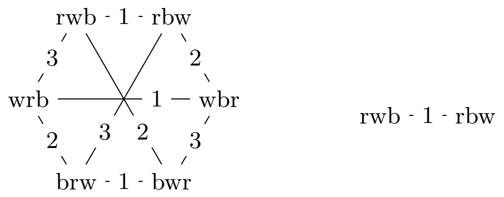


**Figure 1: The full models $\mathcal{M}$ and $\mathcal{M}'$ (reflexive edges are omitted for simplicity)**

i.e., player 1 holds red, she knows it, but 2 does not, and finally, 1 knows that 2 does not know that 1 holds red.

In general, for every state $s$ in $\mathcal{M}$,

$$(\mathcal{M}, s) \models \exists p \left( p \wedge K_i p \wedge \bigwedge_{j \neq i} \neg K_j p \wedge K_i \bigwedge_{j \neq i} \neg K_j p \right)$$

i.e., every player $i$ knows something that the other players do not know (and she knows that they do not), namely the value of the card that $i$ possesses.

Now suppose player 1 announces publicly the card she has. Such an announcement in state rwb leads to the updated model $\mathcal{M}'$ in Fig. 1. Indeed, for $q_i \in \{r_i, w_i, b_i\}$ we have

$$(\mathcal{M}, s) \models q_i \to \exists p \langle K_i p \rangle \left( \bigwedge_{j \neq i} K_j q_i \right)$$

that is, there is some proposition (namely, the value $U = R_i(s)$ of player $i$'s card) that player $i$ can truthfully announce, so that any other player knows the value of $i$'s card.

On the other hand, the mere announcement that player $i$ knows something is not sufficient to derive the same conclusion, as for every state $s \in W$, $(\mathcal{M}, s) \models \exists p K_i p$, and therefore $\mathcal{M}_{|\exists p K_i p} = \mathcal{M}$. Hence,

$$(\mathcal{M}, s) \not\models q_i \to \langle \exists p K_i p \rangle \left( \bigwedge_{j \neq i} K_j q_i \right)$$

Furthermore, the (false) announcement that player $i$ knows everything trivially implies that the other players know her card:

$$(\mathcal{M}, s) \models q_i \to [\forall p K_i p] \left( \bigwedge_{j \neq i} K_j q_i \right)$$

Indeed, $(\mathcal{M}, s) \not\models \forall p K_i p$. Then, $(\mathcal{M}, s) \models [\forall p K_i p](\bigwedge_{j \neq i} K_j q_i)$. However, it is not the case that every truthful announcement pertaining to player $i$'s knowledge entails that the other players know her card:

$$(\mathcal{M}, s) \not\models q_i \to \forall p [K_i p] \left( \bigwedge_{j \neq i} K_j q_i \right)$$

as for proposition $U = W$, $(\mathcal{M}^p_U, s') \models K_i p$ for every $s' \in W$. But $((\mathcal{M}^p_U)_{|K_i p}, s) \not\models \bigwedge_{j \neq i} K_j q_i$, since $(\mathcal{M}^p_U)_{|K_i p} = \mathcal{M}^p_U$.

By comparing the formulas above, we clearly see that quantifying inside or outside (epistemic) announcements allows us to express subtle differences in SOPAL. ■

# 3. COMPARISON WITH APAL

In this section we compare SOPAL with APAL, whose original motivation also included the ability to express arbitrary announcements in PAL. The main result of this section is that SOPAL is capable of capturing APAL at the frame level, while the two logics are incomparable at the model level. But first we state some auxiliary lemmas, that will be routinely applied throughout the paper, which illustrate some features of quantification in SOPAL.

Lemma 3. *Let $q$ and $\psi$ be free for $p$ in $\phi$.*

1. *In $\mathcal{K}_{all}$, $(\mathcal{M}^p_{V(q)}, w) \models \phi$ iff $(\mathcal{M}, w) \models \phi[p/q]$*
2. *For $x = bl$ (resp. $el$, $fl$) and $y = pl$ (resp. $el$, $sopal$), in $\mathcal{K}_x$, $(\mathcal{M}^p_{[\![\psi]\!]}, w) \models \phi$ iff $(\mathcal{M}, w) \models \phi[p/\psi]$, for any $\psi \in \mathcal{L}_y$*
3. *If $p \in fr(\phi)$ implies $\psi \in \mathcal{L}_{sopl}$, then $(\mathcal{M}^p_{[\![\psi]\!]})_{|\phi} = (\mathcal{M}_{|\phi[p/\psi]})^p_{[\![\psi]\!]}$*
4. *If $V(fr(\phi)) = V'(fr(\phi))$ then $(\mathcal{M}, w) \models \phi$ iff $(\mathcal{M}', w) \models \phi$*
5. *If $V(fr(\psi)) = V'(fr(\psi))$ then $\mathcal{M}_\psi = \mathcal{M}'_\psi$*

According to Lemma 3.1-2, the syntactic notion of substitution $\phi[p/\psi]$ corresponds to the semantic concept of reinterpretation $\mathcal{M}^p_{[\![\psi]\!]}$; while Lemma 3.3 specifies the interaction between substitution, reinterpretation and model restriction, namely the restriction $(\mathcal{M}^p_{[\![\psi]\!]})_{|\phi}$ of a reinterpreted model is equal to the reinterpretation $(\mathcal{M}_{|\phi[p/\psi]})^p_{[\![\psi]\!]}$ of the model restricted by the substituted formula $\phi[p/\psi]$, provided that $\psi \in \mathcal{L}_{sopl}$ whenever $p \in fr(\phi)$. Moreover, by Lemma 3.4-5 models built on the same frame and agreeing on the interpretation of free atoms, also satisfy the same formulas, and their model restrictions are equal. These results, which show that quantification in SOPAL is "well-behaved", will be extensively used hereafter.

To compare SOPAL and APAL we recall the clause for interpreting the $\Box$ operator [2]:

$$(\mathcal{M}, w) \models \Box\psi \quad \text{iff} \quad \text{for all } \phi \in \mathcal{L}_{el}, (\mathcal{M}, w) \models [\phi]\psi \quad (5)$$

We now prove that, according to (5), APAL can be captured within SOPAL in the following sense.

Definition 8. *Given a class $\mathcal{K}$ of frames, a logic $L'$ is*

- at least as m-expressive as *logic $L$, or $L \leq_m L'$, iff for any $\phi \in L$, for some $\phi' \in L'$, for any model $\mathcal{M}$ based on some frame in $\mathcal{K}$,*

$$(\mathcal{M}, w) \models \phi \quad \textit{iff} \quad (\mathcal{M}, w) \models \phi'$$

- at least as f-expressive as *logic $L$, or $L \leq_f L'$, iff for any $\phi \in L$, for some $\phi' \in L'$, for any frame $\mathcal{F}$ in $\mathcal{K}$,*

$$(\mathcal{F}, w) \models \phi \quad \textit{iff} \quad (\mathcal{F}, w) \models \phi'$$

Clearly, each relation $\leq$ is a partial order, and we write $L = L'$ iff $L \leq L'$ and $L' \leq L$, and $L < L'$ iff $L \leq L'$ and $L \neq L'$. Also, $L \leq_m L'$ implies $L \leq_f L'$.

To investigate the relation between SOPAL and APAL, we start with some preliminary results. First of all, we test the intuition that the operator $\Box$ can be expressed by quantification and announcements.

Lemma 4. *Let $\mathcal{M}$ be an epistemic model, then*

$$(\mathcal{M}, w) \models \forall p[p]\phi \quad \textit{implies that} \quad (\mathcal{M}, w) \models \Box\phi \qquad (6)$$

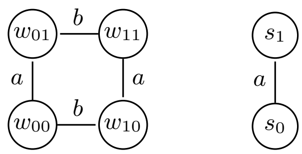


**Figure 2: The full models $\mathcal{M}$ and $\mathcal{M}'$ (reflexive edges are omitted for simplicity)**

PROOF. Indeed, if $(\mathcal{M}, w) \not\models [\psi]\phi$ for some $\psi \in \mathcal{L}_{el}$, then in particular, $[\![\psi]\!] \in D$ by Lemma 1 and for $U = [\![\psi]\!]$, $(\mathcal{M}^p_U, w) \not\models [p]\phi$. That is, $(\mathcal{M}, w) \not\models \forall p[p]\phi$. □

However, the converse of (6) does not always hold. Consider the full model $\mathcal{M}$ in Fig. 2. Formally, we have that $\mathcal{M} = \langle W, R, D, V\rangle$ with $W = \{w_{00}, w_{01}, w_{10}, w_{11}\}$; $R_a = \{(w_{ij}, w_{i'j'}) \mid i = i'\}$; $R_b = \{(w_{ij}, w_{i'j'}) \mid j = j'\}$; $D = 2^W$; and $V(q) = \{w_{ij} \mid j = 0\}$ for every $q \in AP$. We can check that, for every $\psi \in \mathcal{L}_{el}$, $[\![\psi]\!]$ is equal to either $W$, or $\emptyset$, or $\{w_{ij} \mid j = 0\}$, or $\{w_{ij} \mid j = 1\}$. As a consequence, for every $\psi \in \mathcal{L}_{el}$, $(\mathcal{M}, w_{i0}) \models [\psi](K_a q \to K_b K_a q)$, that is, $(\mathcal{M}, w_{i0}) \models \Box(K_a q \to K_b K_a q)$. However, for $U = \{w_{00}, w_{01}, w_{10}\}$ we obtain that $(\mathcal{M}^p_U, w_{10}) \not\models [p](K_a q \to K_b K_a q)$, i.e., $(\mathcal{M}, w_{10}) \not\models \forall p[p](K_a q \to K_b K_a q)$.

Actually, Def. (5) for APAL preserves bisimilarity of structures, while Def. 7 for SOPAL does not. To see this, consider the full model $\mathcal{M}'$ in Fig. 2. We remark without proof that the pointed models $(\mathcal{M}, w_{10})$ and $(\mathcal{M}', s_0)$ are bisimilar [6], and satisfy the same formulas in PAL, and consequently, in APAL. However, we noticed that $(\mathcal{M}, w_{10}) \not\models \forall p[p](K_a q \to K_b K_a q)$, while it is easy to check that $(\mathcal{M}', s_0) \models \forall p[p](K_a q \to K_b K_a q)$.

Incidentally, models $\mathcal{M}$ and $\mathcal{M}'$ above prove that at the level of models, SOPAL is no less expressive than APAL.

THEOREM 5. *In class $\mathcal{K}_{fl}$ of full frames, SOPAL $\not\leq_m$ APAL.*

PROOF. Suppose that SOPAL $\leq_m$ APAL. Then, for $\phi = \forall p[p](K_a q \to K_b K_a q)$ in SOPAL there exists a corresponding $\phi'$ in APAL. However, $(\mathcal{M}, w_{10}) \not\models \phi$ implies $(\mathcal{M}, w_{10}) \not\models \phi'$, which implies $(\mathcal{M}', s_0) \not\models \phi'$ by bisimulation, which finally implies $(\mathcal{M}', s_0) \not\models \phi$. A contradiction. □

We observe that Def. 7 is discussed in [2], but discarded exactly on the ground that it does not preserve bisimulations. Bisimulation-preserving quantification is analysed in [11], the resulting logic is proved as expressive as Epistemic Logic. Here we maintain that in second-order propositional modal logics a stronger notion of bisimulation is needed, which takes into account also quantification, as discussed in [5]. Also, Def. (5) has other issues, in particular, it is not analytic (more below).

Even though $\forall p[p]\psi$ is not equivalent to $\Box\psi$ at the level of models, the two formulas are provably equivalent at the level of frames, under a cardinality assumption.

Consider the following translation $\tau$ from APAL to SOPAL:

$$
\begin{aligned}
\tau(p) &= p\\
\tau(\neg\psi) &= \neg\tau(\psi)\\
\tau(\psi \to \psi') &= \tau(\psi) \to \tau(\psi')\\
\tau(K_a\psi) &= K_a\tau(\psi)\\
\tau([\psi]\psi') &= [\psi]\tau(\psi')\\
\tau(\Box\psi') &= \forall p[p]\tau(\psi')
\end{aligned}
$$

where $p$ does not appear free in $\psi'$.

We can now prove the following result.

LEMMA 6. *In the class of epistemic frames where $|D|$ is enumerable,*

$$\models \phi \quad \textit{iff} \quad \models \tau(\phi)$$

PROOF. The $\Leftarrow$ direction follows from (6) above.

As for the $\Rightarrow$ direction, suppose that for some model $\mathcal{M}$ and state $w$, $(\mathcal{M}, w) \not\models \tau(\phi)$. Consider now a model $\mathcal{M}'$ s.t. $\mathcal{F}' = \mathcal{F}$ and $V'$ coincides with $V$ on all atoms appearing in $\phi$. Further, for every $U \in D$ take $q_U \in AP$ not appearing in $\phi$ and let $V'(q_U) = U$. By Lemma 3.4, $(\mathcal{M}, w) \not\models \tau(\phi)$ implies $(\mathcal{M}', w) \not\models \tau(\phi)$ (the assignment $V'(q)$ for all atoms not appearing in $\phi$ and not assigned to a set $U$ is uninfluential.)

Hereafter we write $\mathcal{N} \subseteq \mathcal{M}$ to express that $\mathcal{N}$ is a submodel of $\mathcal{M}$, i.e., $W_{\mathcal{N}} \subseteq W$; $D_{\mathcal{N}} = \{U \cap W_{\mathcal{N}} \mid U \in D\}$; $R_{\mathcal{N},a} = R_a \cap W^2_{\mathcal{N}}$; and $V_{\mathcal{N}}(p) = V(p) \cap W_{\mathcal{N}}$ for every $p \in AP$. We can now state the following auxiliary result: for every submodel $\mathcal{N}'$ of $\mathcal{M}'$ and subformula $\psi$ of $\phi$,

$$(\mathcal{N}', w) \models \psi \quad \text{iff} \quad (\mathcal{N}', w) \models \tau(\psi) \tag{7}$$

Finally, by (7) $(\mathcal{M}', w) \not\models \tau(\phi)$ implies $(\mathcal{M}', w) \not\models \phi$. □

As a result, whenever the domain $D$ of propositions is enumerable, APAL can be captured within SOPAL at the frame level, by means of translation $\tau$. Specifically, the arbitrary announcement operator $\Box$ can be expressed by quantification and standard announcements. As a corollary, we have the following result.

COROLLARY 7. *In the class of epistemic frames where $|D|$ is enumerable, APAL $\leq_f$ SOPAL.*

We now show that the converse does not hold in general.

THEOREM 8. *In class $\mathcal{K}_{fl}$ of full frames, SOPAL $\not\leq_f$ APAL.*

As an immediate consequence of Corollary 7 and Theorem 8 we obtain the following.

COROLLARY 9. *In class of epistemic frames where $|D|$ is enumerable, APAL $<_f$ SOPAL.*

Finally, we remarked above that Def. 7 is discussed in [2], but dismissed as it does not preserve bisimulations. On the other hand, the APAL semantics is not analytic in the sense that Lemma 3.4 fails: models that agree on the interpretation of free atoms, may differ in the satisfaction of formulas. Consider again model $\mathcal{M}$ in Fig. 2 and $\varphi = \Box(K_a q \to K_b K_a q)$. Then, define model $\mathcal{M}''$ such that for every $U \subseteq W$, $V(p_U) = U$ for some $p_U \neq q$. Clearly, $V$ and $V''$ agree on the only free variable $q$ in $\varphi$. However, $(\mathcal{M}, w_{00}) \models \varphi$ as noticed above, while $(\mathcal{M}'', w_{00}) \not\models \varphi$. In particular, for $U = \{w_{00}, w_{01}, w_{10}\}$, $(\mathcal{M}'', w_{00}) \not\models [p_U](K_a q \to K_b K_a q)$. Therefore, in APAL the satisfaction of formulas does not depend on values assigned to free variables only, but, if the formula contains an operator $\Box$, on all variables in $AP$. The example above also entails the following result.

THEOREM 10. *In class $\mathcal{K}_{fl}$ of full frames, APAL $\not\leq_m$ SOPAL.*

PROOF. If APAL $\leq_m$ SOPAL, then for $\varphi = \Box(K_a q \to K_b K_a q)$ in APAL there exists a corresponding $\varphi'$ in SOPAL. However, $(\mathcal{M}, w_{00}) \models \varphi$ implies $(\mathcal{M}, w_{00}) \models \varphi'$, which implies $(\mathcal{M}'', w_{00}) \models \varphi'$ by Lemma 3.4, which implies $(\mathcal{M}'', w_{00}) \models \varphi$. A contradiction. $\Box$

To summarize the main results proved in this section, SOPAL and APAL are incomparable at the model level, while the former is strictly more expressive than the latter at the frame level.

# 4. EXPRESSIVITY

In this section we explore expressivity in the various classes of Kripke frames, starting with the properties of quantifiers. The main result of this section is that SOPAL is as expressive as Second-order Propositional Epistemic Logic.

LEMMA 11. *In SOPAL we have the following validities, for $x \in \{bl, el, fl\}$ and $y \in \{pl, el, sopal\}$:*

$$\mathcal{K}_{all} \models \forall p\phi \to \phi[p/q] \quad \textit{for every } q \in AP \tag{8}$$

$$\mathcal{K}_{x} \models \forall p\phi \to \phi[p/\psi] \quad \textit{for every } \psi \in \mathcal{L}_y \tag{9}$$

*where $q$ and $\psi$ are free for $p$ in $\phi$.*

*For every class $\mathcal{K}$ of frames,*

$$\mathcal{K} \models \psi \to \phi \quad \textit{implies} \quad \mathcal{K} \models \psi \to \forall p\phi \tag{10}$$

*where $p$ does not appear free in $\psi$.*

We remark the essential use of Lemmas 3.2-5 in this proof. By Lemma 11 we can see that quantifiers in SOPAL satisfy the standard principles of quantification: exemplification (8)-(9) and generalisation (10).

It is of utmost interest to study the interactions between quantification and public announcements in SOPAL. In this respect, we obtain the following key result.

LEMMA 12. *The following validities hold in all classes of frames.*

$$[\psi]\forall p\phi \quad \leftrightarrow \quad \psi \to \forall p[\psi]\phi \tag{11}$$

$$\langle\psi\rangle\exists p\phi \quad \leftrightarrow \quad \psi \wedge \exists p\langle\psi\rangle\phi \tag{12}$$

$$[\psi]\,\exists p\phi \quad \leftrightarrow \quad \psi \to \exists p[\psi]\phi \tag{13}$$

$$\langle\psi\rangle\forall p\phi \quad \leftrightarrow \quad \psi \wedge \forall p\langle\psi\rangle\phi \tag{14}$$

*where $p$ does not appear in $\psi$ (w.l.o.g. bound variables can always be renamed).*

PROOF. As regards (11) observe that,

$$\begin{aligned}(\mathcal{M}, w) \models [\psi]\forall p\phi \text{ iff } & (\mathcal{M}, w) \models \psi \text{ implies } (\mathcal{M}_{|\psi}, w) \models \forall p\phi \\ \text{iff } & (\mathcal{M}, w) \models \psi \text{ implies for all } U' \in D_{|\psi}, \\ & ((\mathcal{M}_{|\psi})^p_{U'}, w) \models \phi\end{aligned}$$

Now, if $U \in D$ then $U' = U \cap W_{|\psi} \in D_{|\psi}$. On the other hand, if $U' \in D_{|\psi}$ then for some $U \in D$, $U' = U \cap W_{|\psi}$. In particular $(V_{|\psi})^p_{U'} = (V^p_U)_{|\psi}$, as $p$ does not appear free in $\psi$. Hence,

$$\begin{aligned}(\mathcal{M}, w) \models [\psi]\forall p\phi \text{ iff } & (\mathcal{M}, w) \models \psi \text{ implies for all } U \in D, \\ & ((\mathcal{M}^p_U)_{|\psi}, w) \models \phi \\ \text{iff } & (\mathcal{M}, w) \models \psi \text{ implies } (\mathcal{M}, w) \models \forall p[\psi]\phi \\ \text{iff } & (\mathcal{M}, w) \models \psi \to \forall p[\psi]\phi\end{aligned}$$

The other equivalences are proved similarly. $\Box$

We recall that Second-order Propositional Epistemic Logic is obtained by removing clause $[\psi]\psi$ from Def. 1. From Lemma 12 and the standard reduction axioms for PAL [18] we derive the following expressivity result:

THEOREM 13. *SOPAL is as expressive as SOPEL.*

This result is extremely relevant, as it allows to apply the model theory and techniques for SOPEL also to SOPAL. As an example, the bisimulations introduced in [5] for Second-order Propositional Modal Logic apply to SOPAL as well. Further consequences of Theorem 13 regard the decidability of model checking SOPAL and its axiomatisability.

COROLLARY 14.

- *The model checking problem for SOPAL is decidable.*
- *SOPAL has a sound and complete axiomatisation.*

These results directly follow from the PSPACE-completeness and axiomatisation of SOPEL in [4].

## 4.1 Knowability

In this section we analyse the notions of preservation and knowability introduced in Example 1, and present successfulness. Such concepts are of interest to understand the epistemic capabilities of agents in response to different types of public announcements.

We start by introducing the *positive* fragment $\mathcal{L}^+_{sopal}$ inductively defined as

$$\psi \quad ::= \quad p \mid \neg p \mid \psi \wedge \psi \mid \psi \vee \psi \mid K_a\psi \mid [\neg\psi]\psi \mid \forall p\psi$$

As anticipated in Example 1, *preserved formulas* keep their truth under arbitrary announcements. Given a class $\mathcal{K}$, they are defined semantically as those $\phi$ in SOPAL for which $\mathcal{K} \models \phi \to \forall p[p]\phi$. We immediately extend the following result for APAL.

LEMMA 15. *Positive formulas are preserved in $\mathcal{K}_{all}$.*

PROOF. We show that for every model $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{M}''$ with $\mathcal{M}'' \subseteq \mathcal{M}' \subseteq \mathcal{M}$, $s \in W''$, and positive $\phi$, $(\mathcal{M}', s) \models \phi$ implies $(\mathcal{M}'', s) \models \phi$. The inductive cases for $\phi \neq \forall p\psi$ follow as in [20]. As for $\phi = \forall p\psi$, Consider $U'' \in W''$ s.t. $(\mathcal{M}''^p_{U''}, s) \models \psi$. Clearly, $\mathcal{M}''^p_{U''} \subseteq \mathcal{M}'^p_{U'}$ for $U' \in D'$ s.t. $U'' = U' \cap W''$. Moreover, hypothesis $(\mathcal{M}', s) \models \forall p\psi$ implies $(\mathcal{M}'^p_{U'}, s) \models \psi$, and by induction hypothesis it follows that $(\mathcal{M}''^p_{U''}, s) \models \psi$. Since $U''$ is arbitrary, $(\mathcal{M}'', s) \models \forall p\psi$. $\Box$

As an immediate consequence of Lemma 15, positive formulas are preserved in every class of frames.

In connection with preserved formulas, in Example 1 we introduced the formulas preserved after arbitrary *epistemic* announcements (in a class $\mathcal{K}$) as those formulas $\phi$ for which $\mathcal{K} \models \phi \to \forall p[K_a p]\phi$. In Example 2 we remarked that Moore's formulas are not preserved under arbitrary announcements, but they are for epistemic announcements. Obviously, positive formulas are also preserved epistemically. So, it would be of interest to characterize exactly the class of formulas preserved under arbitrary epistemic announcements, but this is beyond the scope of the present paper.

Another semantic notion of interest when dealing with public announcements is that of *success*. Formally, a formula $\phi$ is successful in class $\mathcal{K}$ of frames iff $[\phi]\phi$ is valid in $\mathcal{K}$.

LEMMA 16. *Formulas preserved in their own class $\mathcal{K}$ of frames are successful in $\mathcal{K}$.*

Finally, we recall that for a given class $\mathcal{K}$ of frames, *knowable* formulas as those for which, for any agents $a \in Ag$, $\mathcal{K} \models \phi \to \exists p\langle p\rangle K_a\phi$ [22].

LEMMA 17. *Positive formulas are knowable in $\mathcal{K}_{all}$ (always knowable). Formulas preserved (resp. successful) in their own class $\mathcal{K}$ are also knowable in $\mathcal{K}$.*

We clearly see that SOPAL allows for a fine-grained analysis of the epistemic notions of preservation, successfulness, and knowability.

# 5. SUCCINCTNESS OF SOPAL

The fact that SOPAL and SOPEL are equally expressive does not necessarily mean that they are 'the same'. Indeed, we now argue that SOPAL is more succinct than SOPEL, in the sense described below. We will sketch the argument using techniques from [12], where it was proven that PAL is exponentionally more succinct than epistemic logic. For the following we define the length $|\phi|$ of a formula $\phi$ as standard.

DEFINITION 9. *Given two logics $L_1$ and $L_2$ that are equally expressive on a class $\mathcal{K}$ of frames, we say that $L_1$ is exponentially more succinct than $L_2$ on $\mathcal{K}$, written $L_1 \preceq^{exp}_{\mathcal{K}} L_2$, if the following holds: There are sequences $\varphi_{n\in\mathbb{N}} = \varphi_1, \varphi_2, \cdots \in L_1$ and $\psi_{n\in\mathbb{N}} = \psi_1, \psi_2, \cdots \in L_2$ and a polynomial function $f$ such that, for all $n \in \mathbb{N}$,*

1. *$|\varphi_n| \leq f(n)$;*
2. *$|\psi_n| > 2^n$;*
3. *$\psi_n$ is the shortest formula in $L_2$ equivalent to $\varphi_n$ in $\mathcal{K}$.*

In stating the main result below we also consider the class $\mathcal{C}$ of frames with arbitrary accessibility relations.

THEOREM 18.

- *SOPAL $\preceq^{exp}_{\mathcal{C}}$ SOPEL, if $|Ag| \geq 2$*
- *SOPAL $\preceq^{exp}_{\mathcal{K}_{all}}$ SOPEL, if $|Ag| \geq 4$*

We will only argue here for the first item of the theorem. Consider the following two sequences $\varphi_{n\in\mathbb{N}}$ and $\psi_{n\in\mathbb{N}}$.

$$\begin{aligned} \varphi_0 &= \top \\ \varphi_{n+1} &= \langle\varphi_n\rangle(M_a p \vee M_b q) \\ \psi_0 &= \top \\ \psi_n &= M_a(\psi_{n-1} \wedge p) \vee M_b(\psi_{n-1} \wedge q) \end{aligned}$$

It is easy to see that $|\varphi_i| \leq i \cdot 6$ and $|\psi_i| \geq 2^i$. Using PAL equivalences, we also have that $\varphi_i$ and $\psi_i$ are equivalent, for all $i$. So the first two items for succinctness are easily checked, what remains to establish is, that even when we allow for quantification, there are no formulas $\beta_i \in \mathcal{L}_{sopel}$ shorter than $\psi_i \in \mathcal{L}_{sopel}$ equivalent to $\psi_i$.

For propositional epistemic logic, the technique that [12] uses for this is that of Formula Size Games. We now extend such games to deal with quantification.

DEFINITION 10 (FORMULA SIZE GAME). *The rules of the one-person formula size game ($FSG$) for Spoiler are the following. The game is played on a tree, where each node is labeled with a pair $\langle \mathrm{M} \circ \mathrm{N}\rangle$ such that $\mathrm{M}$ and $\mathrm{N}$ are finite sets of finite pointed models. At each step of the game, a node is labeled with one of the symbols from the set $\Sigma = \{\top, \bot, p, \neg, \vee, \wedge, M_i, K_i, \exists p, \forall p\}$ and either it is closed or at most two new nodes are added. Let a node $\langle \mathrm{M} \circ \mathrm{N}\rangle$ be given. Spoiler can make the following moves at this node:*

**$\top$-move** *This can be played only if $\mathrm{N} = \emptyset$. When Spoiler plays this move, the node is closed and labeled with $\top$.*

**atomic-move** *Spoiler chooses an atom $p$ such that every pointed model in $\mathrm{M}$ satisfies $p$, and no pointed model in $\mathrm{N}$ does. After this move, this node is closed and labeled with $p$.*

**not-move** *Spoiler labels the node with symbol $\neg$ and adds one new node denoted $\langle \mathrm{N}\circ\mathrm{M}\rangle$ as a successor to $\langle \mathrm{M}\circ\mathrm{N}\rangle$.*

**or-move** *Spoiler labels the node with symbol $\vee$ and splits $\mathrm{M}$ in two sets $\mathrm{M} = \mathrm{M}_1 \cup \mathrm{M}_2$. Two new nodes are added to the tree as successors to $\langle \mathrm{M} \circ \mathrm{N}\rangle$, i.e., $\langle \mathrm{M}_1 \circ \mathrm{N}\rangle$ and $\langle \mathrm{M}_2 \circ \mathrm{N}\rangle$.*

**$M_a$-move** *Spoiler labels the node with symbol $M_a$ and for each pointed model $(\mathcal{M}, w) \in \mathrm{M}$, he chooses a pointed model $(\mathcal{M}, w')$ such that $wR_iw'$. All such choices are collected in $\mathrm{M}_1$. A set of models $\mathrm{N}_1$ is then constructed as follows. For each pointed model $(\mathcal{N}, v) \in \mathrm{N}$ add to $\mathrm{N}_1$ all pointed models $(\mathcal{N}, v')$ such that $vR_iv'$. If for some pointed model $(\mathcal{N}, v)$, world $v$ does not have an $R_i$-successor, nothing is added to $\mathrm{N}_1$ for $(\mathcal{N}, v)$. A new node $\langle \mathrm{M}_1 \circ \mathrm{N}_1\rangle$ is added as a successor to $\langle \mathrm{M} \circ \mathrm{N}\rangle$.*

**$\exists p$-move** *Spoiler labels the node with symbol $\exists p$, and, for each $(\mathcal{M}, w) \in \mathrm{M}$, Spoiler chooses a set $U \in D_{\mathcal{M}}$ and replaces $(\mathcal{M}, w)$ by $(\mathcal{M}^p_U, v)$. All those choices are collected in $\mathrm{M}_1$. A set $\mathrm{N}_1$ is then constructed as follows. For every $(\mathcal{N}, v) \in \mathrm{N}$ and $U' \in D_{\mathcal{N}}$, add $(\mathcal{N}^p_{U'}, v)$ to $\mathrm{N}$.*

The moves for $\bot$, **and**, $K_a$, and $\forall p$ can be inferred from this: Spoiler acts on $\mathrm{N}$, instead of $\mathrm{M}$. **and**-moves and **or**-moves are collectively called *splitting* moves, while $K_a$ and $M_a$-moves are called *agent* moves.

DEFINITION 11. *Spoiler wins the FSG starting at $\langle \mathrm{M} \circ \mathrm{N}\rangle$ in $n$ moves iff there is a game tree $T$ with root $\langle \mathrm{M} \circ \mathrm{N}\rangle$ and precisely $n$ nodes such that every leaf of $T$ is closed. Otherwise, Spoiler loses the game in $n$ moves.*

THEOREM 19. *Spoiler can win the FSG starting at $\langle \mathrm{M} \circ \mathrm{N}\rangle$ in less than $k$ moves iff there is a SOPEL formula $\psi$ such that $\mathrm{M} \models \psi$, $\mathrm{N} \models \neg\psi$, $|\psi| = n$, and $n < k$.*

PROOF. We briefly sketch the case for quantifiers, for the other cases we refer to [12, Theorem 1]. The 'if' direction is by induction on the formula, so suppose $\varphi = \exists p\psi$, with the claim proven for $\psi$ with $|\psi| = n - 1$. That is, suppose that $\varphi$ has size $n < k$ and that $\mathrm{M} \models \exists p\psi$ while $\mathrm{N} \models \neg\exists p\psi$. Spoiler plays the $\exists p$-move: since $\mathrm{M} \models \exists p\psi$, for every model $(\mathcal{M}, w) \in \mathrm{M}$, Spoiler can choose some $U \in D_{\mathcal{M}}$ such that $(\mathcal{M}^p_U, w) \models \psi$. Collecting all pointed models thus obtained in $\mathrm{M}_1$, we have $\mathrm{M}_1 \models \psi$. Since $\mathrm{N} \models \neg\exists p\psi$, if we put all

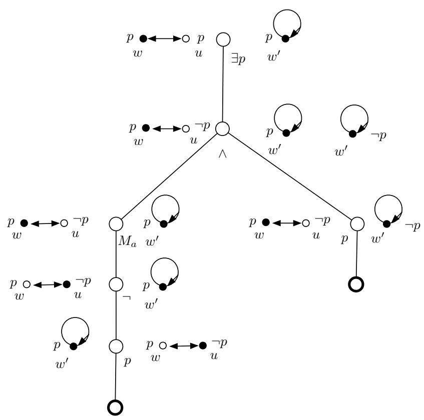

**Figure 3: The game tree from Example 4.**

models $(\mathcal{N}, v)$ in a set $\mathrm{N}_1$, we have $\mathrm{N}_1 \models \neg\psi$. We know that Spoiler can win the sub-game starting in $\langle \mathrm{M}_1 \circ \mathrm{N}_1 \rangle$ in $n-1$ moves, which in turn ensures he wins the game starting in $\langle \mathrm{M} \circ \mathrm{N} \rangle$ in $n$ moves.

For the 'only-if' direction, if Spoiler has won the FSG starting at $\langle \mathrm{M} \circ \mathrm{N} \rangle$ (in $n < k$ moves) then the resulting closed game tree is a parse tree of a formula $\varphi$ of length $n$ such that $\mathrm{M} \models \varphi$ and $\mathrm{N} \models \neg\varphi$. To see this, we label the nodes of the tree with formulae, starting with the leaves. In particular, if a node has a label $\exists p$ and its successor is labeled with $\psi$, then the current node is labeled with $\exists p\psi$. One can verify that for each node $\langle \mathrm{A} \circ \mathrm{B} \rangle$, the formula labelling the node is true in A, and false in B. Hence, the game tree is a parse tree for the formula labelling the root. □

Example 4. Consider Figure 3. This is a game tree for pair pair $\langle \mathrm{M}, \mathrm{N} \rangle$ with $\mathrm{M} = \{(\mathcal{M}, w)\}$ and $\mathrm{N} = \{(\mathcal{N}, w)\}$, depicted, resp. left and right of the root of the tree. Designated points of models are black dots, non-designated points are open dots. Leaves are closed nodes and are depicted with thick perimeters. We further assume that in M and N all atoms are true in all worlds, and there is only one agent, $a$. Notice that the two initial models are bisimilar, and hence have the same epistemic theory. This implies that the FSG starting in $\langle \mathrm{M} \circ \mathrm{N} \rangle$ can only be won if an $\exists p$ or $\forall p$ move is played. Note that the game displayed 'corresponds' to the formula $\exists p(M_a \neg p \wedge p)$. ■

In light of Theorem 19, if we can, for every $n \in \mathbb{N}$, find classes $\mathrm{M}_n$ and $\mathrm{N}_n$ of pointed models such that the following two items hold (details omitted), then we have shown that also item 3 of Definition 9 holds for the three step proof that settles that SOPAL $\preceq_{\mathcal{C}}^{exp}$ SOPEL.

1. $\mathrm{M}_n \models \psi_n$ and $\mathrm{N}_n \models \neg\psi_n$;
2. it takes Spoiler at least $2^n$ moves to win the FSG starting in $\langle \mathrm{M}_n \circ \mathrm{N}_n \rangle$.

We conclude this exercise in succinctness with two remarks. Firstly, we think that the notion of FSG can be extended to be played on *frames*. Secondly, although our argument goes through for structures whose accessibility relations are equivalences, the exact formalisation of the argument is more cumbersome, and can, we think, not be done on equivalence frames.

# 6. CONCLUSIONS

In this paper we introduced Second-order Propositional Announcement Logic: a logic to reason about arbitrary announcements in multi-agent contexts. We presented the language of SOPAL, which extends Public Announcement Logic by means of propositional quantification, and endowed it with a semantics in terms of multi-agent Kripke frames and models. We illustrated the expressivity of SOPAL by analysing relevant notions in knowledge reasoning and representation, such as preservation under arbitrary (epistemic) announcements, knowability, and successfulness. Further, we compared SOPAL with APAL by providing two notions of order between logics. Specifically, we proved that, while SOPAL and APAL are uncomparable at the model level, the former is strictly more expressive than the latter at the frame level. Moreover, we analysed the set of validities in SOPAL and provided reduction equivalences that allow to prove that SOPAL is as expressive as Second-order Propositional Epistemic Logic. As a consequence, SOPAL has a decidable model checking problem and a complete axiomatisation. Announcements make a difference nonetheless. Indeed, SOPAL is exponentially more succinct than SOPEL. We conclude that SOPAL is a succinct, rich logic, strictly more expressive than previous proposals in the area, but with nice computational properties still.

**Related Literature.** This paper draws from two different traditions in knowledge reasoning and representation: Second-order Propositional Modal Logic [8, 10] on one hand, and Dynamic Epistemic Logic [23] and Public Announcement Logic [18, 13] on the other. Both lines of research are well-established, with a rich literature. For reasons of space we only discuss the closest contributions. Second-order Propositional Epistemic Logic has been introduced in [4], where it is called Epistemic Quantified Boolean Logic. The designation 'second-order' used here is standard in the literature [19, 15], referring to features of propositional quantification. In [5] bisimulations for SOPEL are put forward. In the line of Public Announcement Logic, APAL has been presented in [2, 3], with the aim of capturing arbitrary announcements. We share the same motivation, but the formal analysis through propositional quantification is novel. Quantification (on bisimilar models) has been analysed in [11]. However, the resulting logic is as expressive as Epistemic Logic, and therefore strictly weaker than SOPAL.

In future work we plan to develop further the analysis of SOPAL in multi-agent contexts. In this direction it is of interest to study agents performing announcements: which announcements can an agent perform based on her knowledge? How do such announcements modify the epistemic state of other agents (including knowability and preservation)? How is the proposed framework to be modified to accommodate private communication? In this direction contributions such as Group Announcement Logic [1] are certainly relevant. More technically, SOPAL calls for the development of model-theoretic techniques, such as decision methods for satisfiability, in line with the well-known model theory of modal logic [6, 7].

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