Asymptotic Collusion-Proofness of Voting Rules: The Case of Large Number of Candidates

(Extended Abstract)

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ABSTRACT

We study manipulability in elections when the number of candidates is large. Elections with a large number of voters have been studied in the literature and the focus of this paper is on studying election with a large number of candidates. Manipulability, when the number of candidates is large, is significant in the context of computational social choice. Our investigation in this paper covers the impartial culture (IC) assumption as well as a new culture of society which we call impartial scores culture (ISC) assumption, where all score vectors of the candidates are equally likely. Under the IC and ISC models, we study asymptotic collusion-proofness for plurality, veto, k-approval, and Borda voting rules. We provide bounds for the fraction of manipulable profiles when the number of candidates is large. Our results show that the size of the coalition and the tie-breaking rule play a crucial role in determining whether or not a voting rule satisfies asymptotic collusion-proofness.

Categories and Subject Descriptors

I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multiagent Systems*;
J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*

General Terms

Economics, Theory

Keywords

Social choice; Manipulation; Game Theory; Asymptotic strategy proofness; Asymptotic collusion proofness

1. INTRODUCTION

In many real life situations including multiagent systems, agents often need to agree upon a common decision although they may have different preferences over the possible alternatives. A natural tool in these situations is voting.

A fundamental problem of voting rules is strategic manipulation by voters - sometimes voters are better off by voting non-truthfully. Informally, a voter is said to *manipulate* an election if she does not report her true preference. The Gibbard Satterthwaite theorem [3,6] says that every *unanimous* and *non-dictatorial* voting rule with at least three candidates is manipulable. Clearly, manipulation is undesirable and therefore we seek voting rules that are *rarely* manipulable.

Slinko [7] showed that for some common voting rules , under impartial culture (IC) assumption, the probability of drawing a manipulable profile at random goes to zero as the number of voters increases. The IC assumption implies that the voters' preferences are independent and uniformly distributed among all possible linear orders of the candidates. Subsequently researchers [5] studied coalitional manipulation under the IC assumption. All the works in literature study elections with a large number of voters and a fixed number of candidates. This paper investigates elections when we have a large number of candidates and a fixed number of voters. Elections with a large number of candidates are common in application based on multiagent systems, for example, meta search engines. Nitzan [4] empirically showed that the problem of manipulation is severe only in societies with small number of voters. Hence it is the elections with a small number of voters which we should possibly target for preventing manipulation. However elections with only a few candidates are always easily manipulable - manipulators can just try all possible linear orders over the candidates. This leaves only one case open - elections with fixed number of voters but large number of candidates.

Contributions

We study asymptotic manipulability of voting rules when the number of voters is fixed and the number of candidates increases to infinity, thereby filling an important research gap.

We investigate asymptotic manipulability under two societal culture assumptions - IC and ISC. In the ISC assumption, we assume that the score vectors of the candidates are equally likely. This is in contrast to the classical IC assumption where the preference profiles are assumed to be equally likely. We derive our results under the IC assumption as well. We believe the ISC assumption is natural and appropriate in the context of voting rules which are score-based (that is, a winner is determined solely based on scores). Also, the ISC assumption provides a better handle for the coalition manipulability problem with scores.

We define the notions of asymptotic strategy-proofness and asymptotic collusion-proofness with respect to new voters (Definition 1). This makes the notions more directly applicable from the perspective of the computational problem of manipulation.

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The specific contributions of this paper are as follows.

- We show that the existing results on the manipulability for plurality, veto, and k-approval voting rules continue to hold under the proposed definitions that are defined with respect to new voters.
- We provide bounds for the fraction of manipulable profiles for voting rules when the number of candidates is large under the IC and ISC assumptions. These bounds immediately tell us whether or not the given voting rule is asymptotically collusion-proof. We prove asymptotic results for plurality, veto, and k-approval voting rules. We show that the Borda rule is not asymptotic strategy-proof on the number of candidates under the IC assumption (however, the problem is still unresolved under the ISC assumption).

Our results on asymptotic collusion-proofness of voting rules are summarized as in Table 1.

Cases	Plurality for[a]	Plurality against[b]	Veto	k-Approval, $k > 1$	Borda
$c = 1$ IC	$\surd$	$\times$	$\times$	$k = o(m)$: $\times$	$\times$
$c = 1$ ISC	$\surd$	$\times$	$\times$	$k = o(m)$: $\times$	?
$c > 1$ IC	$\times$	$\times$	$\times$	$k = o(m)$: $\times$	$\times$
$c > 1$ ISC	$\times$	$\times$	$\times$	$k = o(m)$: $\times$	?

Table 1: Asymptotic collusion-proofness results on candidates

[a]Lexicographic tie breaking rule for the manipulators
[b]Lexicographic tie breaking rule against the manipulators

2. ASYMPTOTIC COLLUSION-PROOFNESS

Given a voting rule r, a set of candidates $\mathcal{C}$, a set of n truthful voters, and a coalition of size c, we define c-collusion-proof voting profiles as follows.

Definition 1. *(c-Collusion-proof Voting Profile) A voting profile $\succ^n \in \mathcal{L}(\mathcal{C})^n$ is called c-collusion-proof if $\forall \succ_1^c, \succ_2^c \in \mathcal{L}(\mathcal{C})^c$,*

$$r(\succ^n, \succ_1^c) \succ_1^c r(\succ^n, \succ_2^c)$$

For $c = 1$, these profiles are called strategy-proof voting profiles. Given a voting rule r, we denote the set of all c-collusion-proof voting profiles by $T_r^c(\mathcal{C})$. The set of all strategy-proof voting profiles is denoted by $T_r(\mathcal{C})$. Notice that manipulation by c number of new voters is *game theoretically* not possible in the above defined collusion-proof profiles. We define the above notions with respect to *new* voters. Thus we see that the hardness of the computational manipulation (CM) problem [1, 2] at collusion-proof instances is of no use. Previously, Slinko [7] defined strategy-proof voting profiles with respect to the existing voters. Our definition is more aligned with the formulation of the CM problem because in the CM problem, a voting profile of truthful voters is given and it is fixed. Hence the results proved here show in how many instances of the CM problem, manipulation is possible.

A voting rule is called a score based voting rule if winner is determined solely based on scores. Positional scoring rules are examples of such rules. If r is a score based voting rule, then we define strategy-proof and collusion-proof scoring profiles as the scores that each candidate receive in some strategy-proof and collusion-proof voting profiles respectively. The notion of collusion-proof scoring profiles will help us understand manipulability in score based voting rules better. The above notions naturally lead us to study a society where all the scoring profiles are equally likely. We name this assumption the Impartial Scores Culture (ISC) assumption.

With the above definitions, the notions of *asymptotic strategy-proofness* and *asymptotic collusion-proofness* on voters and candidates are defined as follows. Let $\mathcal{D}$ be a probability distribution over the voting profiles.

Definition 2. *(Asymptotic Strategy-proofness on Voters) A voting rule r is called asymptotically strategy-proof on voters if for all fixed and finite $\mathcal{C}$,*

$$\lim_{n\to\infty} Prob_{\succ^n \sim \mathcal{D}}\{\succ^n \in T_r(\mathcal{C})\} = 1$$

In words, a voting rule is called asymptotically strategy-proof on voters if *almost all* the voting profiles are strategy-proof as we increase the number of voters. On similar lines, we define asymptotic strategy-proofness on candidates as follows.

Definition 3. *(Asymptotic Strategy-proofness on Candidates) A voting rule r is called asymptotically strategy-proof on candidates if $\exists N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$,*

$$\lim_{|\mathcal{C}|\to\infty} Prob_{\succ^n \sim \mathcal{D}}\{\succ^n \in T_r(\mathcal{C})\} = 1$$

The above concepts are generalized to asymptotic collusion-proofness as follows.

Definition 4. *(Asymptotic c-Collusion-proofness on Voters) A voting rule r is called asymptotically c-collusion-proof on voters if for all fixed and finite $\mathcal{C}$,*

$$\lim_{n\to\infty} Prob_{\succ^n \sim \mathcal{D}}\{\succ^n \in T_r^c(\mathcal{C})\} = 1$$

Definition 5. *(Asymptotic c-Collusion-proofness on Candidates) A voting rule r is called asymptotically c-collusion-proof on candidates if $\exists N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$,*

$$\lim_{|\mathcal{C}|\to\infty} Prob_{\succ^n \sim \mathcal{D}}\{\succ^n \in T_r^c(\mathcal{C})\} = 1$$

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