Majority Graphs of Assignment Problems and Properties of Popular Random Assignments

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ABSTRACT

Randomized mechanisms for assigning objects to individual agents have received increasing attention by computer scientists as well as economists. In this paper, we study a property of random assignments, called *popularity*, which corresponds to the well-known notion of Condorcet-consistency in social choice theory. Our contribution is threefold. First, we define a simple condition that characterizes whether two assignment problems induce the same majority graph and which can be checked in polynomial time. Secondly, we analytically and experimentally investigate the uniqueness of popular random assignments. Finally, we prove that popularity is incompatible with very weak notions of both strategyproofness and envy-freeness. This settles two open problems by Aziz et al. [3] and reveals an interesting tradeoff between social and individual goals in random assignment.

General Terms

Economics, Theory

Keywords

assignment problem; random assignment; popularity; majority graphs, strategyproofness, envy-freeness

1. INTRODUCTION

Assigning objects to individual agents is a fundamental problem that has received considerable attention by computer scientists as well as economists [e.g., 15, 36, 32, 9]. In its simplest form, the problem is known as the *assignment problem*, the *house allocation problem*, or *two-sided matching with one-sided preferences*. Formally, an assignment problem concerns a set of agents $\mathscr{A}$, a set of houses $\mathscr{H}$, and the agents' (ordinal) preferences over the houses $\succsim$. For simplicity, it is often assumed that $\mathscr{A}$ and $\mathscr{H}$ are of equal size. The central question is how to assign exactly one house to each agent. An important assumption in this setting is that monetary transfers between the agents are not permitted.[1]

[1]Monetary transfers may be impossible or highly undesirable, as is the case if houses are public facilities provided to low-income people. There are a number of settings such as voting, kidney exchange, or school choice in which money cannot be used as compensation due to practical, ethical, or legal constraints [see, e.g., 32].

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Possible applications include assigning dormitories to students, jobs to applicants, rooms to housemates, processor time slots to jobs, parking spaces to employees, offices to workers, kidneys to patients, etc.

In this paper, we focus on the notion of popularity due to Gärdenfors [21]. An assignment is popular if there is no other assignment that is preferred by a majority of the agents. Popular assignments thus correspond to the well-studied notion of (weak) Condorcet winners in social choice theory. Unpopular assignments are unstable in the sense that a proposal to move to another assignment would be supported by a majority of those agents who dot not receive identical houses in both assignments. Unfortunately, the assignment setting is not immune to the Condorcet paradox and there are assignment problems that do not admit a popular assignment [21]. However, Kavitha et al. [28] have shown that existence *can* be guaranteed when allowing randomization and appropriately extending the definition of popular assignments to popular *random* assignments. A random assignment is popular if there does not exist another random assignment that is preferred by an *expected* majority of agents. Randomization is a natural and widespread technique to establish *ex ante* fairness in assignment. It is, for example, easily seen that every deterministic assignment violates 'equal treatment of equals' when all agents have identical preferences. As Hofstee [22] notes, " [. . .] if scarcity arises, lottery is the only just procedure (barring [. . .] the dividing of goods or their denial to everyone, neither of which is appropriate in the present context) [. . .]".

Popular random assignments satisfy a particularly strong notion of economic efficiency called PC-efficiency,[2] unmatched by other common assignment rules, and can be efficiently computed via linear programming. The formulation as a linear program allows one to easily accommodate for additional constraints (such as equal treatments of equals or assignment quotas) [3]. As popularity only takes into account how many agents prefer one assignment over another, it suffices to consider the pairwise majority comparisons between all possible assignments in order to determine popular assignments and popular random assignments. This information can easily be captured by a weighted graph,

[2]Intuitively, the *pairwise comparison (PC)* lottery extension prescribes that one lottery is preferred to another if the former is more likely to yield a more preferred alternative than the latter [see, also, 4]. PC is a strengthening of the well-known stochastic dominance extension.

the *majority graph*, where the set of vertices equals the set of possible deterministic assignments and edge weights are determined via majority comparisons. Such graphs are routinely studied in social choice theory. In fact, as pointed out by Aziz et al. [3], random assignment is 'merely' a special case of the general social choice setting and popular random assignments correspond to so-called maximal lotteries in general social choice [see, also, 12].

In social choice theory, it is well-known that all weighted majority graphs can be induced by some configuration of preferences [34, 18]. Majority graphs induced by assignment problems, on the other hand, constitute only a small subclass of all possible majority graphs. For example, it is easily seen that the number of vertices—i.e., the number of deterministic assignments—is always $n!$ where n is the number of agents and houses. On top of that, assignment problems impose certain structural restrictions on the corresponding majority graphs.

Contributions.

The contribution of this paper is threefold. First, we investigate the relationship between assignment problems and majority graphs. More precisely, we define a natural *decomposition* of assignment problems and show that two assignment problems induce identical majority graphs and thus have identical popular random assignments if and only if their decompositions are *rotation equivalent*. Our proof is constructive in the sense that it is even possible to check whether a given majority graph can be induced by some assignment problem.

We then study the uniqueness of popular random assignments. We prove that if all agents share the same preferences, the resulting popular random assignment is unique if there is an odd number of agents and there are infinitely many popular random assignments if the number of agents is even. Using computer experiments we find that the number of assignment problems giving rise to a unique popular random assignment decreases exponentially with the number of agents. This is in contrast to the general social choice setting where maximal lotteries, a generalization of popular random assignments, are known to be almost always unique [see, e.g., 12]. In order to avoid the problem of non-uniqueness, we propose different ways of narrowing down the set of popular random assignments.

Finally, we are able to answer two open questions posed by Aziz et al. [3]. Aziz et al. show that popularity is incompatible with strong notions of strategyproofness and envy-freeness. We prove that these impossibilities still hold when considering the significantly weaker notions of weak strategyproofness and weak envy-freeness, when the number of agents is at least seven and five, respectively.

Related work.

In the context of deterministic assignments, popularity was first considered by Gärdenfors [21] who also showed that popular assignments need not always exist. Mahdian [31] proved an interesting threshold for the existence of popular assignments: if there are n agents and the number of houses exceeds αn with $\alpha \approx 1.42$, then the probability that there is a popular assignment converges to 1 as n goes to infinity. Abraham et al. [1] proposed a polynomial-time algorithm that can both verify whether a popular assignment exists and find a popular assignment of maximal cardinality if it exists.

A closely related line of research considers popularity in marriage markets, i.e., two-sided matching with two-sided preferences. In this setting, every stable matching is also popular. Kavitha and Nasre [27] further reduced the set of popular assignments by considering "optimal" popular assignments. Biró et al. [6] defined a strong variant of popularity and provided algorithmic results for marriage markets and the more general roommate markets. Huang and Kavitha [23] studied marriage markets with possible inacceptabilities and the problem of finding popular matchings of maximal cardinality. The tradeoff between popularity and cardinality of a matching was investigated by Kavitha [26] who also provided bounds on the size of popular matchings. Cseh et al. [17] considered the complexity of finding popular matchings if one side is allowed to express indifferences in its preferences.

Finally, popular *random* assignments were introduced by Kavitha et al. [28]. Aziz et al. [3] initiated the study of axiomatic properties such as efficiency, fairness, and strategyproofness of popular random assignments. Brandl et al. [13] investigated popular random assignment rules under the assumption that participation is optional.

The two most-studied random assignment rules in the literature are *random serial dictatorship (RSD)* and the *probabilistic serial rule (PS)* [see, e.g., 7], both of which may result in unpopular outcomes [3]. See Section 6 for a more detailed discussion of RSD and PS.

2. PRELIMINARIES

An *assignment problem* is a triple $(\mathscr{A}, \mathscr{H}, \succsim)$ consisting of a set of agents $\mathscr{A}$, a set of houses $\mathscr{H}$, $|\mathscr{A}| = |\mathscr{H}| = n$, and a preference profile $\succsim = (\succsim_1, \dots, \succsim_n)$ containing preferences $\succsim_a \subseteq \mathscr{H} \times \mathscr{H}$ for all $a \in \mathscr{A}$. We assume individual preferences $\succsim_a$ to be antisymmetric, complete and transitive. $\succsim_a$ is denoted as a comma-separated list, i.e., $a\colon h_1, h_2, h_3$ means $h_1 \succsim_a h_2 \succsim_a h_3$. $\succ_a$ represents the strict part of $\succsim_a$, i.e., $h \succ_a h'$ if $h \succsim_a h'$ but not $h' \succsim_a h$.

A *deterministic assignment* (or *matching*) M is a subset of $\mathscr{A} \times \mathscr{H}$ such that $|M| = n$ and all tuples in M are pairwise disjoint, i.e., no two tuples contain the same agent or house. We write $M(a) = h$ and $M(h) = a$ if $(a, h) \in M$. Let $\mathscr{M}(n)$ denote the set of all matchings of size n.

Denote by $[k]$ the set of all natural numbers up to k, i.e., $[k] = \{1, \dots, k\}$. A *random assignment* is a matrix $p \in \mathbb{R}^{n \times n}$ with $p_{i,j} \geq 0$ for all $i, j \in [n]$, $\sum_{i \in [n]} p_{i,j} = 1$ for all $j \in [n]$, and $\sum_{j \in [n]} p_{i,j} = 1$ for all $i \in [n]$. We interpret $p_{i,j}$ as the probability with which agent a_i receives house h_j. Denote by $\mathscr{R}(n)$ the set of all random assignments of size n and by $p_{[i]}$ the vector $(p_{i,1}, \dots, p_{i,n})$. Note that, by the Birkhoff-von Neumann Theorem, we have that every probability distribution over deterministic assignments induces a unique random assignment while every random assignment can be written as a probability distribution over deterministic assignments [see, e.g., 28].

A *random assignment rule* f is a function that returns a random assignment p for all assignment problems $(\mathscr{A}, \mathscr{H}, \succsim)$.

For two deterministic assignments M, M' and an agent a with preferences $\succsim_a$ we define ϕ, a function that compares two houses, as

$$\phi_{\succsim_a}(M(a), M'(a)) = \begin{cases} 1 & \text{if } M(a) \succ_a M'(a), \\ -1 & \text{if } M'(a) \succ_a M(a), \\ 0 & \text{otherwise.} \end{cases}$$

With slight abuse of notation, we also use $\phi_{\succsim_a}(M, M') = \phi_{\succsim_a}(M(a), M'(a))$ whenever suitable. For an assignment problem $(\mathscr{A}, \mathscr{H}, \succsim)$, denote by $\phi_{\succsim}(M, M')$ the natural extension of ϕ to all agents in $\mathscr{A}$, $\phi_{\succsim}(M, M') = \sum_{a \in \mathscr{A}} \phi_{\succsim_a}(M, M')$. When considering random assignments, we define

$$\phi_{\succsim}(p, p') = \sum_{i \in [n]} \sum_{j, j' \in [n]} p_{i,j} p'_{i,j'} \phi_{\succsim_{a_i}}(h_j, h_{j'}).$$

Observe that ϕ is skew-symmetric, i.e., $\phi_{\succsim}(M, M') = -\phi_{\succsim}(M', M)$ as well as $\phi_{\succsim}(p, p') = -\phi_{\succsim}(p', p)$.

In the following, we formally introduce the concepts of popularity, majority graphs, stochastic dominance, envy-freeness, and strategyproofness.

Popularity.

Let $(\mathscr{A}, \mathscr{H}, \succsim)$ be an assignment problem. Then, a deterministic assignment M is *popular* if $\phi_{\succsim}(M, M') \geq 0$ for all $M' \in \mathscr{M}(n)$. Correspondingly, a random assignment p is *popular* if $\phi_{\succsim}(p, p') \geq 0$ for all $p' \in \mathscr{R}(n)$. Popular deterministic assignments need not always exist but the Minimax Theorem implies that every assignment problem admits at least one popular *random* assignment [28].

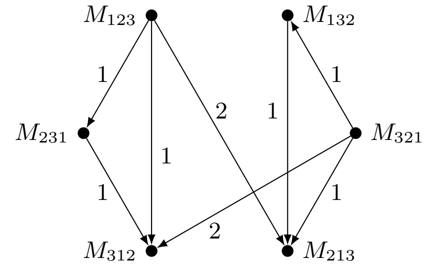
Majority Graph.

For a given assignment problem $(\mathscr{A}, \mathscr{H}, \succsim)$ we define the corresponding *majority graph* $G = (V, E, w)$ by letting the set of vertices be the set of all possible deterministic assignments and setting the edge weights according to the agents' preferences over these assignments, i.e., $V = \mathscr{M}(n)$, $E = \mathscr{M}(n) \times \mathscr{M}(n)$, and $w(M, M') = \phi_{\succsim}(M, M')$. We consequently have a graph with $|V| = n!$ vertices and a directed edge in between every pair of vertices. Note that as $w(M, M) = 0$ and $w(M, M') = -w(M', M)$, it is sufficient to depict edges with positive weight in order to capture all information stored in G.

Different assignment problems may induce identical majority graphs. Consider for instance $(\mathscr{A}, \mathscr{H}, \succsim)$ and $(\mathscr{A}, \mathscr{H}, \succsim')$ with $\mathscr{A} = \{a_1, a_2, a_3\}$, $\mathscr{H} = \{h_1, h_2, h_3\}$, and $\succsim$ and $\succsim'$ as given below.

$$\succsim = \begin{matrix} a_1: & h_1, h_2, h_3 \\ a_2: & h_2, h_1, h_3 \\ a_3: & h_1, h_2, h_3 \end{matrix} \qquad \succsim' = \begin{matrix} a_1: & h_3, h_1, h_2 \\ a_2: & h_3, h_2, h_1 \\ a_3: & h_3, h_1, h_2 \end{matrix}$$

For both assignment problems we obtain identical majority graphs $G = (V, E, w)$ with $V = \mathscr{M}(3)$, $E = V \times V$ and $w(M, M') = \phi_{\succsim}(M, M') = \phi_{\succsim'}(M, M')$ for all $M, M' \in V$ as depicted in Figure 1. Note that in order to determine which random assignments are popular for a given assignment problem $(\mathscr{A}, \mathscr{H}, \succsim)$, it suffices to consider the corresponding majority graph G. All information relevant for the computation—$\mathscr{M}(n)$ and $\phi_{\succsim}(M, M') = w(M, M')$ for all $M, M' \in \mathscr{M}(n)$—can be obtained from G. For the majority graph given above, and thereby for assignment problems $(\mathscr{A}, \mathscr{H}, \succsim)$ and $(\mathscr{A}, \mathscr{H}, \succsim')$, M_{123} and M_{321} are the only



$$\begin{aligned} M_{123} &= \{(a_1, h_1), (a_2, h_2), (a_3, h_3)\} \\ M_{231} &= \{(a_1, h_2), (a_2, h_3), (a_3, h_1)\} \\ M_{312} &= \{(a_1, h_3), (a_2, h_1), (a_3, h_2)\} \\ M_{132} &= \{(a_1, h_1), (a_2, h_3), (a_3, h_2)\} \\ M_{321} &= \{(a_1, h_3), (a_2, h_2), (a_3, h_1)\} \\ M_{213} &= \{(a_1, h_2), (a_2, h_1), (a_3, h_3)\} \end{aligned}$$

Figure 1: Majority graph for $(\mathscr{A}, \mathscr{H}, \succsim)$ and $(\mathscr{A}, \mathscr{H}, \succsim')$.

popular matchings. This can easily be seen when reasoning that popular matchings are weak Condorcet winners in the majority graph. Any randomization between M_{123} and M_{321} constitutes a popular random assignment, for instance,

$$2/3\, M_{123} + 1/3\, M_{321} = \begin{pmatrix} 2/3 & 0 & 1/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

Stochastic Dominance.

So far, agents are only endowed with an ordinal preference relation that allows the comparison of deterministic assignments, but not of random assignments. We therefore propose to extend preferences over houses to preferences over probability distributions based on *stochastic dominance (SD)*. We have that $p_{[i]} \succsim^{\mathrm{SD}}_{a_i} p'_{[i]}$ if

$$\sum_{h_j \in \mathscr{H}, h_j \succsim_{a_i} h} p_{i,j} \geq \sum_{h_j \in \mathscr{H}, h_j \succsim_{a_i} h} p'_{i,j}$$

for all $h \in \mathscr{H}$. In this case, we say that a_i weakly SD-prefers p to p'. With slight abuse of notation, we sometimes also write $p \succsim^{\mathrm{SD}}_{a_i} p'$. This preference extension is of special importance as $p \succsim^{\mathrm{SD}}_{a_i} p'$ if and only if p yields at least as much expected utility than p' with respect to all von Neumann-Morgenstern utility functions consistent with a_i's ordinal preferences $\succsim_{a_i}$ [see, e.g., 7, 25].

Given an assignment problem $(\mathscr{A}, \mathscr{H}, \succsim)$, a random assignment $p \in \mathscr{R}(n)$ is *SD-efficient* if there is no $p' \in \mathscr{R}(n)$ such that $p' \succsim^{\mathrm{SD}}_{a} p$ for all $a \in \mathscr{A}$ and $p' \succ^{\mathrm{SD}}_{a'} p$ for some $a' \in \mathscr{A}$.

While stochastic dominance is the most common preference extension, there are also other natural extensions that can be used to define variants of efficiency, strategyproofness, and envy-freeness. In particular, there is a weakening of stochastic dominance called bilinear dominance (BD) and a strengthening of SD called pairwise comparison (PC). We refer to Aziz et al. [4, 5] for more details.

Strategyproofness and Envy-freeness.

Strategyproofness requires that stating one's true preferences is always at least as good as misrepresenting them, while envy-freeness requires that every agent weakly prefers his allocation to that of all others.

Formally, an assignment rule f is *strategyproof* if for all $(\mathscr{A}, \mathscr{H}, \succsim)$, $a \in \mathscr{A}$, and $(\mathscr{A}, \mathscr{H}, \succsim')$ such that $\succsim_{a'} = \succsim'_{a'}$ for all $a' \in \mathscr{A} \setminus \{a\}$ we have that $f(\mathscr{A}, \mathscr{H}, \succsim) \succsim_a^{\text{SD}} f(\mathscr{A}, \mathscr{H}, \succsim')$. Since the SD preference extension only yields an incomplete preference relation over lotteries, one can also define a weaker notion of strategyproofness that merely requires that no agent benefits by misrepresenting his preferences.[3] f satisfies *weak strategyproofness* if for all $(\mathscr{A}, \mathscr{H}, \succsim)$ and $a \in \mathscr{A}$ there is no $(\mathscr{A}, \mathscr{H}, \succsim')$ with $\succsim_{a'} = \succsim'_{a'}$ for all $a' \in \mathscr{A} \setminus \{a\}$ such that $f(\mathscr{A}, \mathscr{H}, \succsim') \succ_a^{\text{SD}} f(\mathscr{A}, \mathscr{H}, \succsim)$.[4]

A random assignment $p \in \mathscr{R}(n)$ satisfies *envy-freeness* if $p_{[i]} \succsim_{a_i}^{\text{SD}} p_{[j]}$ for all agents $a_i \in \mathscr{A}$ and $j \in [n] \setminus \{i\}$. Similarly as above, one can define a weaker notion of envy-freeness. p satisfies *weak envy-freeness* if there is no agent a_i such that $p_{[j]} \succ_{a_i}^{\text{SD}} p_{[i]}$ for some $j \in [n]$.

3. DECOMPOSITION OF ASSIGNMENT PROBLEMS

This section focuses on the question under which conditions two assignment problems induce the same majority graph. This study is motivated by the fact that the set of popular random assignments depends on the majority graph only, i.e., two assignment problems that have identical majority graphs also have identical popular random assignments. Thus, gaining insights in the structure of majority graphs automatically results in insights into popularity.

We will provide an easily verifiable condition that holds if and only if two assignment problems have identical majority graphs. Furthermore, given a majority graph, it is possible to determine all assignment problems that induce this graph.

Given an assignment problem $(\mathscr{A}, \mathscr{H}, \succsim)$, we say that $(\mathscr{A}, H_k, \succsim^k)_{k \in [m]}$ is a *decomposition* of $(\mathscr{A}, \mathscr{H}, \succsim)$, if conditions (*i*)–(*iii*) hold and there does not exists an $m' > m$ for which (*i*)–(*iii*) can also be satisfied:

(*i*) $\dot{\bigcup}_{k \in [m]} H_k = \mathscr{H}$ with $H_k \neq \emptyset$ for all $k \in [m]$,

(*ii*) $h \succsim_a h'$ implies $h \succsim_a^k h'$ for all $k \in [m]$, $h, h' \in H_k$, $a \in \mathscr{A}$, and

(*iii*) $h \succ_a h'$ for all $h \in H_k, h' \in H_{k'}, 1 \leq k < k' \leq m$, $a \in \mathscr{A}$.

By decomposing $(\mathscr{A}, \mathscr{H}, \succsim)$, we thus partition $\mathscr{H}$ into nonempty subsets such that all agents prefer houses contained in H_k to houses in $H_{k'}$ if and only if $k < k'$. At the same time, agents' preferences over houses contained in the same H_k remain unchanged. It is easy to see that for every assignment problem there exists a unique decomposition. If it holds that $m = 1$, we use the term *trivial decomposition.*

Let $(\mathscr{A}, \mathscr{H}, \succsim)$ and $(\mathscr{A}, \mathscr{H}, \succsim')$ be two assignment problems and $(\mathscr{A}, H_k, \succsim^k)_{k \in [m]}$ and $(\mathscr{A}, H_{k'}, \succsim'^{k'})_{k' \in [m']}$ their corresponding decompositions. We say that $(\mathscr{A}, H_k, \succsim^k)_{k \in [m]}$ and $(\mathscr{A}, H_{k'}, \succsim'^{k'})_{k' \in [m']}$ are *rotation equivalent* if there exists $d \in [m]$ such that $\succsim^k = \succsim'^{((k+d-1) \mod m)+1}$ for all $k \in [m]$. For the sake of readability, we hereafter use mod_1 defined by $k \mod_1 k' = ((k-1) \mod k') + 1$. Rotation equivalence can thus be rewritten as $\succsim^k = \succsim'^{(k+d) \mod_1 m}$. Intuitively, two decompositions are rotation equivalent if they agree on the partitioning of $\mathscr{H}$, agents' preferences within the partition's subsets and the ordering of those subsets *modulo* m.

For better illustration of the concept consider the following brief example with four agents $\mathscr{A} = \{a_1, a_2, a_3, a_4\}$, four houses $\mathscr{H} = \{h_1, h_2, h_3, h_4\}$ and preference profiles $\succsim$, $\succsim'$, and $\succsim''$.

$$
\succsim \; = \;
\begin{array}{l|l|ll}
a_1\colon & h_1, & h_2, & h_3, \; h_4 \\
a_2\colon & h_1, & h_2, & h_4, \; h_3 \\
a_3\colon & h_1, & h_2, & h_3, \; h_4 \\
a_4\colon & h_1, & h_2, & h_4, \; h_3
\end{array}
$$

$$
\succsim' \; = \;
\begin{array}{l|ll|l}
a_1\colon & h_2, & h_3, \; h_4, & h_1 \\
a_2\colon & h_2, & h_4, \; h_3, & h_1 \\
a_3\colon & h_2, & h_3, \; h_4, & h_1 \\
a_4\colon & h_2, & h_4, \; h_3, & h_1
\end{array}
$$

$$
\succsim'' \; = \;
\begin{array}{l|ll|l}
a_1\colon & h_1, & h_3, \; h_4, & h_2 \\
a_2\colon & h_1, & h_4, \; h_3, & h_2 \\
a_3\colon & h_1, & h_3, \; h_4, & h_2 \\
a_4\colon & h_1, & h_4, \; h_3, & h_2
\end{array}
$$

We see that $\mathscr{H}$ is partitioned into the sets $\{h_1\}, \{h_2\}, \{h_3, h_4\}$ in all three decompositions with agents' preferences over the houses within those sets being identical in all cases. For better exposition, dotted lines are added in between the components. However, only the decompositions of $(\mathscr{A}, \mathscr{H}, \succsim)$ and $(\mathscr{A}, \mathscr{H}, \succsim')$ are rotation equivalent. Our first theorem links rotation equivalent decompositions to identical majority graphs.

THEOREM 1. *Let $(\mathscr{A}, \mathscr{H}, \succsim)$ and $(\mathscr{A}, \mathscr{H}, \succsim')$ be two assignment problems that induce majority graphs G and G', respectively. Then, $G = G'$ if and only if the decompositions of $(\mathscr{A}, \mathscr{H}, \succsim)$ and $(\mathscr{A}, \mathscr{H}, \succsim')$ are rotation equivalent.*

Due to space constraints, we omit the proof of Theorem 1; it can be found in the workshop version of this paper [14]. Since the proof is constructive, it is easy to develop an algorithm that, given a majority graph, finds all assignment problems that induce this graph. This algorithm can also answer the question whether a given graph is induced by an assignment problem.

It is worth noting that, as the set of popular random assignments only depends on the majority graph, Theorem 1 directly implies that two assignment problems admit an identical set of popular random assignments if their decompositions are rotation equivalent. Thus, when we are interested in the question whether two assignment problems give rise to identical sets of popular random assignments, we do not always have to compute them explicitly. Whether two assignment problems are rotation equivalent can easily be checked. Going back to the example above, we hence have that $(\mathscr{A}, \mathscr{H}, \succsim)$ and $(\mathscr{A}, \mathscr{H}, \succsim')$ admit identical popular random assignments.

[3]In other words, a manipulation only counts as a manipulation if it leads to more expected utility *for all* expected utility representations of the agent's ordinal preferences.

[4]Note that what we call strategyproofness and weak strategyproofness are often also referred to as *strong strategyproofness* and *strategyproofness* in the literature.

n	$N(n)$	$n!^n$	$N(n)/n!^n$
1	1	1	1
2	3	4	0.75
3	194	216	0.898
4	329 898	331 776	0.994
5	24 841 082 904	24 883 200 000	0.998

Table 1: Number of inducible majority graphs relative to the number of assignment problems of size n.

As a second consequence we see that assignment problems that only admit the trivial decomposition induce a unique majority graph.

We conclude this section with an observation regarding the number of different majority graphs that can be induced by assignment problems of size n. Directly counting the number of majority graphs is not possible because we lack a suitable characterization thereof. Still, by Theorem 1, we know that two assignment problems induce identical majority graphs if and only if their decompositions are rotation equivalent. We make use of this correspondence and actually sum up the number of assignment problems of size n not having rotation equivalent decompositions:

$$N(n) = \sum_{i\in[n]} \frac{(-1)^{i+1}}{i} \cdot \left(\sum_{\substack{x_0,\dots,x_i\in\mathbb{N}_0 \\ 0=x_0<\dots<x_i=n}} \prod_{j\in[i]} \binom{n-x_{j-1}}{x_j-x_{j-1}} \cdot ((x_j-x_{j-1})!)^n \right)$$

It turns out that $N(n)$ is roughly equivalent to $n!^n$; see Table 1 for the values of $N(n)$ and $n!^n$ up to $n = 5$. Note that the total number of assignment problems of size n is exactly $n!^n$, which implies that a nontrivial decomposition is impossible for a vast majority of profiles.

As most assignment problems hence induce different majority graphs, the question remains which ratio of the possible majority graphs may be induced. Regarding majority graphs in the context of social choice, observe that the total number of directed, weighted graphs (V, E, w) with edge weights $|w(e)| \leq n$ for all $e \in E$ is $(2n+1)^{1/2\ n!(n!-1)}$. The fraction of those graphs that can be induced by an assignment problem of size n is comparatively small, it can easily be upper-bounded by $n!^n/n^{n!}$. Given the many interdependencies of edge weights due to the fact that agents only have preferences over n houses but we have $n!$ vertices, this result confirms the naive intuition that most majority graphs cannot be induced by an assignment problem.

In this context, it is worth noting that the empty graph, i.e., the majority graph with $w(e) = 0$ for all $e \in E$, cannot be induced by any assignment problem of size $n > 2$. This can easily be seen when considering two matchings $M, M' \in \mathscr{M}(n)$ where three agents 'rotate' their houses, i.e., $M(h) = M'(h')$, $M(h') = M'(h'')$, $M(h'') = M'(h)$, and $M(h''') = M'(h''')$ for all $h''' \in \mathscr{H} \setminus \{h, h', h''\}$. Here, $\phi_{\succsim}(M, M') \in \{3, 1, -1, -3\}$. We consequently obtain that whenever $n > 2$, it is impossible that all random assignments are popular, or, put differently, popularity always imposes a restriction on the set of random assignments.

4. UNIQUENESS OF POPULAR RANDOM ASSIGNMENTS

We started our study of popularity by looking at the parts of an assignment problem relevant for computing the corresponding popular random assignments: the majority graph. Now, we want to have a closer look at the concept of popularity and see which restrictions it imposes on the set of random assignments.

As already briefly discussed, popular deterministic assignments need not always exist [21]. When considering random assignments instead, Kavitha et al. [28] have shown that there always is at least one popular random assignment. It is easy to show that the set of popular random assignments is convex, i.e., if there are at least two different popular random assignments, there are infinitely many.

Hence, a natural question is which assignment problems admit a unique popular random assignment. In other words, under which circumstances does popularity already restrict the set of desirable random assignments to a singleton?

In order to tackle this question, we first focus on the setting where all agents have identical preferences, and completely characterize the set of popular random assignments for arbitrary n. A similar property was also studied in the context of stable matching by Irving et al. [24], who considered cases where either one or both sides of the matching share identical preferences over the opposite side, so-called *master lists*, except possible unacceptabilities. Note that these situations are not particularly unlikely, for instance if objects are consistently evaluated by size or monetary value [see, also, 8].

For this restricted case we are able to show that there is a unique popular random assignment if n is odd and infinitely many if n is even. To get a better idea about the frequency of unique popular random assignments, we present the results of computer experiments. We conclude the section with some ideas how to further narrow down the set of popular random assignments.

4.1 Identical Preferences

In this subsection we consider assignment problems $(\mathscr{A}, \mathscr{H}, \succsim)$ where all agents have identical preferences. Without loss of generality let us assume agents always prefer houses with a lower index, i.e., $h_k \succ_a h_{k'}$ for all h_k, $h_{k'} \in \mathscr{H}, 1 \leq k < k' \leq n$, and $a \in \mathscr{A}$. As the preferences $\succsim$ only depend on the number of agents in this subsection, we simplify notation by writing $\phi(p, p') = \phi_{\succsim}(p, p')$.

The upcoming theorem builds on a *left shift* of probabilities. The left shift function $L(p)$ maps the probability an agent a receives for house h_k to the probability he receives for the next less preferred house h_{k+1}. We define the function $L\colon \mathscr{R}(n) \to \mathscr{R}(n)$, $(L(p))_{i,j} = p_{i,(j \mod n)+1}$.

It holds that the set of all popular random assignments consists of exactly those random assignments, that are invariant under double application of L.

Theorem 2. *Let $(\mathscr{A}, \mathscr{H}, \succsim)$ be an assignment problem where all agents have identical preferences. Then, a random assignment $p \in \mathscr{R}(n)$ is popular if and only if $L(L(p)) = p$.*

Proof. For the sake of brevity, we only give a proof outline without details for the upcoming arguments. The full proof is available in [14].

Let $(\mathscr{A}, \mathscr{H}, \succsim)$ be a random assignment problem where all agents have identical preferences. We start the proof by showing that for every popular random assignment $p' \in \mathscr{R}(n)$ we have $L(L(p')) = p'$. First, some tedious transformations yield that $\phi(L(p), p) \geq 0$ for all $p \in \mathscr{R}(n)$, i.e., every random assignment p is at most as popular as its corresponding left shift. We furthermore deduce that $\phi(L(p), p) = 0$ holds if and only if $p_{i,j} - p_{i,(j+2) \bmod_1 n} = 0$ for all $i, j \in [n]$. Put differently, $\phi(L(p), p) = 0$ if $L(L(p)) = p$ and $\phi(L(p), p) > 0$ if $L(L(p)) \neq p$. Hence, we can find for every random assignment p which does not satisfy $L(L(p)) = p$ another random assignment which is strictly more popular. This gives that every popular random assignment p' satisfies $L(L(p')) = p'$.

For the converse direction, we distinguish between odd and even n. If n is odd, there is only one $p \in \mathscr{R}(n)$ that possibly satisfies $L(L(p)) = p$, namely $p_{i,j} = 1/n$ for all $i, j \in [n]$. Since we know that there has to exist at least one popular random assignment, we directly deduce that it must be p.

In order to show that every random assignment p that satisfies $L(L(p)) = p$ is popular in the case of even n, we first define the set of *extremal* random assignments $E(n)$:

$$E(n) = \{e^I \in \mathscr{R}(n) \colon I \subseteq [n], |I| = n/2\}$$

$$e^I_{i,j} = \begin{cases} 2/n & \text{if either } i \in I \text{ and } j \text{ odd,} \\ & \quad \text{or } i \notin I \text{ and } j \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$E(n)$ thus consists of random assignments p with $p_{i,j} \in \{0, 2/n\}$ for all $i, j \in [n]$ where for every agent probabilities alternate throughout his preference list. First, we can show that $\phi(e^{[n/2]}, p') = 0$ for all $p' \in \mathscr{R}(n)$. A second step proves that $\phi(e^I, p') = 0$ for all $p' \in \mathscr{R}(n), e^I \in E(n)$, essentially giving that all extremal random assignments are popular. Lastly, we show that all $p \in \mathscr{R}(n)$ that satisfy $L(L(p)) = p$ can be represented as convex combination of random assignments in $E(n)$. Using the convexity of the set of popular random assignments, we get that every $p \in \mathscr{R}(n)$ that satisfies $L(L(p)) = p$ is popular as well for even n, which completes the proof. □

The following corollary precisely characterizes the set of popular random assignments for the case of identical preferences.

Corollary 1. *Let $(\mathscr{A}, \mathscr{H}, \succsim)$ be a random assignment problem where all agents have identical preferences. If n is odd, there exists a unique popular random assignment p, namely $p_{i,j} = 1/n$ for all $i, j \in [n]$. If n is even, there exist multiple popular random assignments, namely* $\mathrm{conv}(E(n))$ *with $E(n)$ defined as above.*

We see that popularity implies a certain degree of randomness: given n is odd every agent receives positive probability for all houses while for even n, agents receive positive probability for either half or all of the houses.

4.2 Experimental Results

Of course, there are plenty of assignment problems with a unique popular random assignment, even when the preferences of agents are not identical. Consider for instance the assignment problem $(\mathscr{A}, \mathscr{H}, \succsim)$ with $n = 5$ and $\succsim$ as depicted below. We see that neither do all agents share identical preferences nor is there any obvious inherent structure within the preference profile. Nevertheless, there is a unique popular random assignment $p \in \mathscr{R}(5)$.

$$\succsim \; = \; \begin{array}{rl} a_1\colon & h_2, h_5, h_4, h_3, h_1 \\ a_2\colon & h_5, h_2, h_4, h_3, h_1 \\ a_3\colon & h_2, h_1, h_4, h_3, h_5 \\ a_4\colon & h_2, h_1, h_3, h_5, h_4 \\ a_5\colon & h_2, h_5, h_1, h_3, h_4 \end{array} \qquad p = \begin{pmatrix} 0 & 1/7 & 0 & 5/7 & 1/7 \\ 0 & 0 & 1/7 & 1/7 & 5/7 \\ 2/7 & 3/7 & 1/7 & 1/7 & 0 \\ 3/7 & 2/7 & 2/7 & 0 & 0 \\ 2/7 & 1/7 & 3/7 & 0 & 1/7 \end{pmatrix}$$

Our next goal is to determine the fraction of profiles that admit a unique popular random assignment, depending on n. We can compute this number exactly as long as n is relatively small. However, as the number of preference profiles is $(n!)^n$, exact computation quickly becomes infeasible, even when exploiting symmetries with respect to both agents and houses. Note that for instance for $n = 6$, we already have more than $1.3 \cdot 10^{17}$ different profiles. The exact number of profiles admitting a unique popular random assignment for $n \leq 4$ is given in Table 2.

To overcome the intractability of computing the exact fraction of profiles admitting a unique popular random assignment but still obtain a quantitative insight, we automatically sample preference profiles and verify whether they admit multiple popular random assignments.

For the sampling process, we focus on two common parameter-free stochastic models. First, we choose each agent's preferences uniformly at random, which is known as the *impartial culture (IC)* model [see, e.g., 19, for early use of IC].

In the *spatial model*, we sample a point in the unit square for every $a \in \mathscr{A}$ and $h \in \mathscr{H}$ and determine agents' preferences by their proximity to each house, i.e., the Euclidian distance between the corresponding points [see, e.g., 35, 2]. For a more profound discussion of stochastic preference models in general see for instance Critchlow et al. [16] and Marden [33].

n	1	2	3	4
Unique	1	2	54	35 904
Total	1	4	216	331 776
Fraction	1	0.5	0.25	0.108

Table 2: Number of profiles that admit a unique popular random assignment and the total number of profiles for $n \leq 4$.

For both models, Table 3 summarizes the results for 10 000 samples each. Figure 2 provides a visualization where the probability that a randomly picked assignment problem admits a unique popular random assignment is plotted on a logarithmic scale. We see that this probability decreases exponentially in n, where the decreases are slightly more distinctive when going from an odd n to an even one compared to from an even n to an odd one. A possible explanation might be related to Theorem 2.

Note that this exponential decrease stands in sharp contrast to results obtained in the social choice setting. Recall that popular random assignments correspond directly to maximal lotteries. Maximal lotteries are unique in many cases [29, 30] and the set of preference profiles admitting a unique maximal lottery is open and dense [12]. The set of profiles that admit multiple maximal lotteries is therefore nowhere dense and thus negligible.

n	1	2	3	4	5	6	7
IC	1	0.49	0.25	0.11	0.044	0.020	0.0088
Spatial	1	0.43	0.26	0.14	0.078	0.040	0.027

Table 3: Fraction of preference profiles admitting a unique popular random assignment when preferences are sampled according to either IC or the spatial model; $10\,000$ samples for each n.

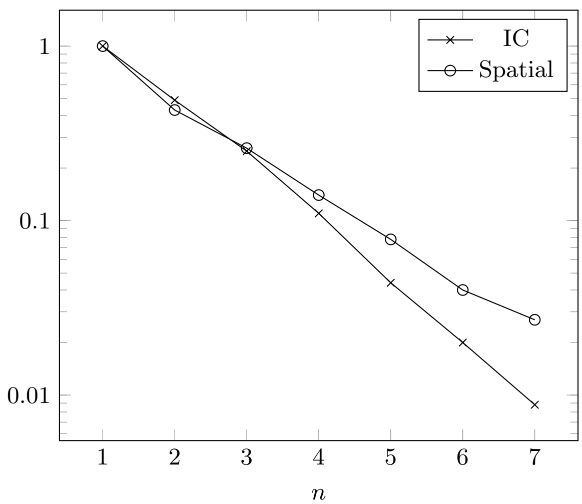


Figure 2: Probability that a randomly selected assignment problem of size n admits a unique popular random assignment. Preferences are sampled according to either IC or the spatial model.

4.3 Selecting Popular Random Assignments

Since our experiments suggest that the fraction of assignment problems admitting a unique popular random assignment decreases exponentially in n, a natural question to ask is whether we can somehow further narrow down the set of popular random assignments in a meaningful way. In this subsection, we propose three different methods to achieve this [see, also, 20].

Minimize envy.
A natural idea is to require envy-freeness in addition to popularity. Unfortunately, Theorem 3 shows that there exist assignment problems for which no popular random assignment satisfies even weak envy-freeness. Rather than disallowing envy, one can try to *minimize* it. As popular random assignments can be computed via a linear feasibility problem, it is relatively easy to include the necessary constraints and minimization objective [see, also, 28, 3]. However, computer experiments show that even though this increases the fraction of assignment problems returning a unique solution, there are still many cases where we are left with infinitely many solutions.

Minimize randomness.
Depending on the application setting, it may be desirable to either maximize or minimize the amount of 'randomness'. For one possible definition of randomness—$\sum_{i,j\in[n]} p_{i,j}(1-p_{i,j})$—we obtain a unique solution if we desire maximal randomness but still have multiple assignments when randomness is minimized.

Barycenter.
A natural unique choice from the polytope of random assignment seems to be its barycenter. It intuitively feels 'fair' and can be shown to satisfy equal treatment of equals. However, computing the barycenter of the polytope of popular random assignments is a challenging computational problem which may be infeasible.

5. ENVY-FREENESS AND STRATEGYPROOFNESS

In this section, we investigate to which extent popularity is compatible with envy-freeness and strategyproofness. Put differently, we want to know whether for every assignment problem there exists a popular random assignment that satisfies envy-freeness and whether there exists a random assignment rule that satisfies both popularity and strategyproofness. Prior research in this direction by Aziz et al. [3] has established the following results. First, it was shown that there exists a profile with $n = 3$ for which no popular assignment satisfies envy-freeness. Secondly, popularity was proven to be incompatible with strategyproofness when $n \geq 3$.[5] Whether both results also hold for *weak* envy-freeness and *weak* strategyproofness, respectively, was left as an open problem.

We are able to answer this question in the affirmative: We provide a profile with $n = 5$ for which no popular random assignment satisfies weak envy-freeness and that can easily be extended to $n \geq 5$. In addition, we show that no random assignment rule can satisfy popularity and weak strategyproofness simultaneously whenever $n \geq 7$.

It is worth mentioning that weak envy-freeness and weak strategyproofness are significantly weaker than their stronger counterparts. While, for example, a failure of strategyproofness means that there exists one von Neumann-Morgenstern utility function consistent with the agent's preferences for which manipulation is possible, a failure of *weak* strategyproofness means that manipulation is possible for *all* von Neumann-Morgenstern utility representations.

THEOREM 3. *There exist assignment problems for which no popular random assignment satisfies weak envy-freeness when $n \geq 5$.*

PROOF. Consider the assignment problem $(\mathscr{A}, \mathscr{H}, \succsim)$ with five agents $\mathscr{A} = \{a_1, \ldots, a_5\}$, five houses

[5] When agents' actions are limited to strategic abstention instead of misrepresentation of preferences, Brandl et al. [13] have shown that manipulation is not possible, even when considering deviations by groups of agents.

$\mathscr{H} = \{h_1, \ldots, h_5\}$, and

$$\succsim = \begin{array}{ll} a_1\colon & h_1, h_2, h_3, h_4, h_5 \\ a_2\colon & h_1, h_2, h_3, h_4, h_5 \\ a_3\colon & h_1, h_2, h_3, h_4, h_5 \\ a_4\colon & h_4, h_1, h_2, h_3, h_5 \\ a_5\colon & h_1, h_4, h_2, h_5, h_3 \end{array}.$$

It can be shown that for all popular random assignments $p \in \mathscr{R}(n)$, $p_{i,j} = 1/3$ for all $i, j \in [3]$.

Consequently, only a_4 and a_5 are competing for houses h_4 and h_5. Even though they share the strict preference $h_4 \succ h_5$, a_4 ranks h_4 higher and h_5 lower in comparison to a_5. We compute that popularity of p implies $2/3 \leq p_{5,5} \leq 1$. Thus, every popular random assignment p is of the form

$$p = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1-\lambda \\ 0 & 0 & 0 & 1-\lambda & \lambda \end{pmatrix}$$

with $2/3 \leq \lambda \leq 1$. For all such assignments p, a_5 SD-prefers a_4's allocation to his own.

Note that similar profiles can also be constructed for all $n \geq 5$. This can be done by adding agents a_i, $i \geq 6$, who each have house h_i as first preference while the preferences of a_1 to a_5 over h_1 to h_5 remain as given above. The allocation for agents a_1 to a_5 is not affected by those additional agents following the SD-efficiency of popular random assignments. □

Theorem 4. *No popular random assignment rule satisfies weak strategyproofness when $n \geq 7$.*

Proof. Consider the assignment problem $(\mathscr{A}, \mathscr{H}, \succsim)$ with seven agents $\mathscr{A} = \{a_1, \ldots, a_7\}$, seven houses $\mathscr{H} = \{h_1, \ldots, h_7\}$, and

$$\succsim = \begin{array}{ll} a_1\colon & h_1, h_2, h_3, h_6, h_4, h_5, h_7 \\ a_2\colon & h_1, h_2, h_3, h_6, h_4, h_5, h_7 \\ a_3\colon & h_1, h_2, h_3, h_6, h_4, h_5, h_7 \\ a_4\colon & h_4, h_5, h_1, h_2, h_3, h_6, h_7 \\ a_5\colon & h_4, h_5, h_1, h_2, h_3, h_6, h_7 \\ a_6\colon & h_1, h_6, h_4, h_3, h_5, h_2, h_7 \\ a_7\colon & h_1, h_4, h_6, h_7, h_2, h_5, h_3 \end{array}.$$

One can compute the vertices of the convex polytope containing all popular random assignments p. For all those, we deduce that $1/2 \leq p_{7,7} = 1 - p_{7,6} \leq 1$. Put differently, a_7 receives h_7 with probability at least $1/2$ and h_1 to h_5 with probability 0.

Now, let a_7 alter his preferences in a way such that h_6 shall be his most preferred house while h_7 becomes the least preferred one leaving everything else unchanged, i.e.,

$$a_7'\colon \quad h_6, h_1, h_4, h_2, h_5, h_3, h_7.$$

For the new assignment problem $(\mathscr{A}, \mathscr{H}, \succsim')$ with $\succsim'_a = \succsim_a$ for all $a \in \mathscr{A} \setminus \{a_7\}$ we once more compute all popular random assignments p'. Now, we obtain that $0 \leq p'_{7,7} = 1 - p'_{7,6} \leq 2/5$. Hence, in all random assignments $p' \in \mathscr{R}(7)$ that are popular with respect to $(\mathscr{A}, \mathscr{H}, \succsim')$, a_7 receives h_6 with strictly more probability than in p while getting h_7 less frequently. Consequently, $p'_{[7]} \succ^{\text{SD}}_{a_7} p_{[7]}$.

Introducing additional agents and houses such that each agent a_i has house h_i as first preference, $i \geq 8$, allows us to construct preference profiles for $n \geq 8$, each admitting the same manipulation beneficial for a_7. Thus, no random assignment rule can satisfy popularity and weak strategyproofness at the same time when $n \geq 7$. □

The results presented do not only hold for the SD-extension, but also for *bilinear dominance*, leading to even weaker notions of strategyproofness and envy-freeness (see Section 2).

6. CONCLUSION AND DISCUSSION

We have analyzed the structure of majority graphs induced by assignment problems and investigated the uniqueness, envy-freeness, and strategyproofness of popular random assignments and popular random assignment rules, respectively. It has turned out that most assignment problems admit more than one popular random assignment and that popularity does not align well with individual incentives as popularity is incompatible with weak envy-freeness and also with weak strategyproofness. On the other hand, it is known that popular random assignments satisfy a very strong notion of efficiency (PC-efficiency) and even maximize social welfare according to the canonical skew-symmetric bilinear (SSB) utility functions induced by the agents' preferences [see 10]. This hints at an interesting tradeoff between social goals (such as efficiency and popularity) and individual goals (such as envy-freeness and strategyproofness) in random assignment. For comparison, the two most-studied assignment rules RSD and PS fail to satisfy PC-efficiency (and thus popularity). In fact, RSD does not even satisfy SD-efficiency. On the other hand, these rules fare better in terms of individual incentives of agents. RSD satisfies strategyproofness and PS satisfies envy-freeness.

This tradeoff has been observed before. For example, Bogomolnaia and Moulin [7] have shown that SD-efficiency and strategyproofness are incompatible. When allowing ties in individual preferences, Katta and Sethuraman [25] proved that no random assignment rule simultaneously satisfies SD-efficiency, weak strategyproofness, and weak envy-freeness. Recently, Brandl et al. [11] gave a computer-aided proof that shows the incompatibility of SD-efficiency and weak strategyproofness in the more general domain of social choice. It is open whether the same statement also holds for random assignment (when agents have weak preferences). For the case of strict preferences, it would be interesting to see whether Theorem 4 can be strengthened by replacing popularity with PC-efficiency.

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REFERENCES

[1] D. K. Abraham, R. W. Irving, T. Kavitha, and K. Mehlhorn. Popular matchings. *SIAM Journal on Computing*, 37(4):1030–1034, 2007.

[2] D. Austen-Smith and J. S. Banks. *Positive Political Theory I: Collective Preference*. University of Michigan Press, 1999.

[3] H. Aziz, F. Brandt, and P. Stursberg. On popular random assignments. In *Proceedings of the 6th International Symposium on Algorithmic Game*

Theory (SAGT), volume 8146 of *Lecture Notes in Computer Science (LNCS)*, pages 183–194. Springer-Verlag, 2013.

[4] H. Aziz, F. Brandl, and F. Brandt. On the incompatibility of efficiency and strategyproofness in randomized social choice. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI)*, pages 545–551. AAAI Press, 2014.

[5] H. Aziz, F. Brandl, and F. Brandt. Universal Pareto dominance and welfare for plausible utility functions. *Journal of Mathematical Economics*, 60:123–133, 2015.

[6] P. Biró, R. W. Irving, and D. Manlove. Popular matchings in the marriage and roommates problems. In *Proceedings of the 7th Italian Conference on Algorithms and Complexity (CIAC)*, pages 97–108, 2010.

[7] A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, 2001.

[8] A. Bogomolnaia and H. Moulin. A simple random assignment problem with a unique solution. *Economic Theory*, 19(3):623–635, 2002.

[9] S. Bouveret, Y. Chevaleyre, and N. Maudet. Fair allocation of indivisible goods. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 12. Cambridge University Press, 2016.

[10] F. Brandl, F. Brandt, and J. Hofbauer. Welfare maximization entices participation. Technical report, http://arxiv.org/abs/1508.03538, 2015.

[11] F. Brandl, F. Brandt, and C. Geist. Proving the incompatibility of efficiency and strategyproofness via SMT solving. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 116–122. AAAI Press, 2016.

[12] F. Brandl, F. Brandt, and H. G. Seedig. Consistent probabilistic social choice. *Econometrica*, 84(5): 1839–1880, 2016.

[13] F. Brandl, F. Brandt, and J. Hofbauer. Random assignment with optional participation. In *Proceedings of the 16th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 2017. Forthcoming.

[14] F. Brandt, J. Hofbauer, and M. Suderland. Majority graphs of assignment problems and properties of popular random assignments. In *Proceedings of the 6th International Workshop on Computational Social Choice (COMSOC)*, 2016.

[15] Y. Chevaleyre, P. E. Dunne, U. Endriss, J. Lang, M. Lemaître, N. Maudet, J. Padget, S. Phelps, J. A. Rodríguez-Aguilar, and P. Sousa. Issues in multiagent resource allocation. *Informatica*, 30:3–31, 2006.

[16] D. E. Critchlow, M. A. Fligner, and J. S. Verducci. Probability models on rankings. *Journal of Mathematical Psychology*, 35:294–318, 1991.

[17] Á. Cseh, C.-C. Huang, and T. Kavitha. Popular matchings with two-sided preferences and one-sided ties. In *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP)*, volume 9134 of *Lecture Notes in Computer Science (LNCS)*, pages 367–379, 2015.

[18] B. Debord. Caractérisation des matrices des préférences nettes et méthodes d'agrégation associées. *Mathématiques et sciences humaines*, 97:5–17, 1987.

[19] F. DeMeyer and C. R. Plott. The probability of a cyclical majority. *Econometrica*, 38(2):345–354, 1970.

[20] M. Engeßer. Tie-breaking for popular random assignments. Master's thesis, Technische Universität München, 2016.

[21] P. Gärdenfors. Match making: Assignments based on bilateral preferences. *Behavioral Science*, 20(3): 166–173, 1975.

[22] W. K. B. Hofstee. Allocation by lot: a conceptual and empirical analysis. *Social Science Information*, 29(4): 745–763, 1990.

[23] C.-C. Huang and T. Kavitha. Popular matchings in the stable marriage problem. In *Proceedings of the 38th International Colloquium on Automata, Languages, and Programming (ICALP)*, volume 6755 of *Lecture Notes in Computer Science (LNCS)*, pages 666–677, 2011.

[24] R. W. Irving, D. F. Manlove, and S. Scott. The stable marriage problem with master preference lists. *Discrete Applied Mathematics*, 156:2959–2977, 2008.

[25] A.-K. Katta and J. Sethuraman. A solution to the random assignment problem on the full preference domain. *Journal of Economic Theory*, 131(1):231–250, 2006.

[26] T. Kavitha. A size-popularity tradeoff in the stable marriage problem. *SIAM Journal on Computing*, 43 (1):52–71, 2014.

[27] T. Kavitha and M. Nasre. Optimal popular matchings. *Discrete Applied Mathematics*, 157:3181–3186, 2009.

[28] T. Kavitha, J. Mestre, and M. Nasre. Popular mixed matchings. *Theoretical Computer Science*, 412(24): 2679–2690, 2011.

[29] G. Laffond, J.-F. Laslier, and M. Le Breton. A theorem on symmetric two-player zero-sum games. *Journal of Economic Theory*, 72(2):426–431, 1997.

[30] M. Le Breton. On the uniqueness of equilibrium in symmetric two-player zero-sum games with integer payoffs. *Économie publique*, 17(2):187–195, 2005.

[31] M. Mahdian. Random popular matchings. In *Proceedings of the 7th ACM Conference on Electronic Commerce (ACM-EC)*, pages 238–242, 2006.

[32] D. Manlove. *Algorithmics of Matching Under Preferences*. World Scientific Publishing Company, 2013.

[33] J. I. Marden. *Analyzing and Modeling Rank Data*. Number 64 in Monographs on Statistics and Applied Probability. Chapman & Hall, 1995.

[34] D. C. McGarvey. A theorem on the construction of voting paradoxes. *Econometrica*, 21(4):608–610, 1953.

[35] P. C. Ordeshook. The spatial analysis of elections and committees: four decades of research. Technical report, California Institute of Technology, 1993.

[36] T. Sönmez and M. U. Ünver. Matching, allocation, and exchange of discrete resources. In J. Benhabib, M. O. Jackson, and A. Bisin, editors, *Handbook of Social Economics*, volume 1, chapter 17, pages 781–852. Elsevier, 2011.