Minimal Retentive Sets in Tournaments

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ABSTRACT

Many problems in multiagent decision making can be addressed using *tournament solutions*, i.e., functions that associate with each complete and asymmetric relation on a set of alternatives a non-empty subset of the alternatives. For any given tournament solution S, there is another tournament solution $\mathring{S}$, which returns the union of all inclusion-minimal sets that satisfy S-*retentiveness*, a natural stability criterion with respect to S. Schwartz's *tournament equilibrium set* (TEQ) is then defined as $TEQ = \mathring{TEQ}$. Due to this unwieldy recursive definition, preciously little is known about TEQ. Contingent on a well-known conjecture about TEQ, we show that $\mathring{S}$ inherits a number of important and desirable properties from S. We thus obtain an infinite hierarchy of attractive and efficiently computable tournament solutions that "approximate" TEQ, which itself is intractable. This hierarchy contains well-known tournament solutions such as the top cycle (TC) and the minimal covering set (MC). We further prove a weaker version of the conjecture mentioned above, which establishes $\mathring{TC}$ as an attractive new tournament solution.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*

General Terms

Theory, Economics

Keywords

Social Choice Theory, Tournament Solutions, Retentiveness, Tournament Equilibrium Set

1. INTRODUCTION

Many problems in multiagent decision making can be addressed using tournament solutions, i.e., functions that associate with each complete and asymmetric relation on a set of alternatives a non-empty subset of the alternatives. For instance, tournament solutions play an important role in social choice theory, where the binary relation is typically defined via pairwise majority voting [22, 21]. Other application areas include multi-criteria decision analysis [2, 3], zero-sum games [14, 19, 10], coalition formation [6], and argumentation theory [11, 12].

Examples of well-studied tournament solutions are the Copeland set, the minimal covering set, the Banks set, and the Slater set [21]. Recent years have witnessed an increasing interest in these concepts by the multiagent systems and theoretical computer science communities, particularly with respect to their computational complexity. For example, the Copeland set and the minimal covering set of a tournament can be computed in polynomial time [7, 5], whereas computing the Banks set and the Slater set is computationally intractable [25, 1, 9].

The *tournament equilibrium set* (TEQ), introduced by Schwartz [24], ranks among the most intriguing, but also among the most enigmatic, tournament solutions. For a given tournament solution S, Schwartz calls a set of alternatives S-*retentive* if it satisfies a natural stability criterion with respect to S. He then recursively defines TEQ as $\mathring{TEQ}$, the union of all inclusion-minimal TEQ-retentive sets. Unfortunately, and somewhat surprisingly, it is unknown whether TEQ satisfies several important properties proposed in the literature on tournament solutions, namely *monotonicity*, *independence of unchosen alternatives*, and the *weak superset property*. However, Laffond et al. [18] and Houy [16, 17] have shown that TEQ satisfies any one of these properties if and only if it satisfies all of them. They moreover showed that TEQ satisfying any of the properties is equivalent to the statement that every tournament contains a *unique* minimal TEQ-retentive set. This statement had already been conjectured by Schwartz [24] and also implies that TEQ is strictly contained in the minimal covering set. Apart from these implications, the only known facts about TEQ are that it is contained in the Banks set [24], satisfies composition-consistency [20], and is NP-hard to compute [8].

In this paper, we approach the matter from a more general perspective and study tournament solutions that are defined via Schwartz's notion of retentiveness, i.e., we consider $\mathring{S}$ for any given tournament solution S. For tournament solutions S that always admit a unique minimal S-retentive set, we show that most desirable properties are inherited from S to $\mathring{S}$ (and also from $\mathring{S}$ to S). Composition-consistency is a notable exception as we prove that TEQ is the *only* composition-consistent tournament solution defined via retentiveness.

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Starting with the trivial tournament solution that always returns all alternatives, one can define an infinite sequence of tournament solutions $S_1, S_2, \ldots$ such that $S_{i+1} = \mathring{S}_i$. Assuming Schwartz's conjecture, we show that these tournament solutions are strictly contained in each other, strictly contain *TEQ*, and share most of the desirable properties of *TEQ*. The sequence converges in a well-defined way to *TEQ* and yields an infinite sequence of weaker versions of Schwartz's conjecture. The first statement of this sequence was shown by Good [15], and we conclude the paper by proving the second one.

2. PRELIMINARIES

In this section, we provide the terminology and notation required for our results (see Laslier [21] for an excellent overview of tournament solutions and their properties).

2.1 Tournaments

Let X be a universe of alternatives, and assume for notational convenience that $\mathbb{N} \subseteq X$. The set of all *finite* subsets of X will be denoted by $\mathcal{F}_0(X)$, the set of all *non-empty* finite subsets of X by $\mathcal{F}(X)$. A (finite) *tournament* T is a pair $(A, \succ)$, where $A \in \mathcal{F}(X)$ and $\succ$ is an asymmetric and complete (and thus irreflexive) binary relation on X, usually referred to as the *dominance relation*.[1] Intuitively, $a \succ b$ signifies that alternative a is preferable to alternative b. The dominance relation can be extended to sets of alternatives by writing $A \succ B$ when $a \succ b$ for all $a \in A$ and $b \in B$. We further write $\mathcal{T}(X)$ for the set of all tournaments on X.

For a set $B \subseteq X$, a relation $R \subseteq X \times X$, and an element a, we denote by $D_{B,R}(a)$ the *dominion* of a in B, i.e.,

$$D_{B,R}(a) = \{\, b \in B : a \; R \; b\},$$

and by $\overline{D}_{B,R}(a)$ the *dominators* of a in B, i.e.,

$$\overline{D}_{B,R}(a) = \{\, b \in B : b \; R \; a\}.$$

Whenever the tournament $(A, \succ)$ is known from the context and R is the dominance relation $\succ$ or B is the set of all alternatives A, the respective subscript will be omitted to improve readability.

For a tournament $T = (A, \succ)$ and a subset $B \subseteq A$ of alternatives, we further write $T|_B = (B, \{(a, b) \in B \times B : a \succ b\})$ for the restriction of T to B.

The *order* of a tournament $T = (A, \succ)$ refers to the cardinality of A. A *tournament isomorphism* of two tournaments $T = (A, \succ)$ and $T' = (A', \succ')$ is a bijection $\pi \colon A \to A'$ such that for all $a, b \in A$, $a \succ b$ if and only if $\pi(a) \succ' \pi(b)$.

2.2 Components and Decompositions

An important structural notion in the context of tournaments is that of a *component*. A component is a subset of alternatives that bear the same relationship to all alternatives not in the set.

Definition 1. *Let $T = (A, \succ)$ be a tournament. A non-empty subset B of A is a* component *of T if for all $a \in A \setminus B$, either $B \succ a$ or $a \succ B$. A* decomposition *of T is a set of pairwise disjoint components $\{B_1, \ldots, B_k\}$ of T such that $A = \bigcup_{i=1}^{k} B_i$.*

For a given tournament $\tilde{T}$, a new tournament can be constructed by replacing each alternative with a component.

Definition 2. *Let $B_1, \ldots, B_k \subseteq X$ be pairwise disjoint sets and $\tilde{T} = (\{1, \ldots, k\}, \tilde{\succ})$, $T_1 = (B_1, \succ_1)$, $\ldots$, $T_k = (B_k, \succ_k)$ tournaments. The* product *of $T_1, \ldots, T_k$ with respect to $\tilde{T}$, denoted by $\Pi(\tilde{T}, T_1, \ldots, T_k)$, is the tournament $(A, \succ)$ such that $A = \bigcup_{i=1}^{k} B_i$ and for all $b_1 \in B_i, b_2 \in B_j$,*

$$b_1 \succ b_2 \text{ if only if } i = j \text{ and } b_1 \succ_i b_2, \text{ or } i \neq j \text{ and } i \mathrel{\tilde{\succ}} j.$$

2.3 Tournament Solutions

Consider the *maximum* function max : $\mathcal{T}(X) \to \mathcal{F}_0(X)$ given by $\max((A, \succ)) = \{a \in A : a \succ b \text{ for all } b \in A \setminus \{a\}\}$. Due to the asymmetry of the dominance relation, this function returns at most one alternative in any tournament. Moreover, maximal—i.e., undominated—and maximum elements coincide. In social choice theory, the maximum of a majority tournament is commonly referred to as the *Condorcet winner*.

Since the dominance relation may contain cycles and thus fail to have a maximal element, a variety of concepts have been suggested to take over the role of singling out the "best" alternatives of a tournament. Formally, a *tournament solution* S is defined as a function that associates with each tournament $T = (A, \succ)$ a non-empty subset $S(T)$ of A. Following Laslier [21], we require a tournament solution to be independent of alternatives outside the tournament, invariant under tournament isomorphisms, and to select the maximal element whenever it exists.

Definition 3. *A* tournament solution *is a function $S : \mathcal{T}(X) \to \mathcal{F}(X)$ such that*

(i) $S(T) = S(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $T|_A = T'|_A$;

(ii) $S((\pi(A), \succ')) = \pi(S((A, \succ)))$ for all tournaments $(A, \succ)$, $(A', \succ')$, and every tournament isomorphism $\pi : A \to A'$ of $(A, \succ)$ and $(A', \succ')$; and

(iii) $\max(T) \subseteq S(T) \subseteq A$ *for all tournaments $T = (A, \succ)$.*

Laslier [21] is slightly more stringent here as he requires the maximum to be the only element in $S(T)$ whenever it exists. We will call a tournament solution *proper* if it satisfies this additional requirement.

The conditions of Definition 3 are trivially satisfied if one invariably selects the set of all alternatives. The corresponding tournament solution *TRIV* is obtained by letting $\mathit{TRIV}((A, \succ)) = A$ for every tournament $(A, \succ)$. Among the tournament solutions considered in this paper, *TRIV* is the only one that is not proper. The *top cycle* $\mathit{TC}(T)$ of a tournament $T = (A, \succ)$ is defined as the smallest set $B \subseteq A$ such that $B \succ A \setminus B$. Uniqueness of such a set is straightforward and was first shown by Good [15].

For two tournament solutions S and S', we write $S' \subseteq S$, and say that S' is a *refinement* of S, if $S'(T) \subseteq S(T)$ for all tournaments T. To avoid cluttered notation, we write $S(A, \succ)$ instead of $S((A, \succ))$ for a tournament $T = (A, \succ)$. Furthermore, we frequently write $S(B)$ instead of $S(B, \succ)$ for a subset $B \subseteq A$ of alternatives, if the dominance relation $\succ$ is known from the context.

2.4 Retentive Sets

Motivated by cooperative majority voting, Schwartz [24] introduced a tournament solution based on a notion he calls

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[1] This definition slightly diverges from the common graph-theoretic definition where $\succ$ is defined on A rather than X. However, it facilitates the sound definition of tournament solutions.

retentiveness. The intuition underlying retentive sets is that alternative a is only "properly" dominated by alternative b if b is chosen among a's dominators by some underlying tournament solution S. A set of alternatives is then called S-retentive if none of its elements is properly dominated by some alternative outside the set with respect to S.

Definition 4. *Let S be a tournament solution and $T = (A, \succ)$ a tournament. Then, $B \subseteq A$ is* S-retentive *in T if $B \neq \emptyset$ and $S(\overline{D}(b)) \subseteq B$ for all $b \in B$ such that $\overline{D}(b) \neq \emptyset$. The set of S-retentive sets for a given tournament $T = (A, \succ)$ will be denoted by $\mathcal{R}_S(T)$, i.e., $\mathcal{R}_S(T) = \{B \subseteq A : B$ is S-retentive in $T\}$.*

Fix an arbitrary tournament solution S. Since the set A of all alternatives is trivially S-retentive in $(A, \succ)$, S-retentive sets are guaranteed to exist. If a Condorcet winner exists, it must clearly be contained in any S-retentive set. The union of all (inclusion-)minimal S-retentive sets thus defines a tournament solution.

Definition 5. *Let S be a tournament solution. Then, the tournament solution $\mathring{S}$ is given by*

$$\mathring{S}(T) = \bigcup \min_{\subseteq}(\mathcal{R}_S(T)).$$

Consider for example the tournament solution $TRIV$, which always selects the set of all alternatives. It is easily verified that there always exists a *unique* minimal $TRIV$-retentive set, and that in fact $\mathring{TRIV} = TC$.

For a tournament solution S, we say that $\mathcal{R}_S$ is *pairwise intersecting* if for each tournament T and for all sets $B, C \in \mathcal{R}_S(T)$, $B \cap C \neq \emptyset$. Observe that the non-empty intersection of two S-retentive sets is itself S-retentive. We thus have the following.

Proposition 1. *For every tournament solution S, $\mathcal{R}_S$ admits a unique minimal element if and only if $\mathcal{R}_S$ is pairwise intersecting.*

Schwartz introduced retentiveness in order to recursively define the *tournament equilibrium set* (TEQ) as the union of minimal TEQ-retentive sets. This recursion is well-defined because the order of the dominator set of any alternative is strictly smaller than the order of the original tournament.

Definition 6 (Schwartz [24]). *The* tournament equilibrium set *(TEQ) is defined recursively as $TEQ = \mathring{TEQ}$.*

In other words, TEQ is the unique fixed point of the $\circ$-operator. Schwartz conjectured that every tournament admits a *unique* minimal TEQ-retentive set.

Conjecture 1 (Schwartz [24]). *$\mathcal{R}_{TEQ}$ is pairwise intersecting.*

Despite several attempts to prove or disprove this statement (e.g., [18, 16]), it has remained an open problem. A recent computer analysis failed to find a counter-example in all tournaments of order 12 or less and a fairly large number of random tournaments [8].

It turns out that the existence of a *unique* minimal S-retentive set is quintessential for showing that $\mathring{S}$ satisfies several important properties to be defined in the next section.

2.5 Properties of Tournament Solutions

In order to compare tournament solutions with each other, a number of desirable properties for tournament solutions have been identified. In this section, we will review six of the most common properties.[2] Moulin [23], in a more general context, distinguishes between *monotonicity* and *independence* conditions, where a monotonicity condition describes the positive association of the solution with some parameter, and an independence condition characterizes the invariance of the solution under the modification of some parameter. Properties of tournament solutions can further be distinguished depending on whether they are defined via the dominance relation or via set inclusion.

We first consider a monotonicity and an independence property defined in terms of the dominance relation. A tournament solution is called *monotonic* if a chosen alternative remains in the choice set when extending its dominion and leaving everything else unchanged.

Definition 7. *A tournament solution S satisfies* monotonicity *(MON) if $a \in S(T)$ implies $a \in S(T')$ for all tournaments $T = (A, \succ)$, $T' = (A, \succ')$, and $a \in A$ such that $T|_{A\setminus\{a\}} = T'|_{A\setminus\{a\}}$ and $D_{\succ}(a) \subseteq D_{\succ'}(a)$.*

A solution satisfies *independence of unchosen alternatives* if the choice set is invariant under any modification of the dominance relation between unchosen alternatives.

Definition 8. *A tournament solution S is* independent of unchosen alternatives *(IUA) if $S(T) = S(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $T|_{S(T)\cup\{a\}} = T'|_{S(T)\cup\{a\}}$ for all $a \in A$.*

With respect to set inclusion, we consider a monotonicity property to be called the *weak superset property* and an independence property known as the *strong superset property*. A tournament solution satisfies the weak superset property if an unchosen alternative remains unchosen when other unchosen alternatives are removed.

Definition 9. *A tournament solution S satisfies the* weak superset property *(WSP) if $S(B) \subseteq S(A)$ for all tournaments $T = (A, \succ)$ and $B \subseteq A$ such that $S(A) \subseteq B$.*

The strong superset property requires that a choice set is invariant under the removal of alternatives not in the choice set.

Definition 10. *A tournament solution S satisfies the* strong superset property *(SSP) if $S(B) = S(A)$ for all tournaments $T = (A, \succ)$ and $B \subseteq A$ such that $S(A) \subseteq B$.*

The four properties defined above (MON, IUA, WSP, and SSP) will be called *basic* properties of tournament solutions. Observe that SSP implies WSP. Furthermore, the conjunction of MON and SSP implies IUA. It is therefore sufficient to show MON and SSP in order to prove that a tournament solution satisfies all basic properties. An additional property considered in this paper is *composition-consistency*. A tournament solution is composition-consistent if it chooses the "best" alternatives from the "best" components.

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[2] Our terminology slightly differs from the one by Laslier [21] and others. *Independence of unchosen alternatives* is also called *independence of the losers* or *independence of non-winners*. The *weak superset property* has been referred to as ϵ^+ or as the *Aïzerman property*.

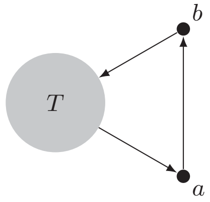

Figure 1: Tournament $C(T, I_a, I_b)$ for a given tournament T. The gray circle represents a component isomorphic to the original tournament T. An edge incident to a component signifies that there is an edge of the same direction incident to each alternative in the component.

Definition 11. *A tournament solution S is* composition-consistent *(COM) if for all tournaments T, $T_1, \dots, T_k$, and $\tilde{T}$ such that $T = \Pi(\tilde{T}, T_1, \dots, T_k)$, $S(T) = \bigcup_{i \in S(\tilde{T})} S(T_i)$.*

The properties defined in this section are not easily satisfied by discriminative tournament solutions. While *TRIV* trivially satisfies all of the properties, the Slater set only satisfies MON and the Banks set only satisfies MON, WSP, and COM. The minimal covering set satisfies all of the properties. The same holds for *TEQ* if Conjecture 1 is correct.

3. INHERITANCE OF PROPERTIES

In this section, we investigate which of the properties defined in the previous section are inherited from S to $\mathring{S}$ or from $\mathring{S}$ to S.

We begin by looking at a particular type of decomposable tournament that will be useful in the following. Let C_3 denote the tournament $C_3 = (\{1,2,3\}, \succ)$ with $1 \succ 2 \succ 3 \succ 1$, and write, for $a \in X$, I_a for the unique tournament on $\{a\}$. For three tournaments T_1, T_2, and T_3 on disjoint sets of alternatives, let $C(T_1, T_2, T_3) = \prod(C_3; T_1, T_2, T_3)$. The structure of $C(T, I_a, I_b)$ for a given tournament T is illustrated in Figure 1. We have the following lemma.

Lemma 1. *Let S be a proper tournament solution. Then, for each tournament T on A and all $a, b \notin A$,*

$$\mathring{S}(C(T, I_a, I_b)) = \{a,b\} \cup S(T).$$

Proof. Let $B = \mathring{S}(C(T, I_a, I_b))$ and observe that $B \cap A \neq \emptyset$, because neither $\{a,b\}$ nor any subset of it is S-retentive. Since a is the Condorcet winner in $\overline{D}(b) = \{a\}$ and b is the Condorcet winner in $\overline{D}(c)$ for any $c \in B \cap A$, by S-retentiveness of B we have that $a \in B$ and $b \in B$. Also by retentiveness of B, we have $S(\overline{D}(a)) = S(T) \subseteq B$. We have thus shown that every S-retentive set must contain $\{a,b\} \cup S(T)$, and that $\{a,b\} \cup S(T)$ is itself S-retentive. ☐

We are now ready to show that a number of desirable properties (including efficient computability) are inherited from $\mathring{S}$ to S and from S to $\mathring{S}$.

Theorem 1. *Let S be a proper tournament solution. Then the following holds:*

(i) Each of the four basic properties is satisfied by S if it is satisfied by $\mathring{S}$.

(ii) If $\mathcal{R}_S$ is pairwise intersecting, each of the following properties is satisfied by S if and only if it is satisfied by $\mathring{S}$: (MON $\wedge$ SSP), SSP, WSP, IUA.

(iii) $\mathring{S}$ is efficiently computable if and only if S is efficiently computable.

Proof. For (i), we show the following: If S violates one of the four basic properties MON, SSP, WSP, or IUA, then $\mathring{S}$ violates the same property. Observe that for each of these properties, the fact that S violates the property can be witnessed by a pair of tournaments $T_1 = (A_1, \succ_1)$ and $T_2 = (A_2, \succ_2)$: In the case of SSP (or WSP), $S(T_1) \subseteq A_2 \subset A_1$, $T_2|_{A_2} = T_1|_{A_2}$, and $S(T_2) \neq S(T_1)$ (or $S(T_2) \not\subseteq S(T_1)$). In the case of MON and IUA, $A_2 = A_1$ and the only difference between the dominance relations is that $D_{\succ_2}(a) = D_{\succ_1}(a) \cup \{b\}$ for some alternatives $a, b \in A_1$. For MON, $a \in S(T_1)$ and $a \notin S(T_2)$; for IUA, $a, b \notin S(T_1)$ and $S(T_1) \neq S(T_2)$.

The pair (T_1, T_2) will be called a counterexample. We go on to show how a counterexample for S can be transformed into a counterexample for $\mathring{S}$. For $a, b \notin A_1$, define $T_1' = C(T_1, I_a, I_b)$ and $T_2' = C(T_2, I_a, I_b)$. Lemma 1 implies that $\mathring{S}(T_1') = \{a,b\} \cup S(T_1)$ and $\mathring{S}(T_2') = \{a,b\} \cup S(T_2)$. Hence, the pair (T_1', T_2') constitutes a counterexample for $\mathring{S}$.

For (ii), assume that $\mathcal{R}_S$ is pairwise intersecting. We need to show that each of the properties (MON$\wedge$SSP), SSP, WSP, and IUA is satisfied by S if and only if it is satisfied by $\mathring{S}$. The direction from right to left follows from (i). We now show that the properties are inherited from S to $\mathring{S}$.

Assume S satisfies SSP. Let $T = (A, \succ)$ be a tournament, and consider an alternative $x \in A \setminus \mathring{S}(T)$. We need to show that $\mathring{S}(T') = \mathring{S}(T)$, where $T' = (A \setminus \{x\}, \succ)$. Since $\mathcal{R}_S$ is pairwise intersecting, it suffices to show that for all $a \in \mathring{S}(T)$, $S(\overline{D}_A(a)) = S(\overline{D}_{A \setminus \{x\}}(a))$. To this end, consider an arbitrary $a \in \mathring{S}(T)$. If $x \notin \overline{D}_A(a)$, then obviously $\overline{D}_A(a) = \overline{D}_{A \setminus \{x\}}(a)$ and thus $S(\overline{D}_A(a)) = S(\overline{D}_{A \setminus \{x\}}(a))$. Assume on the other hand that $x \in \overline{D}_A(a)$. Since $a \in \mathring{S}(T)$ and $x \notin \mathring{S}(T)$, it follows that $x \notin S(\overline{D}_A(a))$. Now, since S satisfies SSP, we obtain $S(\overline{D}_A(a)) = S(\overline{D}_{A \setminus \{x\}}(a))$ as desired.

Assume that S satisfies WSP. Let $T = (A, \succ)$ be a tournament, and consider an alternative $x \in A \setminus \mathring{S}(T)$. We need to show that $\mathring{S}(T') \subseteq \mathring{S}(T)$, where $T' = (A \setminus \{x\}, \succ)$. Since $\mathcal{R}_S$ is pairwise intersecting, it suffices to show that $\mathring{S}(T)$ is also S-retentive in T'. To this end, consider an arbitrary $a \in \mathring{S}(T)$. Since S satisfies WSP, we have that $S(\overline{D}_{A \setminus \{x\}}(a)) \subseteq S(\overline{D}_A(a))$. Furthermore, by S-retentiveness of $\mathring{S}(T)$, $S(\overline{D}_A(a)) \subseteq \mathring{S}(T)$ and thus $S(\overline{D}_{A \setminus \{x\}}(a)) \subseteq \mathring{S}(T)$.

Assume that S satisfies IUA. Let $T = (A, \succ)$ and $T' = (A, \succ')$ be tournaments such that $T|_{A \setminus \{x,y\}} = T'|_{A \setminus \{x,y\}}$ and consider $x, y \in A \setminus \mathring{S}(T)$. We need to show that $\mathring{S}(T) = \mathring{S}(T')$. Since $\mathcal{R}_S$ is pairwise intersecting, it suffices to show that for all $a \in \mathring{S}(T)$, $S(\overline{D}_{\succ}(a), \succ) = S(\overline{D}_{\succ'}(a), \succ')$. To this end, consider an arbitrary $a \in \mathring{S}(T)$. By assumption, $a \neq x$ and $a \neq y$. First consider the case when both $x \in \overline{D}_{\succ}(a)$ and $y \in \overline{D}_{\succ}(a)$. Then, $\overline{D}_{\succ}(a) = \overline{D}_{\succ'}(a)$ and, by S-retentiveness of $\mathring{S}(T)$, $x, y \notin S(\overline{D}_{\succ}(a), \succ)$. Since S satisfies IUA, $S(\overline{D}_{\succ}(a), \succ) = S(\overline{D}_{\succ'}(a), \succ')$ as required. Now consider the case when $x \notin \overline{D}_{\succ}(a)$ or $y \notin \overline{D}_{\succ}(a)$. Then, $T|_{\overline{D}_{\succ}(a)} = T'|_{\overline{D}_{\succ'}(a)}$, and the claim follows immediately.

Assume that S satisfies MON and SSP. Since we have already shown that SSP is inherited, it remains to be shown that $\mathring{S}$ satisfies MON. Let $T = (A, \succ)$ be a tournament, and consider two alternatives $a, b \in A$ such that $a \in \mathring{S}(T)$ and $b \succ a$. Let $T' = (A, \succ')$ be the tournament such that $T|_{A\setminus\{a\}} = T'|_{A\setminus\{a\}}$ and $D_{\succ'}(a) = D_{\succ}(a) \cup \{b\}$. We have to show that $a \in \mathring{S}(T')$. To this end, we claim that for all $c \in A \setminus \{a\}$,

$$\begin{aligned} &a \notin S(\overline{D}_{\succ'}(c), \succ') \quad \text{implies} \\ &\qquad\qquad S(\overline{D}_{\succ}(c), \succ) = S(\overline{D}_{\succ'}(c), \succ'). \end{aligned} \tag{1}$$

Consider the case when $c \neq b$ and assume that $a \notin S(\overline{D}_{\succ'}(c), \succ')$. It follows from monotonicity of S that $a \notin S(\overline{D}_{\succ}(c), \succ)$. To see this, observe that monotonicity of S implies that $a \in S(\overline{D}_{\succ'}(c), \succ')$ whenever $a \in S(\overline{D}_{\succ}(c), \succ)$. Now, since S satisfies SSP,

$$\begin{aligned} S(\overline{D}_{\succ'}(c), \succ') &= S(\overline{D}_{\succ'}(c) \setminus \{a\}, \succ') \quad \text{and} \\ S(\overline{D}_{\succ}(c), \succ) &= S(\overline{D}_{\succ}(c) \setminus \{a\}, \succ). \end{aligned}$$

It is easily verified that $(\overline{D}_{\succ'}(c) \setminus \{a\}, \succ') = (\overline{D}_{\succ}(c) \setminus \{a\}, \succ)$, thus we have $S(\overline{D}_{\succ'}(c), \succ') = S(\overline{D}_{\succ}(c), \succ)$.

If $c = b$, then $a \notin S(\overline{D}_{\succ'}(b), \succ')$ together with SSP of S implies $S(\overline{D}_{\succ'}(b), \succ') = S(\overline{D}_{\succ'}(b) \setminus \{a\}, \succ')$. Furthermore, by definition of T and T', $(\overline{D}_{\succ'}(b) \setminus \{a\}, \succ') = (\overline{D}_{\succ}(b), \succ)$ and thus $S(\overline{D}_{\succ'}(b) \setminus \{a\}, \succ') = S(\overline{D}_{\succ}(b), \succ)$. This proves (1).

We proceed to show that $a \in \mathring{S}(T')$. Assume for contradiction that this is not the case. We claim that this implies that

$$\mathring{S}(T') \text{ is } S\text{-retentive in } T. \tag{2}$$

To see this, consider $c \in \mathring{S}(T')$. We have to show that $S(\overline{D}_{\succ}(c), \succ) \subseteq \mathring{S}(T')$. Since, by assumption, $a \notin \mathring{S}(T')$, we have that $a \notin S(\overline{D}_{\succ'}(c), \succ')$. We can thus apply (1) and get

$$S(\overline{D}_{\succ}(c), \succ) = S(\overline{D}_{\succ'}(c), \succ') \text{ for all } c \in \mathring{S}(T'),$$

which, together with the S-retentiveness of $\mathring{S}(T')$ in T', implies (2).

Since the minimal S-retentive set is unique, it follows from (2) that $\mathring{S}(T) \subseteq \mathring{S}(T')$. Hence, $a \notin \mathring{S}(T)$, a contradiction. This shows that $\mathring{S}$ satisfies MON and completes the proof of *(ii)*.

For *(iii)*, we show that the computation of S and the computation of $\mathring{S}$ are equivalent under polynomial-time reductions.

To see that $\mathring{S}$ can be reduced to S, consider an arbitrary tournament $T = (A, \succ)$ and define the relation $R = \{(a, x) : x \in S(\overline{D}(a))\}$. It is easily verified that $\mathring{S}(T)$ is the union of all minimal R-undominated sets[3] or, equivalently, the maximal elements of the asymmetric part of the transitive closure of R. Observing that both R and the minimal R-undominated sets can be computed in polynomial time (see, e.g., [7], for the latter) completes the reduction.

For the reduction from S to $\mathring{S}$, consider a tournament T on A and define $T^* = C(T, I_a, I_b)$ for $a, b \notin A$. By Lemma 1, $S(T) = \mathring{S}(T^*) \setminus \{a, b\}$. Clearly, T^* can be computed in polynomial time from T, and $S(T)$ can be computed in polynomial time from $\mathring{S}(T^*)$. ☐

We conclude this section by showing that, among all tournament solutions that are defined as a minimal retentive set with respect to some proper tournament solution, *TEQ* is the only one that is composition consistent.

Proposition 2. *Let S be a proper tournament solution such that $\mathcal{R}_S$ is pairwise intersecting. Then, $\mathring{S}$ satisfies* COM *if and only if $S = \mathit{TEQ}$.*

Proof. It is well-known that *TEQ* is composition-consistent [20]. For the direction from left to right, let S be a tournament solution different from *TEQ*, and assume that $\mathring{S}$ is composition-consistent. Since *TEQ* is the only tournament solution S' such that $S' = \mathring{S}'$, there has to exist a tournament T on A such that $S(T) \neq \mathring{S}(T)$. Let $a, b \notin A$, and define $T^* = C(T, I_a, I_b)$. By Lemma 1,

$$\mathring{S}(T^*) = \{a, b\} \cup S(T).$$

On the other hand, by composition-consistency of $\mathring{S}$,

$$\mathring{S}(T^*) = \mathring{S}(T) \cup \mathring{S}(I_a) \cup \mathring{S}(I_b) = \{a, b\} \cup \mathring{S}(T).$$

It follows that $S(T) = \mathring{S}(T)$, a contradiction. ☐

Although *TRIV* is not proper, it is easily seen that all the statements of Theorem 1 and Proposition 2 also hold for *TRIV*. This is due to the fact that Lemma 1 trivially holds for $S = \mathit{TRIV}$.

4. CONVERGENCE

In this section, we study the iterated application of the $\circ$-operator. Inductively define

$$S^{(0)} = S \quad \text{and} \quad S^{(k+1)} = \mathring{S^{(k)}},$$

and consider the sequence $(S^{(n)})_{n\in\mathbb{N}} = (S^{(0)}, S^{(1)}, S^{(2)}, \dots)$. We say that $(S^{(n)})_{n\in\mathbb{N}}$ *converges* to a tournament solution S' if for each tournament T, there exists $k_T \in \mathbb{N}$ such that $S^{(n)}(T) = S'(T)$ for all $n \geq k_T$.

A perhaps surprising result is the following.

Theorem 2. *Every tournament solution converges to TEQ.*

Proof. Let S be a tournament solution. We show by induction on n that

$$S^{(n-1)}(T) = \mathit{TEQ}(T).$$

for all tournaments $T = (A, \succ)$ of order $|A| \leq n$. The case $n = 1$ is trivial. For the induction step, let $T = (A, \succ)$ be a tournament of order $|A| = n + 1$. We have to show that $S^{(n)}(T) = \mathit{TEQ}(T)$. Since $S^{(n)}$ is defined as the union of all minimal $S^{(n-1)}$-retentive sets, it suffices to show that a subset $B \subseteq A$ is $S^{(n-1)}$-retentive if and only if it is *TEQ*-retentive. We have the following chain of equivalences:

$$\begin{aligned} B \text{ is } S^{(n-1)}\text{-retentive} \quad &\text{iff} \quad \text{for all } b \in B,\ S^{(n-1)}(\overline{D}(b)) \subseteq B \\ &\text{iff} \quad \text{for all } b \in B,\ \mathit{TEQ}(\overline{D}(b)) \subseteq B \\ &\text{iff} \quad B \text{ is } \mathit{TEQ}\text{-retentive}. \end{aligned}$$

In particular, the second equivalence follows from the induction hypothesis, since obviously $|\overline{D}(a)| \leq n$ for all $a \in A$. ☐

We proceed by identifying properties of $S^{(k)}$ that are equivalent to Conjecture 1. The following lemma will be useful.

---

[3] A set $B \subseteq A$ is R-undominated if $(a, b) \in R$ for no $b \in B$ and $a \in A \setminus B$.

LEMMA 2. *Let* S_1 *and* S_2 *be tournament solutions such that* $S_1 \subseteq S_2$ *and* $\mathcal{R}_{S_1}$ *is pairwise intersecting. Then,* $\mathcal{R}_{S_2}$ *is pairwise intersecting and* $\mathring{S}_1 \subseteq \mathring{S}_2$*.*

PROOF. First observe that $S_1 \subseteq S_2$ implies that every S_2-retentive set is S_1-retentive. Now assume for contradiction that $\mathcal{R}_{S_2}$ does not intersect pairwise and consider a tournament $T = (A, \succ)$ with two disjoint S_2-retentive sets $B, C \subseteq A$. Then, by the above observation, B and C are S_1-retentive, which contradicts the fact that $\mathcal{R}_{S_1}$ is pairwise intersecting.

Furthermore, for every tournament T, $\mathring{S}_2(T)$ is S_1-retentive and thus contains the unique minimal S_1-retentive set, i.e., $\mathring{S}_1(T) \subseteq \mathring{S}_2(T)$. $\square$

THEOREM 3. *Let* S *be a tournament solution with* $TEQ \subseteq S$ *that satisfies* WSP *or* IUA*. Then, the following statements are equivalent:*

(i) For all $k \in \mathbb{N}$, $\mathcal{R}_{S^{(k)}}$ *is pairwise intersecting.*

(ii) For all $k \in \mathbb{N}$, $S^{(k)}$ *satisfies each of the following properties if* S *does:* $(\mathit{MON} \wedge \mathit{SSP})$, SSP, WSP, IUA.

(iii) Conjecture 1 holds.

PROOF. To see that *(i)* implies *(ii)*, assume that $\mathcal{R}_{S^{(k)}}$ is pairwise intersecting. Then, by Theorem 1, the properties $(\mathsf{MON} \wedge \mathsf{SSP})$, SSP, WSP, and IUA are inherited from $S^{(k)}$ to $S^{(k+1)}$.

For the implication from *(ii)* to *(iii)*, let $\mathsf{P} \in \{\mathsf{MON}, \mathsf{SSP}, \mathsf{WSP}, \mathsf{IUA}\}$ be a basic property such that $S^{(k)}$ satisfies P for all $k \in \mathbb{N}$ and assume for contradiction that Conjecture 1 does not hold. We know from the work of Laffond et al. [18] and Houy [16, 17] that this assumption is equivalent to TEQ not satisfying any of the four basic properties. In particular, the latter has to be true for P. Let T_1 and T_2 be two tournaments showing that TEQ indeed violates P, and let n be the order of T_1. In the proof of Theorem 2, we have shown that $S^{(n-1)}(T) = TEQ(T)$ for all tournaments T of order at most n. Thus T_1 and T_2 serve as an example that for some k, $S^{(k)}$ violates P.

Finally, for the implication from *(iii)* to *(i)*, assume that Conjecture 1 holds. We first prove by induction on k that $TEQ \subseteq S^{(k)}$ for all $k \in \mathbb{N}$. The case $k = 1$ holds by assumption. Now let T be a tournament and suppose that $TEQ(T) \subseteq S^{(k)}(T)$ for some $k \in \mathbb{N}$. By definition, $S^{(k+1)}(T)$ is $S^{(k)}$-retentive. We can thus apply the induction hypothesis to obtain that $S^{(k+1)}(T)$ is TEQ-retentive. Since the minimal TEQ-retentive set is unique, it is contained in any TEQ-retentive set, and we have that $TEQ(T) \subseteq S^{(k+1)}(T)$. We can now apply Lemma 2 with $S_1 = TEQ$ and $S_2 = S^{(k)}$ to show that $\mathcal{R}_{S^{(k)}}$ is pairwise intersecting for all $k \in \mathbb{N}$. $\square$

Among the tournament solutions that satisfy the requirements of Theorem 3 are $TRIV$, TC, the uncovered set UC, and the Banks set BA (see, e.g., Laslier [21] for definitions of the latter two).

4.1 Contracting Sequences

Theorem 2 showed that every tournament solution converges to TEQ. From a practical point of view, monotonic convergence that either yields smaller and smaller supersets of TEQ or larger and larger subsets of TEQ would be particularly desirable. The latter is somewhat problematic as no refinement of TEQ is known and it is doubtful whether any such refinement would be efficiently computable. The former type of convergence turns out to be particularly useful. Call a sequence $(S^{(n)})_{n \in \mathbb{N}}$ of tournament solutions *contracting* if for all $k \in \mathbb{N}$, $S^{(k+1)} \subseteq S^{(k)}$. The elements of such a sequence constitute better and better "approximations" of TEQ. The following proposition identifies a sufficient condition for a sequence to be contracting.

PROPOSITION 3. *Let* S *be a tournament solution with* $TEQ \subseteq S$*. If Conjecture 1 holds and* $\mathring{S} \subseteq S$*, then* $S^{(k+1)} \subseteq S^{(k)}$ *for all* $k \in \mathbb{N}$*.*

PROOF. We prove the statement by induction on k. Let T be an arbitrary tournament. $\mathring{S}(T) \subseteq S(T)$ holds by assumption. Now suppose that $S^{(k)}(T) \subseteq S^{(k-1)}(T)$ for some $k \in \mathbb{N}$. As in the proof of Theorem 3, one can show that $TEQ \subseteq S^{(k)}$. Applying Lemma 2 with $S_1 = TEQ$ and $S_2 = S^{(k)}$ yields that $\mathcal{R}_{S^{(k)}}$ is pairwise intersecting. Therefore, we can apply Lemma 2 again, this time with $S_1 = S^{(k)}$ and $S_2 = S^{(k-1)}$, which gives $S^{(k+1)} \subseteq S^{(k)}$. $\square$

For example, the well-known tournament solutions $TRIV$, TC, UC, and MC give rise to contracting sequences. For $TRIV$ and $TC = \mathring{TRIV}$, the assumptions of Proposition 3 are obviously satisfied. For MC, Laffond et al. [18] have shown that Conjecture 1 implies $TEQ \subseteq MC$ and Brandt [4] has shown that $\mathring{MC} \subseteq MC$. Finally, $TEQ \subseteq UC$ was shown by Schwartz [24] and $\mathring{UC} \subseteq UC$ follows from Conjecture 1 and the observation that $UC(T)$ is UC-retentive for all tournaments T.

The sequences $(TRIV^{(n)})_{n \in \mathbb{N}}$ and $(MC^{(n)})_{n \in \mathbb{N}}$ may be of particular interest. Under the assumption that Conjecture 1 holds, those sequences are contracting and all tournament solutions in those sequences satisfy all basic properties. Furthermore, by Theorem 1 *(iii)*, $TRIV^{(k)}$ as well as $MC^{(k)}$ can be computed in polynomial time for any fixed $k \in \mathbb{N}$. Observe that this does not imply that TEQ can be computed efficiently due to the fact that there exists no $k \in \mathbb{N}$ such that $TRIV^{(k)} = TEQ$, which follows from Proposition 4 below. In fact, Brandt et al. [8] have shown that it is NP-hard to decide whether a given alternative is in TEQ.

One might wonder if MC is contained in the sequence $(TRIV^{(n)})_{n \in \mathbb{N}}$. Actually, it is easy to see that this is not the case: While MC is known to be composition-consistent (see [20]), Proposition 2 establishes that this is not the case for any $TRIV^{(k)}$ with $k \geq 1$.

4.2 Rate of Convergence

We may ask how many iterated applications of the $\circ$-operator are needed until we arrive at TEQ. While we have seen that every tournament solution converges to TEQ, it turns out that no solution other than TEQ itself does so in a finite number of steps.

For a tournament solution S, let $k_n(S)$ be the smallest $k \in \mathbb{N}$ such that $S^{(k)}(T) = TEQ(T)$ for all tournaments T of order at most n.

PROPOSITION 4. *Let* $S \neq TEQ$ *be a proper tournament solution. For each* $n \in \mathbb{N}$ *with* $n \geq n_0$,

$$\left\lfloor \frac{n - n_0}{2} \right\rfloor < k_n(S) \leq n - 1,$$

where n_0 *is the order of a smallest tournament* T *with* $S(T) \neq TEQ(T)$*.*

PROOF. The upper bound follows immediately from the fact that $S^{(n-1)}(T) = TEQ(T)$ for every tournament solution S and every tournament T of order at most n. This was shown in the proof of Theorem 2.

For the lower bound, let $S \neq TEQ$ be a tournament solution. We inductively define a family $T_0, T_1, T_2, \ldots$ of tournaments such that $S^{(k)}(T_k) \neq TEQ(T_k)$. Let $T_0 = (A_0, \succ)$ be a smallest tournament such that $S(T_0) \neq TEQ(T_0)$. Given $T_{k-1} = (A_{k-1}, \succ)$, let $T_k = C(T_{k-1}, I_{a_k}, I_{b_k})$, where $a_k, b_k \notin A_{k-1}$ are two new alternatives. Observe that $A_k = A_0 \cup \bigcup_{\ell=1}^{k}\{a_\ell, b_\ell\}$.

Repeated application of Lemma 1 yields

$$\begin{aligned} S^{(k)}(T_k) &= \{a_k, b_k\} \cup S^{(k-1)}(T_{k-1}) \\ &= \{a_k, b_k\} \cup \{a_{k-1}, b_{k-1}\} \cup S^{(k-2)}(T_{k-2}) \\ &= \cdots = \bigcup_{\ell=1}^{k}\{a_\ell, b_\ell\} \cup S(T_0). \end{aligned}$$

Since $S(T_0) \neq TEQ(T_0)$, we have $S^{(k)}(T_k) \neq TEQ^{(k)}(T_k) = TEQ(T_k)$.

We have thus shown that $k_{n_k}(S) > k$, where $n_k = |A_k|$ is the order of tournament T_k. By definition of T_k, $n_k = n_0 + 2k$, and therefore $k_{n_k}(S) > k$ implies $k_n(S) > \frac{n-n_0}{2}$ for all n such that $n - n_0$ is even.

If $n - n_0$ is odd, i.e., $n = n_0 + 2k + 1$ for some $k \in \mathbb{N}$, consider the tournament $T'_k = (A_{k+1} \setminus \{a_{k+1}\}, \succ)$. This tournament has order n and it is easy to see that $S^{(k)}(T'_k) = S^{(k)}(T_k) \neq TEQ(T_k) = TEQ(T'_k)$. Thus, $k_{n_k+1}(S) > k$, or, equivalently, $k_n(S) > \lfloor \frac{n-n_0}{2} \rfloor$. $\square$

As it was the case for the results in Section 3, Proposition 4 also holds for $TRIV$ even though $TRIV$ is not a proper tournament solution. Since $TRIV$ and TEQ differ for every tournament with two alternatives, we immediately have $k_n(TRIV) > \frac{n}{2} - 1$. Furthermore, Dutta [13] constructed a tournament T of order 8 for which $TEQ(T) \neq MC(T)$, and thus $k_n(MC) > \frac{n}{2} - 4$.

Interestingly, the tournaments T_k constructed in the proof of Proposition 4 show that it might be impossible to *recognize* convergence within less than $k_n(S)$ iterations.

5. THE MINIMAL TC-RETENTIVE SET

As mentioned in Section 1, it is known from earlier work that Conjecture 1 is equivalent to TEQ satisfying any of the basic properties, and the attractiveness of TEQ thus hinges on the resolution of this conjecture. In Section 3 we have looked more generally at tournament solutions $\mathring{S}$, defined as the union of all minimal S-retentive sets for arbitrary tournament solutions S. It turned out that uniqueness of minimal retentive sets again plays an important role: If $\mathcal{R}_S$ is pairwise intersecting, then $\mathring{S}$ inherits many desirable properties from S. We now prove the equivalent of Conjecture 1 for $\mathcal{R}_{TC}$, thus establishing $\mathring{TC}$ as an efficiently computable refinement of TC that satisfies all basic properties. Note that this result is a weaker version of Conjecture 1.

THEOREM 4. *$\mathcal{R}_{TC}$ is pairwise intersecting.*

PROOF. Consider an arbitrary tournament T on A, and assume for contradiction that B and C are two disjoint TC-retentive sets of T. Let $b_0 \in B$ and $c_0 \in C$. Without loss of generality we may assume that $c_0 \succ b_0$. Then, $c_0 \in \overline{D}(b_0)$ and by TC-retentiveness of B there has to be some $b_1 \in B$ with $b_1 \in TC(\overline{D}(b_0))$ and $b_1 \succ c_0$. We claim that for each $m \geq 1$ there are $c_1, \ldots, c_m \in C$ such that for all i and j with $0 \leq i < j \leq m$,

(*i*) $c_{i+1} \in TC(\overline{D}(c_i))$,

(*ii*) $b_0 \succ c_i$ and $c_i \succ b_1$ if i is odd, and $b_1 \succ c_i$ and $c_i \succ b_0$ otherwise, and

(*iii*) $c_j \succ c_i$ if $j - i$ is odd, and $c_i \succ c_j$ otherwise,

Let us first show that this claim implies the theorem. For this, consider i and j with $0 \leq i < j \leq m$. If $j - i$ is odd, then $c_j \succ c_i$ by (*iii*). If $j - i$ is even, then $c_j \succ c_{j-1}$ by (*i*) and $c_{j-1} \succ c_i$ by (*iii*). Since the dominance relation is irreflexive and anti-symmetric, c_i and c_j must be distinct alternatives in both cases. This in turn implies that the size of C is unbounded, contradicting finiteness of A. The situation is illustrated in Figure 2.

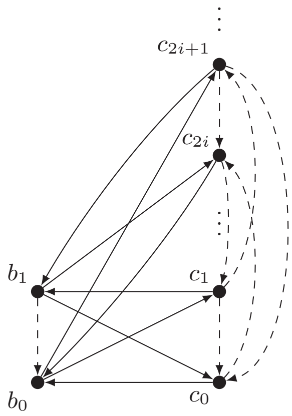

Figure 2: Structure of a tournament with two disjoint TC-retentive sets. A dashed edge (a, b) indicates that $a \in TC(\overline{D}(b))$.

The claim itself can be proved by induction on m. First consider the case $m = 1$. Since $b_1 \succ c_0$, and by TC-retentiveness of C, there has to be some $c_1 \in C$ with $c_1 \in TC(\overline{D}(c_0))$ and $c_1 \succ b_1$, showing (*i*). Furthermore, by TC-retentiveness of B, $c_1 \notin TC(\overline{D}(b_0))$ and thus $b_0 \succ c_1$. It is now easily verified that (*ii*) and (*iii*) hold as well.

Now assume that the claim holds for all k with $1 \leq k \leq m$. We show that it also holds for $m + 1$.

Consider the case when m is odd; the case when m is even is analogous. By the induction hypothesis, $b_0 \succ c_m$. Hence, by TC-retentiveness of C, there has to exist some $c_{m+1} \in C$ with $c_{m+1} \in TC(\overline{D}(c_m))$ and $c_{m+1} \succ b_0$, which together with the induction hypothesis implies (*i*).

Moreover, since $b_1 \in TC(\overline{D}(b_0))$ and $c_{m+1} \succ b_0$, TC-retentiveness of B yields $b_1 \succ c_{m+1}$. With the induction hypothesis this proves (*ii*).

For (*iii*), consider an arbitrary i with $1 \leq i \leq m$, and first assume that i is odd. If $i = m$, then immediately $c_{i+1} \succ c_i$. If $i < m$, then by the induction hypothesis, $c_i \succ c_m$, $b_0 \succ c_i$, and $b_0 \succ c_m$. Hence, $\{c_{m+1}, c_i, b_0\} \subseteq \overline{D}(c_m)$. Moreover, as we have already shown, $c_{m+1} \succ b_0$. Assuming for contradiction that $c_i \succ c_{m+1}$, the three alternatives c_{m+1}, c_i, and b_0 would constitute a cycle in $\overline{D}(c_m)$. Since $c_{m+1} \in TC(\overline{D}(c_m))$, we would then have that $b_0 \in TC(\overline{D}(c_m))$, contradicting TC-retentiveness of C. As $c_{m+1} \succ b_0$ and $b_0 \succ c_i$, also $c_{m+1} \neq c_i$, and it follows that $c_{m+1} \succ c_i$.

Now assume that i is even. By the induction hypothesis, $c_m \succ c_i$ and $b_1 \succ c_i$. Assume for contradiction that $c_{m+1} \succ c_i$ and thus $c_{m+1} \in \overline{D}(c_i)$. Since $i+1$ is odd, we already know that $c_{m+1} \succ c_{i+1}$. Furthermore, $c_{i+1} \in TC(\overline{D}(c_i))$, and thus $c_{m+1} \in TC(\overline{D}(c_i))$. However, $b_1 \succ c_{m+1}$ and $b_1 \in \overline{D}(c_i)$, and thus $b_1 \in TC(\overline{D}(c_i))$. This contradicts TC-retentiveness of C. Since $c_{m+1} \succ c_m$ and $c_m \succ c_i$, $c_{m+1} \neq c_i$, and we may conclude that $c_1 \succ c_{m+1}$. By virtue of the induction hypothesis we are done. $\Box$

6. DISCUSSION

Assuming Schwartz's conjecture and starting with the trivial tournament solution, we have defined an infinite sequence of efficiently computable tournament solutions that are strictly contained in each other, strictly contain TEQ, and share most of its desirable properties. The implications of these findings are both of theoretical and practical nature.

From a practical point of view, we have outlined an anytime algorithm for computing TEQ that returns smaller and smaller supersets of TEQ, which are furthermore consistent according to standard properties suggested in the literature. Previous algorithms for TEQ (see, e.g., [8]) are incapable of providing *any* useful information when stopped prematurely.

From a theoretical point of view, the new perspective on TEQ as the limit of an infinite sequence of tournament solutions may prove useful for showing Schwartz's conjecture. In particular, it yields an infinite sequence of increasingly difficult conjectures, each of which is a weaker version of Schwartz's conjecture. We proved the second statement of this sequence. Our inheritance results can be interpreted as alternative proofs for the fact that Schwartz's conjecture implies that TEQ satisfies all basic properties. A natural way to prove Schwartz's conjecture would be to prove all statements of the above mentioned sequence by induction, i.e., by showing that $\mathcal{R}_{\mathring{S}}$ is pairwise intersecting if $\mathcal{R}_S$ is. Both proving and disproving that $\mathcal{R}_S$ is pairwise intersecting for some reasonable solution concept S turns out to be surprisingly difficult. So far, we have only found degenerate examples of tournament solutions that admit disjoint retentive sets.

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