j,2}$, $w_{j,3}$, where all the pairs are proper. Finally, if literal v_i appears in clause C_j then t_i and $m_{j,1}$ is a proper pair and if literal $\overline{v_i}$ appears in C_j then f_i and $m_{j,1}$ form a proper pair. The core of the variable gadget G_i is a length four preference cycle, with the following pairwise comparisons.

$$t_i \succ_{a_i} f_i; \overline{a_i} \succ_{t_i} a_i; f_i \succ_{\overline{a_i}} t_i; a_i \succ_{f_i} \overline{a_i}$$

For this sub-instance, we have two stable matchings, $T_i = \{a_it_i, \overline{a_i}f_i\}$ and $F_i = \{a_if_i, \overline{a_i}t_i\}$, which will correspond to the cases of setting v_i to be true or false in the truth assignment, respectively.

For the clause gadget G'_j, we set the preferences as in the classical Gale-Shapley instance [9] with three men, three women and three stable matchings. Namely, let the linear preferences of these agents to be

$w_{j,1} \succ_{m_{j,1}} w_{j,2} \succ_{m_{j,1}} w_{j,3}$; $m_{j,2} \succ_{w_{j,1}} m_{j,3} \succ_{w_{j,1}} m_{j,1}$;
$w_{j,2} \succ_{m_{j,2}} w_{j,3} \succ_{m_{j,2}} w_{j,1}$; $m_{j,3} \succ_{w_{j,2}} m_{j,1} \succ_{w_{j,2}} m_{j,2}$;
$w_{j,3} \succ_{m_{j,3}} w_{j,1} \succ_{m_{j,3}} w_{j,2}$; $m_{j,1} \succ_{w_{j,3}} m_{j,2} \succ_{w_{j,3}} m_{j,3}$;
We have three stable matchings:
$\mu_j^1 = \{m_{j,1}w_{j,1}, m_{j,2}w_{j,2}, m_{j,3}w_{j,3}\}$,
$\mu_j^2 = \{m_{j,1}w_{j,2}, m_{j,2}w_{j,3}, m_{j,3}w_{j,1}\}$, and
$\mu_j^3 = \{m_{j,1}w_{j,3}, m_{j,2}w_{j,1}, m_{j,3}w_{j,2}\}$.

Now we set the preferences over the linking pairs of the forms $t_im_{j,1}$ and $f_im_{j,1}$. For the agent in the variable gadget, these partners are in the middle of the preferences, as follows:

$$a_i \succ_{t_i} m_{j,1}; m_{j,1} \succ_{t_i} \overline{a_i}; a_i \succ_{f_i} m_{j,1}; m_{j,1} \succ_{f_i} \overline{a_i}.$$

Note that these pairwise comparisons are intransitive for t_i regarding partners a_i, $\overline{a_i}$ and $m_{j,1}$. For $m_{j,1}$, if the literal v_i appears in the kth place in the clause (for $k \in \{1,2,3\}$) then let $m_{j,1}$ have the following pairwise comparisons:

$$t_i \succ_{m_{j,1}} w_{j,k}; w_{j,k+1(mod3)} \succ_{m_{j,1}} t_i; w_{j,k+2(mod3)} \succ_{m_{j,1}} t_i.$$

The intuition behind this setting is the following. If variable v_i is set to be true and v_i appears as the kth literal in clause C_j then we can choose matching μ_j^k in gadget G'_j without making pair $t_im_{j,1}$ blocking. Similarly, if literal $\overline{v_i}$ appears in C_j (i.e., when f_i is linked to $m_{j,1}$) then we set the preferences of $m_{j,1}$ as follows:

$$f_i \succ_{m_{j,1}} w_{j,k}; w_{j,k+1(mod3)} \succ_{m_{j,1}} f_i; w_{j,k+2(mod3)} \succ_{m_{j,1}} f_i.$$

Therefore here as well, if variable v_i is set to be false and $\overline{v_i}$ appears as the k-th literal in clause C_j then we can choose matching μ_j^k in gadget G'_j without making pair $f_im_{j,1}$ blocking. Furthermore, the other pairs of forms $t_im_{j,1}$ and $f_im_{j,1}$ won't be blocking either, as they correspond to the other two literals, for which $m_{j,1}$ prefers $w_{j,k}$ over t_i (or f_i).

With regard to the proper pairs of form $t_it'_i$ and $f_if'_i$, we only use them to forbid any pair of form $t_im_{j,1}$ and $f_im_{j,1}$ to be included in a stable matching. Namely, we set

$$a_i \succ_{t_i} t'_i; \overline{a_i} \succ_{t_i} t'_i$$

and for any pair $t_im_{j,1}$ we set

$$t'_i \succ_{t_i} m_{j,1}.$$

Similarly, we set

$$a_i \succ_{f_i} f'_i; \overline{a_i} \succ_{f_i} f'_i$$

and for any pair $f_im_{j,1}$ we set

$$f'_i \succ_{f_i} m_{j,1}.$$

Finally, we let t'_i and f'_i form a proper pair, with new agents t''_i and f''_i, respectively. We let t'_i prefer t_i to t''_i and f'_i prefer f_i to f''_i. All the other pairs are not proper and we suppose that all agents want to get a proper partner and their preferences over the not proper partners are arbitrary.

We are ready to show that B has a truth assignment if and only if there exist a (certainly) stable matching in I. Suppose first that we are given a truth assignment, then we create the matching μ in I as we already described above. Namely, if v_i is set to be true then we add T_i to μ in the variable gadget G_i and when v_i is false then we add F_i to μ. Regarding the clause gadgets G'_j, we choose a literal that is true according to the truth assignment (there must be one, say, the kth one), then we add μ_j^k to μ. Finally we match t_i' to t_i'' and f_i' to f_i'' for every i. This matching is stable, as argued above.

In the other direction, suppose that I has a (certainly) stable matching μ. As we discussed when describing the construction, no pair of forms $t_i m_{j,1}$ and $f_i m_{j,1}$ may be in μ, because a corresponding pair $t_i t_i'$ or $f_i f_i'$ would be blocking. Similarly, no pair of forms $t_i t_i'$ and $f_i f_i'$ can be present in μ, as otherwise some pair in E_i would be blocking. After ignoring these pairs we can observe that every agent is the first choice of some proper partner, therefore the matching must consist of proper pairs. Thus either T_i or F_i must be in μ, and also one of the three possible stable matching, μ_j^1, μ_j^2 or μ_j^3 must be chosen in G_j' for μ. We then assign the values in the Boolean formula according to the matchings in the variable gadgets, namely, we set v_i to be true if $T_i \subset \mu$ and false if $F_i \subset \mu$. This will be a truth assignment, since if μ_j^k is chosen for any clause gadget G_j' then the kth literal must be true in clause C_j, as otherwise the corresponding pair, $t_i m_{j,1}$ or $f_i m_{j,1}$ would block matching μ. □

4.2 Tractability for transitivity on one side

We present an efficient algorithm for ExistsCertainlyStableMatching for the case when one side must have transitive certainly preferred relations, but there is no constraint on the preferences of the other side. In Aziz et al. [2] it was shown that certainly stable matchings are closely related to the notion of super-stable matchings widely studied in the literature [14]. In fact they were able to provide a definition of certainly stable matchings using a terminology similar to that of super-stability. Given a matching μ and an unmatched pair (m, w), a pair (m, w) *very weakly blocks (blocks)* μ if $\mu(m) \not\succ_m^{\text{cert}} w$ and $\mu(w) \not\succ_w^{\text{cert}} m$.

Proposition 1 ([2]). *A matching is certainly stable iff it admits no very weakly blocking pair.*

It directly follows from a result by Aziz et al. [2] (Theorem 1) that if the certainly preferred relation is transitive for both sides of the market, then employing the SUPER-SMP algorithm of [14] leads to a polynomial time algorithm for ExistsCertainlyStableMatching. Here we show that by making changes to SUPER-SMP we can relax our assumption to require transitivity on only one side of the market. Without loss of generality we assume that the certainly preferred relation is transitive for men. Algorithm CERTAINLY-STABLE depicted as Algorithm 1 is a modification of SUPER-SMP. The main ideas behind the algorithm and the proofs are similar to those presented in [14], but we have adapted them to our problem as well as managing to shorten and simplify some of the proofs.

We begin with providing an intuitive high-level description of what Algorithm 1 does. Let p_m denote the certainly preferred relation for each man m. Since every agent finds all candidates acceptable, he or she certainly prefers every candidate to remaining unmatched. To start with, every agent is set free, i.e. not engaged. The algorithm conducts a series of proposals by men to women. A man m proposes to a woman w if w is at the *head* of p_m (defined in the next paragraph) and m and w are not engaged to each other. If w certainly prefers m to her current fiancé (if she has one) and every man she as rejected so far, then w breaks her current engagement (if any exists) and m and w become engaged. Otherwise, w rejects m. Furthermore, if w has a fiancé m', then she will break her engagement with m' if she does not certainly prefer m' to m. Notice that when a woman rejects a man, or breaks her engagement with a man, it has become evident that they cannot be matched in a certainly stable matching. Also note that during the execution of the algorithm, each woman is engaged to at most one man, but a man may be engaged to several women. When there is no more proposals to be made—i.e., when every man m is engaged to all women at the head of p_m—the current engagement relation is examined to determine whether it qualifies as a certainly stable matching or not. It qualifies if no man is engaged to more than one women—i.e., if it is a matching, and there is no woman who has received a proposal and is remained unmatched.

We now define the terms used in Algorithm 1. When we say *delete* the pair (m, w), we mean that w should be deleted from the certainly preferred relation p_m (deletion of m from w's preferences is not necessary for the purpose of Algorithm 1). A pair (m, w) is deleted if w rejects m immediately after he has made a proposal, or breaks her engagement with m later on. For the ease of the presentation we say that w rejects m in either case. For any agent x, we refer to x's preferences at the termination of Algorithm 1 as x's *reduced preferences*. At any stage of the algorithm, we say a woman w is at the *head* of a man m's certainly preferred relation if there is no other woman in m's remaining certainly preferred relation (i.e., no woman who is not deleted from p_m) whom m certainly prefers to w. Note that as the certainly preferred relation is transitive, and therefore acyclic, for men, as long as there is a woman who has not yet been deleted from p_m, there is at least one woman at the head of m's certainly preferred relation. Also note that more than one woman can be at the head of m's certainly preferred relation.

Theorem 5. ExistsCertainlyStableMatching *can be solved in polynomial time, if the certainly preferred relation is transitive for at least one side of the market.*

We prove Theorem 5 by showing that if an instance I admits a certainly stable matching then Algorithm 1 returns one, and if no matching is certainly stable in I then Algorithm 1 returns false. Several lemmas will come in handy.

Lemma 1. *If the pair (m, w) is deleted during an execution of the algorithm, then the pair cannot block any matching output by the algorithm.*

Proof. Assume that CERTAINLY-STABLE outputs μ as a certainly stable matching, and suppose that (m, w) is deleted during an execution of the algorithm. Since w has received proposals, the algorithm does not output a matching in which she is unmatched. Since we assume that the algorithm outputs a matching μ, hence w is matched to say m'. We show that w must certainly prefer m' to m. To see this, consider the following two cases that can occur. Case 1: If m' proposes to w after (m, w) is deleted, or when w is engaged to m, then w would not have accepted m' if she does not certainly prefer m' to m (see the IF-THEN-ELSE condition of line 11). Case 2: Else, m must have proposed to w when m' and w are engaged. In this case w would have broken the engagement if she does not certainly prefer m' to m (see the IF-THEN-ELSE condition of line 18). So we have established the fact that w certainly prefers m', her partner in μ, to m, implying that (m, w) cannot block μ. □

Algorithm 1 Algorithm CERTAINLY-STABLE

Input: $I = (M, W, L)$ where the certainly preferred relation is transitive for men. We assume that every agent certainly prefers every candidate over null. Let p_m denote the certainly preferred relation for each man m. We say that a woman w is at the head of a man m's certainly preferred relation if there is no other woman whom m certainly prefers to w.

Output: Matching μ that is certainly stable in I, or **false**

1 $\mu(w) \longleftarrow \varnothing$ for all $w \in W$ {Fiancé of w, null to start with.}
2 $F(m) \longleftarrow \emptyset$ for all $m \in M$ {Set of women engaged to m}
3 $R(w) \longleftarrow \emptyset$ for all $w \in W$ {Set of men rejected by w}
4 $R(m) \longleftarrow \emptyset$ for all $m \in M$ {Set of women who have rejected m}
5 **for all** $w \in W$ **do**
6   proposed(w) $\leftarrow$ **false**
7 **end for**
8 **while** some man m has a woman w at the head of p_m to whom he is not engaged **do**
9   m proposes to w
10   proposed(w) $\leftarrow$ **true**
11   **if** $m \succ_w^{\text{cert}} m'$ for each $m' \in R(w) \cup \{\mu(w)\}$ **then**
12   **if** $\mu(w) \neq \varnothing$ **then**
13       $F(\mu(w)) = F(\mu(w)) \setminus \{w\}$; $R(w) = R(w) \cup \{\mu(w)\}$; $R(\mu(w)) = R(\mu(w)) \cup \{w\}$; $delete(\mu(w), w)$ {$delete(\mu(w), w)$ means that w should be deleted from p_m.}
14   **end if**
15     $F(m) = F(m) \cup \{w\}$; $\mu(w) = m$
{If m is certainly more preferred to all the men w has rejected so far, as well as her current fiancé if she has one, then (i) if w is engaged she breaks her current engagement and subsequently is removed from her current fiancé's certainly preferred relation, and (ii) becomes engaged to m.}
16   **else**
17     $R(w) = R(w) \cup \{m\}$; $R(m) = R(m) \cup \{w\}$; delete(m,w)
{m's proposal is rejected by w and subsequently w is removed from $p(m)$.}
18   **if** $\mu(w) \neq \varnothing$ **and** $m' = \mu(w) \not\succ_w^{\text{cert}} m$ **then**
19       $F(m') = F(m') \setminus \{w\}$; $\mu(w) = \varnothing$; $R(w) = R(w) \cup \{m'\}$; $R(m') = R(m') \cup \{w\}$; $delete(m', w)$
20   **end if**
{If w is engaged and she does not certainly prefer her fiancé m' to m, she breaks the engagement and subsequently is removed from $p(m')$.}
21   **end if**
22 **end while**
23 **if** (there is a man m such that $|F(m)| > 1$) **or** (there is a woman w such that $\mu(w) = \varnothing$ and proposed(w) =**true**) **then**
24   **return false**
25 **else**
26   **return** μ
27 **end if**

We say that a pair (m, w) is a certainly-stable pair if m and w are matched together in at least one certainly stable matching.

Lemma 2. *No certainly-stable pair is deleted during the execution of the algorithm.*

Proof. Assume otherwise for a contradiction. Let (m, w) be the first certainly-stable pair deleted during the execution of the algorithm. Let μ^* be a certainly stable matching in which m and w are matched together.

Deletion of (m, w) can only occur under the following three scenarios. Case 1: w is engaged to m and receives a proposal from a man m' whom she strictly prefers to m and all the men she has already rejected, in which case the deletion occurs in line 13. Case 2: w receives a proposal from m, and there is a man $m' \in R(w) \cup \{\mu(w)\}$ to whom w does not certainly prefer m, in which case the deletion occurs in line 17. (ase 3: w is engaged to m and receives a proposal from a man m' to whom she does not strictly prefer m, in which case the deletion occurs in line 19. In all three cases, m' cannot have a certainly-stable partner w' whom he certainly prefers to w; for then the certainly-stable pair (m', w') would have been deleted before (m, w) during the execution of the algorithm, a contradiction. Therefore m' does not certainly prefer $\mu^*(m')$ to w. Hence (m', w) blocks μ^*, a contradiction. $\square$

Claim 6. *If Algorithm 1 returns a matching μ then μ is certainly stable in I.*

Proof. Assume for a contradiction that μ is not certainly stable. Therefore there exits a pair (m, w) that blocks μ; i.e., $\mu(m) \not\succ_m^{\text{cert}} w$ and $\mu(w) \not\succ_w^{\text{cert}} m$. It follows Lemma 1 that (m, w) is not deleted. If $w \succ_m^{\text{cert}} \mu(m)$, then for m to propose to $\mu(m)$ he must have already been rejected by w (or else $\mu(m)$ is not at the head of m's certainly preferred relation), and consequently (m, w) is deleted, a contradiction. Hence neither $\mu(m)$ nor w is certainly preferred to one another by m. If after the execution of the while loop w is at the head of m's certainly preferred relation, then m must have proposed to her and be engaged to her as well, a contradiction (as m is only engaged to one woman or else the algorithm returns "false"). So it remains that w is not at the head of m's certainly preferred relation. Therefore, as certainly preferred relation is transitive for men and $\mu(m) \not\succ_m^{\text{cert}} w$, there must be a woman w^*, besides $\mu(m)$, at the head of m's certainly preferred relation, implying that m must have proposed to her and be engaged to her as well, a contradiction. $\square$

Claim 7. *If I admits a certainly stable matching, then Algorithm 1 returns one.*

Proof. By Lemma 2 no certainly-stable pair is deleted during the execution of the algorithm. Therefore, at the termination of the while loop, each man m has in his reduced preferences all women w where (m, w) is a certainly-stable pair. Hence each m who is matched in some certainly stable matching is engaged in F to all women at the head of p_m.

Each woman w is engaged to at most one man as no woman keeps more than one fiancé.

We show that if either of the conditions in line 23 hold then I cannot admit any certainly stable matching. Assume, for a contradiction, that a certainly stable matching μ^* exists and the algorithm returns "false".

First assume that there is a woman w who has received at least one proposal and is not engaged ($\mu(w) = \varnothing$). Say man m proposed to her during the execution of the algorithm and was rejected. Suppose first that w is unmatched in μ^*. As m has proposed to her, thus there is no woman in m's reduced preferences (which includes all his certainly-stable partners and hence $\mu^*(m)$) whom he certainly prefers to w. Therefore, (m, w) blocks μ^*, a contradiction. Now suppose that w is matched in μ^* to say m_1 ($m_1 \neq m$ as (m, w) is removed and no certainly-stable pair is removed). As w is not engaged hence she is not at the head of $p(m_1)$. Let w_1 be a woman at the head of $p(m_1)$, implying that m_1 does not certainly prefer w to w_1. If w_1 is unmatched in μ^* then (m_1, w_1) blocks μ^*, a contradiction, thus w_1 is matched in μ^* to say m_2 and she certainly prefers m_2 to m_1 (or else (m_1, w_1) blocks μ^*). As each woman is engaged to at most one man, and w_1 is engaged to m_1, hence w_1 is not engaged to m_2 implying that she is not at the head of $p(m_2)$. Thus there must be a woman, say w_2, who is at the head of $p(m_2)$. Applying the same arguments as we used for w_1, we deduce that w_2 is matched in μ^* to say m_3, she certainly prefers m_3 to m_2 and she is not at the head of $p(m_3)$, implying that there is another woman at the head of $p(m_3)$. We can continue with this argument, and as the number of men and women are finite we shall at some point encounter a man or a woman twice. Take the first time an agent is visited for the second time. If it's a man, this implies that he is matched to two different women in μ^*, a contradiction to μ^* being a matching. If it's a woman, this implies that she is engaged to more than one man, a contradiction.

Now assume that each woman who has received at least one proposal is engaged at the end of the while loop. Denote the set of men and women engaged by M_F and W_F respectively. Suppose that $|F(m)| > 1$ for at least one man m (i.e., m is engaged to two or more women), thus $|M_F| < |W_F|$. As no certainly-stable pair is removed (Lemma 2), any man who is not in M_F cannot be matched in any certainly stable matching. This fact, combined by $|M_F| < |W_F|$, implies that in any certainly stable matching some woman in W_F remains unmatched. Suppose $w \in W_F$ is unmatched in μ^*. Let m be the man to whom w is engaged ($w \in F(m)$). As m is engaged to w, w must be at the head of $p(m)$. Therefore m does not certainly prefer any of his certainly-stable partners, including $\mu^*(m)$, to w. Thus, (m, w) blocks μ^*, a contradiction. □

Proof of Theorem 5. At most n^2 proposals are made during the execution of Algorithm 1. Using suitable data structures, the computation of p, computation of the head of men's certainly preferred relations, and deletions can be done in polynomial time. Hence Claims 6 and 7 complete the proof of Theorem 5. □

5. COROLLARY ON GRAPH KERNELS

In this section we describe how Theorem 4 directly implies the NP-hardness of the graph kernel checking problem for a special graph class. For a directed graph $D = (V, A)$, a kernel is a set of vertices $K \subseteq V$ which is both *independent* and *absorbant*, i.e., no arc exists between any two vertices in K (independence) and for any vertex u outside K there is a vertex $v \in K$ such that $uv \in A$ (absorbancy). In 1983, Berge and Duchet conjectured that an undirected graph G is perfect if and only if the following condition is satisfied: "If D is any orientation of G such that every clique of D has a kernel, then D has a kernel." Maffray [11] proved that the conjecture holds when G is the line-graph of another graph H, i.e., when G represents the incidence between the edges of H. Thus, for a bipartite graph H, whose line graph G is always perfect, the statement implies that for acyclic orientations in each clique D, when D trivially has a kernel, G also has a kernel.

As Fleiner [7] pointed out in his thesis, the above argument implies the existence of a stable matching for bipartite graphs, a theorem proved by Gale and Shapley [9]. To see this connection, let H be the two-sided graph of a stable marriage instance and let G be the line-graph of H, where each clique corresponds to a set of potential partnerships of an agent. The acyclic orientation of D is created according to the preferences of the agents, if agent u prefers b to a in the stable marriage instance then we have two corresponding vertices, ub and ua in G and we orient this edge towards ub in D. It is now straightforward to show that the kernels of D are in a one-to-one correspondence with the stable matchings of the marriage problem.

In a similar way, the pairwise comparison model with certain but intransitive preferences can be translated to the problem of finding a kernel in a line-graph of a bipartite graph. The orientation of the directed graph D should be as described above. The problem of deciding whether a certainly stable matching exists in this model is equivalent to the problem of deciding whether a kernel exists in the corresponding directed graph. However, here, as the existence of a certainly stable matching is not guaranteed, a kernel may not exist either. The problem of finding a kernel in a directed graph is NP-hard in general [4], and also for some special graphs [8]. However, we are not aware of any result on the complexity of this problem for the line-graph of bipartite graphs. Note that this problem family was described in the Open problem section of the EGRES research group [5]. We state this hardness result in the following theorem.

Theorem 8. *The problem of deciding whether a directed graph D has a kernel is NP-complete, even if D is a directed line-graph of a bipartite graph.*

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