Games					Lemke		Descent 0.1 LS		Descent 0.001 LS		
Game	Graph	# Payoff	LCP	p	Time	% T	Time	ϵ	Time	ϵ	% T
Coord-Zero	Complete	26010	32400	0	1.270	0.0	0.034	2.103e-02	0.760	9.951e-04	0.0
				0.25	63.407	4.0	0.033	2.115e-02	0.748	1.026e-03	0.0
				0.5	337.443	45.0	0.034	1.859e-02	0.750	1.070e-03	0.0
				0.75	522.207	74.0	0.033	1.604e-02	0.725	1.076e-03	0.0
				1	116.354	0.0	0.034	4.844e-03	0.598	5.087e-04	0.0
	Cycle	25920	136900	0	18.430	2.0	0.103	3.352e-02	3.612	1.093e-03	0.0
				0.25	184.451	21.0	0.103	3.157e-02	3.534	1.167e-03	0.0
				0.5	412.947	55.0	0.105	2.859e-02	3.430	1.136e-03	0.0
				0.75	593.414	96.0	0.103	2.626e-02	3.206	1.121e-03	0.0
				1	600.097	100.0	0.107	1.906e-02	2.712	6.557e-04	0.0
	Grid	26136	93636	0	35.447	3.0	0.072	3.143e-02	2.257	1.023e-03	0.0
				0.25	260.455	35.0	0.069	3.239e-02	2.233	1.137e-03	0.0
				0.5	451.699	61.0	0.072	2.955e-02	2.254	1.170e-03	0.0
				0.75	552.286	82.0	0.072	2.786e-02	2.106	1.186e-03	0.0
				1	599.349	99.0	0.070	2.159e-02	1.802	6.489e-04	0.0
	Tree	25992	152100	0	0.276	0.0	0.060	1.012e-02	0.818	1.175e-03	0.0
				0.25	0.542	0.0	0.062	1.997e-02	0.806	1.220e-03	0.0
				0.5	73.443	5.0	0.062	2.139e-02	0.814	1.246e-03	0.0
				0.75	165.418	4.0	0.061	2.150e-02	0.796	1.084e-03	0.0
				1	162.420	0.0	0.063	1.469e-03	0.778	7.686e-04	0.0
Group Zero	Complete	20250	25600	2	368.032	36.0	0.025	1.976e-02	0.564	1.093e-03	0.0
				3	495.919	66.0	0.025	1.762e-02	0.550	1.129e-03	0.0
				5	435.207	29.0	0.025	1.308e-02	0.525	9.926e-04	0.0
		26010	32400	2	438.650	59.0	0.034	1.855e-02	0.760	1.068e-03	0.0
				3	576.439	91.0	0.034	1.583e-02	0.731	1.120e-03	0.0
				5	582.924	88.0	0.033	1.186e-02	0.677	9.738e-04	0.0
		36000	44100	2	545.997	84.0	0.052	1.564e-02	1.073	1.049e-03	0.0
				3	598.616	99.0	0.051	1.396e-02	1.037	1.110e-03	0.0
				5	600.088	100.0	0.051	1.101e-02	0.969	9.721e-04	0.0
Strict	Complete	20250	25600	5	356.009	17.0	0.024	1.878e-02	0.552	1.054e-03	0.0
	Cycle	20480	108900	5	580.891	85.0	0.087	1.729e-02	2.102	1.068e-03	0.0
	Grid	20184	72900	5	551.795	77.0	0.066	1.612e-02	1.428	1.108e-03	0.0
	Tree	20808	122500	5	79.560	0.0	0.048	2.571e-03	0.664	8.111e-04	0.0
Weighted Cooperation	Complete	995000	1440000	2	194.233	8.0	8.455	1.446e-02	86.116	9.603e-04	0.0
				3	410.118	38.0	6.551	1.909e-02	73.485	1.047e-03	0.0
				5	552.583	81.0	4.957	2.585e-02	64.676	1.130e-03	0.0
	Cycle	17500	4410000	2	103.403	0.0	0.227	1.334e-01	577.984	3.094e-02	73.0
				3	90.062	0.0	0.172	1.412e-01	529.581	4.199e-02	48.0
				5	78.883	0.0	0.156	1.427e-01	438.707	3.950e-02	28.0
	Grid	27200	3006756	2	116.157	0.0	0.864	8.298e-02	275.839	5.461e-03	1.0
				3	81.933	0.0	0.464	1.110e-01	480.608	1.368e-02	15.0
				5	58.054	0.0	0.131	1.384e-01	358.662	3.028e-02	2.0
	Tree	24950	9000000	2	240.215	0.0	0.750	0.000e+00	2.768	3.253e-04	0.0
				3	220.510	0.0	0.709	0.000e+00	2.533	1.194e-04	0.0
				5	204.919	0.0	0.653	0.000e+00	2.345	8.947e-05	0.0

Table 5: Results for multi-player (non-Bayesian) polymatrix games. The underlying graphs are complete graphs, cycles, grids and star graphs. $\%T$ is the proportion of the timed out instances. On Cooperation-Zerosum games, the value of p represents the proportion of games which are coordination games, for group zero-sum games, it represents the number of groups, and for weighted cooperation games, it represents the multiplier dictating the total number of colours available, i.e. if there are k colours per player, then there are $k \cdot p$ total colours available.

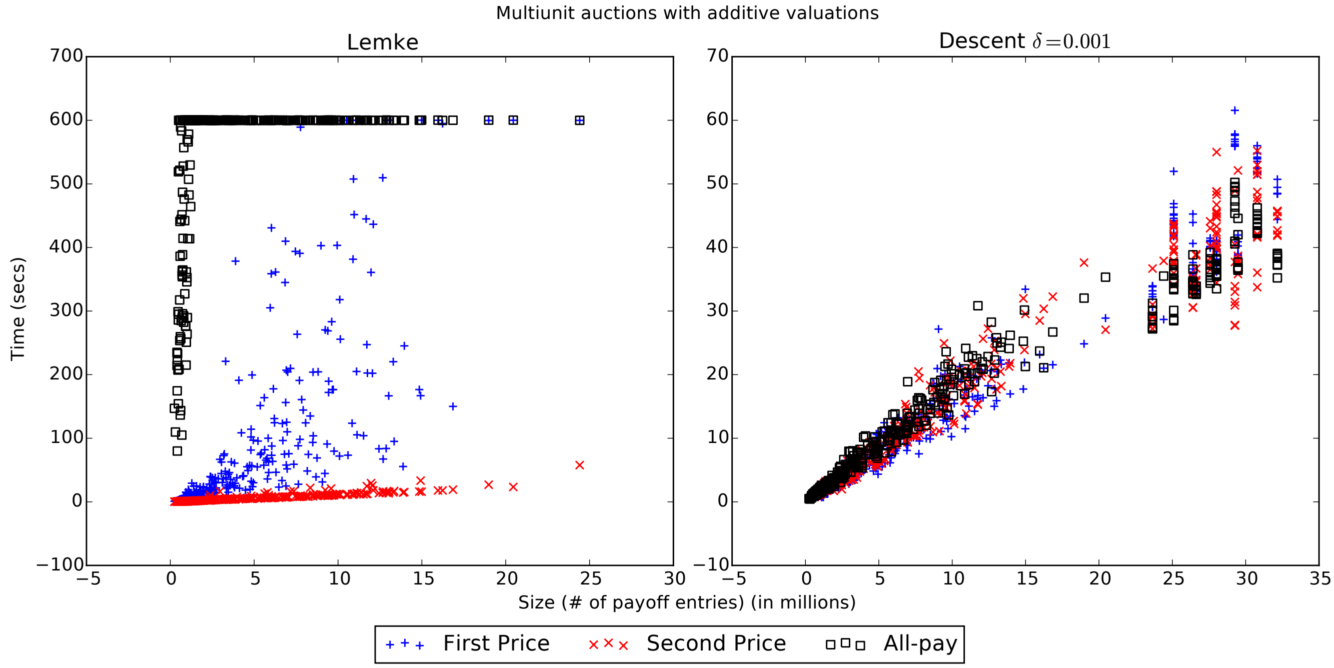

Figure 1: Plots showing the performance of the algorithms on multi-unit auctions with first price, second price, and all-pay payment rules. The left plot shows the performance of Lemke's algorithm. It can clearly be seen that the allocation rule impacts the performance of the algorithm. The right chart right shows the performance of Descent with $\delta = 0.001$. The y-axis scales on the two charts are not equal: Descent is much faster than Lemke.

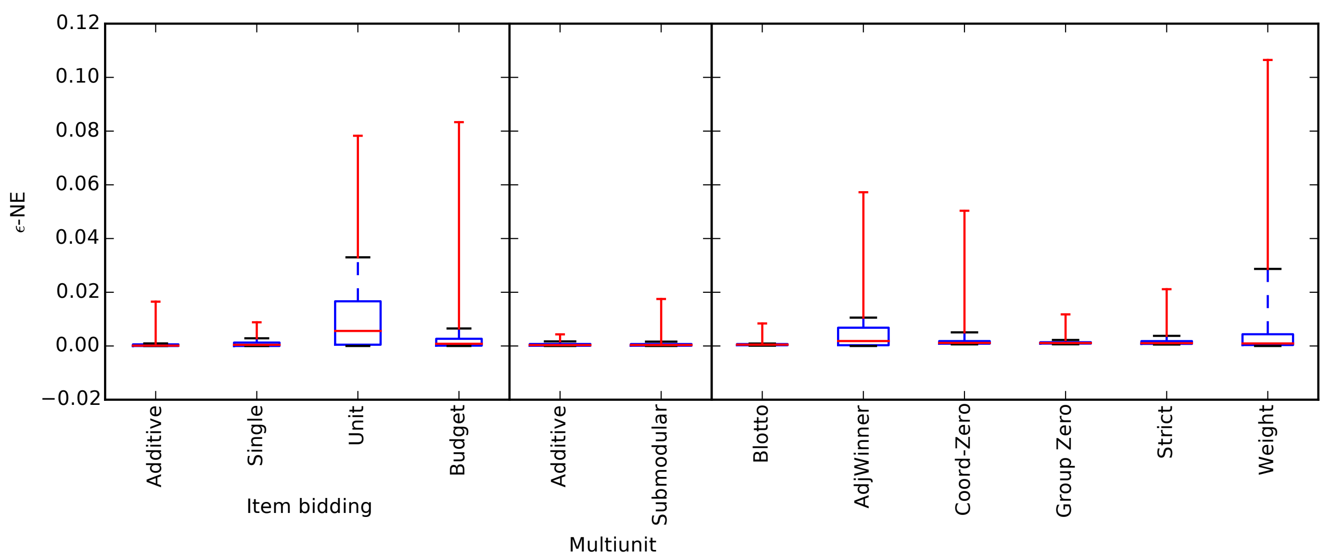

Figure 2: Box and whisker plots showing the approximation quality of the approximate equilibria found by Descent with $\delta = 0.001$. The results show that Descent almost always finds a high quality approximate equilibrium. It can be seen that on many classes of games, even the worst approximation quality over all test cases is very good.

REFERENCES

[1] T. Adamo and A. Matros. A Blotto game with incomplete information. *Economics Letters*, 105(1):100 – 102, 2009.

[2] A. Ahmadi, M. Hajiaghayi, B. Lucier, H. Mahini, and S. Seddighin. From duels to battlefields: Computing equilibria of Blotto and other games. In *Proc. of AAAI*, 2016.

[3] K. R. Apt, M. Rahn, G. Schäfer, and S. Simon. Coordination games on graphs. *CoRR*, abs/1501.07388, 2015.

[4] C. Audet, S. Belhaiza, and P. Hansen. Enumeration of all the extreme equilibria in game theory: Bimatrix and polymatrix games. *Journal of Optimization Theory and Applications*, 129(3):349–372, 2006.

[5] D. Avis, G. Rosenberg, R. Savani, and B. von Stengel. Enumeration of Nash equilibria for two-player games. *Economic Theory*, 42(1):9–37, 2010.

[6] S. Barman, K. Ligett, and G. Piliouras. Approximating nash equilibria in tree polymatrix games. In *Proc. of SAGT*, pages 285–296, 2015.

[7] N. A. R. Bhat and K. Leyton-Brown. Computing Nash equilibria of action-graph games. In *Proc. of UAI*, pages 35–42, 2004.

[8] K. Bhawalkar and T. Roughgarden. Welfare guarantees for combinatorial auctions with item bidding. In *Proc. of SODA*, pages 700–709, 2011.

[9] H. Bosse, J. Byrka, and E. Markakis. New algorithms for approximate Nash equilibria in bimatrix games. *Theoretical Computer Science*, 411(1):164–173, 2010.

[10] S. J. Brams. Fair division. In *Computational Complexity*, pages 1073–1080. Springer, 2012.

[11] P. Briest, P. W. Goldberg, and H. Röglin. Approximate equilibria in games with few players. *CoRR*, abs/0804.4524, 2008.

[12] Y. Cai and C. Daskalakis. On minmax theorems for multiplayer games. In *Proc. of SODA*, pages 217–234, 2011.

[13] Y. Cai and C. H. Papadimitriou. Simultaneous bayesian auctions and computational complexity. In *Proc. of EC*, pages 895–910, 2014.

[14] X. Chen, X. Deng, and S.-H. Teng. Settling the complexity of computing two-player Nash equilibria. *Journal of the ACM*, 56(3):14:1–14:57, 2009.

[15] X. Chen, D. Paparas, and M. Yannakakis. The complexity of non-monotone markets. In *Proc. of STOC*, pages 181–190, 2013.

[16] G. Christodoulou, A. Kovács, and M. Schapira. Bayesian combinatorial auctions. In *Proc. of ICALP*, pages 820–832, 2008.

[17] C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou. The complexity of computing a Nash equilibrium. *SIAM Journal on Computing*, 39(1):195–259, 2009.

[18] C. Daskalakis, A. Mehta, and C. H. Papadimitriou. A note on approximate Nash equilibria. *Theoretical Computer Science*, 410(17):1581–1588, 2009.

[19] C. Daskalakis and C. H. Papadimitriou. Continuous local search. In *Proc. of SODA*, pages 790–804, 2011.

[20] C. Daskalakis, G. Schoenebeck, G. Valiant, and P. Valiant. On the complexity of Nash equilibria of action-graph games. In *Proc. of SODA*, pages 710–719, 2009.

[21] A. Deligkas, J. Fearnley, T. P. Igwe, and R. Savani. An Empirical Study on Computing Equilibria in Polymatrix Games. *CoRR*, abs/1602.06865, 2016.

[22] A. Deligkas, J. Fearnley, R. Savani, and P. Spirakis. Computing approximate Nash equilibria in polymatrix games. *Algorithmica*, pages 1–28, 2015.

[23] S. Dobzinski, H. Fu, and R. D. Kleinberg. On the complexity of computing an equilibrium in combinatorial auctions. In *Proc. of SODA*, pages 110–122, 2015.

[24] K. Etessami and M. Yannakakis. On the complexity of Nash equilibria and other fixed points. *SIAM J. Comput.*, 39(6):2531–2597, 2010.

[25] J. Fearnley, T. P. Igwe, and R. Savani. An empirical study of finding approximate equilibria in bimatrix games. In *Proc. of SEA*, pages 339–351, 2015.

[26] U. Feige and I. Talgam-Cohen. A direct reduction from k-player to 2-player approximate Nash equilibrium. In *Proc. of SAGT*, pages 138–149, 2010.

[27] M. Feldman, H. Fu, N. Gravin, and B. Lucier. Simultaneous auctions are (almost) efficient. In *Proc. of STOC*, pages 201–210, 2013.

[28] S. Govindan and R. Wilson. Computing Nash equilibria by iterated polymatrix approximation. *Journal of Economic Dynamics and Control*, 28(7):1229–1241, 2004.

[29] S. Govindan and R. Wilson. A decomposition algorithm for n-player games. *Economic Theory*, 42(1):97–117, 2010.

[30] S. Hémon and M. Santha. Approximate Nash equilibria for multi-player games. In *Proc. of SAGT*, pages 267–278, 2008.

[31] J. Howson, Joseph T. and R. W. Rosenthal. Bayesian equilibria of finite two-person games with incomplete information. *Management Science*, 21(3):pp. 313–315, 1974.

[32] A. X. Jiang, K. Leyton-Brown, and N. A. R. Bhat. Action-graph games. *Games and Economic Behavior*, 71(1):141–173, 2011.

[33] D. Kovenock and B. Roberson. A Blotto game with multi-dimensional incomplete information. *Economics Letters*, 113(3):273 – 275, 2011.

[34] C. E. Lemke. Bimatrix equilibrium points and mathematical programming. *Management Science*, 11(7):pp. 681–689, 1965.

[35] C. E. Lemke and J. Howson, J. T. Equilibrium points of bimatrix games. *Journal of the Society for Industrial and Applied Mathematics*, 12(2):pp. 413–423, 1964.

[36] D. A. Miller and S. W. Zucker. Copositive-plus lemke algorithm solves polymatrix games. *Operations Research Letters*, 10(5):285 – 290, 1991.

[37] J. Nash. Non-cooperative games. *The Annals of Mathematics*, 54(2):286–295, 1951.

[38] E. Nudelman, J. Wortman, Y. Shoham, and K. Leyton-Brown. Run the gamut: A comprehensive approach to evaluating game-theoretic algorithms. In *Proc. of AAMAS*, pages 880–887, 2004.

[39] G. Polevoy, S. Trajanovski, and M. de Weerdt. Nash

equilibria in shared effort games. In *Proc. of AAMAS*, pages 861–868, 2014.

[40] R. Porter, E. Nudelman, and Y. Shoham. Simple search methods for finding a Nash equilibrium. In *Proc. of AAAI*, pages 664–669, 2004.

[41] T. Roughgarden. The price of anarchy in games of incomplete information. In *Proc. of EC*, pages 862–879, 2012.

[42] A. Rubinstein. Inapproximability of Nash equilibrium. In *Proc. of STOC*, pages 409–418, 2015.

[43] T. Sandholm, A. Gilpin, and V. Conitzer. Mixed-integer programming methods for finding Nash equilibria. In *Proc. of AAAI*, pages 495–501, 2005.

[44] Y. Shoham and K. Leyton-Brown. Protocols for multiagent resource allocation: Auctions. In *Multiagent Systems*, pages 315–366. Cambridge University Press, 2008. Cambridge Books Online.

[45] H. Tsaknakis and P. G. Spirakis. An optimization approach for approximate Nash equilibria. *Internet Mathematics*, 5(4):365–382, 2008.

[46] H. Tsaknakis, P. G. Spirakis, and D. Kanoulas. Performance evaluation of a descent algorithm for bi-matrix games. In *Proc. of WINE*, pages 222–230, 2008.