Complexity of Control by Partition of Voters and of Voter Groups in Veto and Other Scoring Protocols

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ABSTRACT

In order to model maliciously resizing election districts (a.k.a. gerrymandering), Bartholdi et al. [2] introduced control by partition of voters, which means that an election chair can influence the outcome of an election to her advantage by partitioning the voters such that two first-round subelections are created whose winners will take part in a final run-off. Control by partition of voter groups, due to Erdélyi et al. [10], refers to the same model with the additional constraint that a partition of voters into groups is given beforehand, which the chair's control action must respect: either all voters of a group take part in one of the two first-round subelections, or none of them does. Maushagen and Rothe [25] recently classified some problems of control by partition of either voters or candidates for veto elections in terms of their computational complexity, leaving some other problems open. We solve all these remaining cases. In addition, we generalize a result of Erdélyi et al. [10] for constructive control by partition of voter groups from plurality elections to all nontrivial pure scoring protocols by showing NP-hardness, and we also obtain the analogous result for its destructive variant.

Keywords

Computational social choice; electoral control; scoring rules; veto

1. INTRODUCTION

During the last decade, voting has been studied intensively in computational social choice. Its applications range from political elections over recommender systems (designed by Ghosh et al. [17] for selecting movies according to a number of criteria) and multi-agent planning (Ephrati and Rosenschein [8]) to webpage ranking algorithms (Dwork et al. [7]). In particular, it has been suggested to use computational complexity as a barrier to tampering with election outcomes via manipulation, control, and bribery. An overview of the large body of related literature is given in the recent book chapters by Conitzer and Walsh [6], Faliszewski and Rothe [15], and Baumeister and Rothe [3].

We here consider control scenarios in elections only, focusing in the first part of the paper on various types of control by partition of voters for veto elections. Bartholdi et al. [2] introduced this control scenario in order to model maliciously resizing voting districts (a.k.a. gerrymandering)—though in a rather simplified setting: Only one district will be divided into two.[1]

In the second part of the paper, we turn to an interesting new model proposed by Erdélyi et al. [10], which can be considered more natural to model gerrymandering:[2] *control by partition of voter groups*, which we will study for nontrivial pure scoring protocols.

1.1 Related Work

Control for plurality, Condorcet, and approval voting has first been studied by Bartholdi et al. [2] in the constructive variant and by Hemaspaandra et al. [19] in the destructive variant. Later on, control results have been obtained by Faliszewski et al. [14] for Copeland, by Erdélyi et al. [9] for Bucklin and fallback voting, by Parkes and Xia [27] for Schulze voting, by Erdélyi et al. [11] and Menton [26] for certain variants of approval and range voting. A dichotomy result for constructive control by adding voters is due to Hemaspaandra et al. [22]. Faliszewski et al. [13] introduced and studied multimode control attacks on elections that combine various standard control scenarios. A study of online voter and candidate control in sequential elections is due to Hemaspaandra et al. [21, 20]. More results on control (and bribery) of elections can be found in recent book chapters [15, 3].

The veto rule has been studied, for example, by Conitzer et al. [5] with respect to coalitional weighted manipulation in terms of classical complexity, by Walsh [28] with respect to the phase transition when manipulating veto empirically (i.e., in terms of typical-case complexity), by Faliszewski et al. [12] with respect to bribery, and by Lin [24] and Chen et al. [4] with respect to control by adding or deleting candidates or voters in terms of both classical and parameterized complexity. Recently, Maushagen and Rothe [25] also obtained results on control by partition of voters or candidates for veto elections in terms of classical complexity, see Table 1 for an overview.

1.2 Our Contribution

In Section 3, we settle all cases of control by partition of either voters or candidates for veto elections that have been left open by Maushagen and Rothe [25]. In particular, this refers to control by partition of voters in the ties-promote model and to certain cases of the unique-winner versus the nonunique-winner model for control by partition of candidates (see Section 2.2 for definitions). Table 1

[1]Note, however, that the NP-hardness lower bounds that we will show in this simple setting will be inherited by more involved models of gerrymandering such as those proposed by Erdélyi et al. [10].
[2]For other natural models of gerrymandering, we refer to the recent work of Bachrach et al. [1] and Lewenberg and Lev [23] who analyze this problem under *geographic* constraints.

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control type	CPC-TE		CPC-TP		CRPC-TE		CRPC-TP		CPV-TE		CPV-TP	
	C	D	C	D	C	D	C	D	C	D	C	D
unique-winner model	R❄	R❄	R❄	**R**◇	R❄	R❄	R❄	**R**♮	V❄	V❄	**R**♠	**R**‡
nonunique-winner model	R❄	R❄	R❄	R❄	R❄	R❄	**R**♣	R❄	V❄	V❄	**R**♡	**R**†

Table 1: Overview of complexity results for control by partition in veto elections. Control types are denoted as is standard [3, 15]; they are defined in Section 2.2. "R" means that veto is resistant to this type of control and "V" means it is vulnerable to this type. Results in boldface are established in this paper: Thm. 1 (marked by ♠), Thm. 2 (♡), Thm. 3 (♣), Thm. 4 (◇), Cor. 1 (†), Cor. 2 (‡), and Cor. 3 (♮). The other results are due to Maushagen and Rothe [25] (marked by ❄). (Note that DCRPC-TE = DCPC-TE in the unique-winner model and DCRPC-TE = DCPC-TE and DCRPC-TP = DCPC-TP in the nonunique-winner model [18, Thm. 8].)

gives an overview of results on the computational complexity of control by partition for veto elections.

In Section 4, we turn to the above-mentioned interesting model of control by partition of voter groups recently introduced by Erdélyi et al. [10]. Our main findings here are that their result for plurality in the constructive case and in the unique-winner model can be extended to the class of all nontrivial pure scoring protocols in both the constructive and the destructive case and in both the unique-winner and the nonunique-winner model.

2. PRELIMINARIES

While we assume readers to be familiar with the basic notions of computational complexity (such as the complexity classes P and NP and the notions of NP-hardness and NP-completeness with respect to the polynomial-time many-one reducibility), we present the needed background on social choice theory in Section 2.1 and define the problems of interest, which capture the scenarios of partition of voters and voter groups and of partition of candidates, in Section 2.2 below.

2.1 Elections and Voting Systems

An election is described by a pair (C,V), where C is a set of candidates and V a list of votes specifying the voters' preferences over the candidates. Specifically, each vote is a linear order (a strict ranking) over C, where the left-most candidate is the most preferred one and the rightmost candidate is the least preferred one. For example, if there is a vote of the form $c\ d\ b\ a$, that means that this voter prefers c to d, d to b, and b to a.

We here consider only scoring protocols (also referred to as scoring rules), an important class of voting systems. A *scoring protocol* for m candidates is specified by a scoring vector $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$, where each σ_i is a nonnegative integer specifying the number of points a candidate receives from being ranked in a vote's ith position. Summing up all points a candidate $c \in C$ receives from all votes in V, we obtain c's score in (C,V), denoted by $score_{(C,V)}(c)$. If the election (C,V) is clear from context, we drop the subscript and simply write $score(c)$. Candidates with the highest score win the election.

Prominent scoring protocols include *plurality* with scoring vector $\sigma = (1,0,\ldots,0)$, *veto* (a.k.a. *antiplurality*) with scoring vector $\sigma = (1,\ldots,1,0)$, *k-veto* (a.k.a. $(m-k)$*-approval* when there are $m \geq k$ candidates) with scoring vector $\sigma = (1,\ldots,1,\underbrace{0,\ldots,0}_{k})$, and the *Borda Count* with scoring vector $\sigma = (m-1, m-2, \ldots, 0)$. We will focus on veto elections in Section 3 and will deal with the class of nontrivial, pure scoring protocols in Section 4. A scoring protocol is said to be *pure* if we can obtain the scoring vector for m candidates from the scoring vector for $m-1$ candidates for each $m \geq 2$ by inserting one additional score value at any position subject to satisfying the constraint $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$. This allows us to let, e.g., veto refer to the class $\{(0),(1,0),(1,1,0),(1,1,1,0),\ldots\}$ of pure scoring protocols for any number of candidates. A scoring protocol $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ is said to be *trivial* if $\sigma_1 = \sigma_2 = \cdots = \sigma_m$, i.e., a *nontrivial* scoring protocal satisfies $\sigma_1 > \sigma_m$. For example, if we were to consider, say, k-veto for k or fewer candidates, we would obtain the trivial scoring vector $(0,\ldots,0)$.

2.2 Partition of Voters and Candidates

The following problem has been defined by Erdélyi et al. [10], for each given voting system $\mathscr{E}$, as a generalization of the original problem of constructive control by partition of voters due to Bartholdi et al. [2].

$\mathscr{E}$-CONSTRUCTIVE-CONTROL-BY-PARTITION-OF-VOTER-GROUPS	
Given:	An election (C,V), a partition of V into any number of groups $G_1, G_2, \ldots, G_k$, and a distinguished candidate $p \in C$.
Question:	Is there a partition of V into V_1 and V_2 such that each group G_i is completely contained either in V_1 or in V_2 and p is the unique winner of the two-stage election where the winners of subelection (C,V_1) surviving the tie-handling rule compete against the winners of subelection (C,V_2) surviving the tie-handling rule (with voting system $\mathscr{E}$ being used in both first-stage subelections and in the final run-off)?

The following two tie-handling models have been proposed by Hemaspaandra et al. [19]: In the *ties-eliminate (TE) model*, winners of the two first-stage subelections (C,V_1) and (C,V_2) proceed to the final run-off only when they are unique, whereas in the *ties-promote (TP) model*, all winners of these two first-stage subelections proceed to the final run-off. Note that in the run-off subelections, the votes from V are restricted to the candidates participating in these run-offs. Depending on which model is applied, we abbreviate the above problem by either $\mathscr{E}$-CCPVG-TE or $\mathscr{E}$-CCPVG-TP. These problems will be considered in Section 4 for pure scoring protocols, while the problems defined below will be studied in Section 3 for veto.

The special case where each group G_i is a singleton yields the original problem $\mathscr{E}$-CONSTRUCTIVE-CONTROL-BY-PARTITION-OF-VOTERS as introduced by Bartholdi et al. [2] (again, depending on the tie-handling model used, abbreviated by either $\mathscr{E}$-CCPV-TE or $\mathscr{E}$-CCPV-TP).

In the destructive variants of the four problems defined above, instead of asking whether distinguished candidate p can be made the sole winner, we ask whether p can be precluded from being a sole winner, and we denote these problems by $\mathscr{E}$-DCPVG-TE, $\mathscr{E}$-DCPVG-TP, $\mathscr{E}$-DCPV-TE, and $\mathscr{E}$-DCPV-TP.

Constructive and destructive control by run-off partition of candidates is defined similarly to CCPV-TE and DCPV-TE, except that the two first-round subelections result from a partition of the candidate set C into C_1 and C_2, where the votes in V are restricted to the corresponding subset of candidates, and the winners of both subelections move forward (according to the tie-handling model used) to the final-run-off. In our notation scheme, this gives the problems CCRPC-TE, CCRPC-TP, DCRPC-TE, and DCRPC-TP.

There is also another variant, dubbed control by partition of candidates (without "run-off"), where the winner(s) of the first first-round subelection will face *all* candidates from the second first-round subelection (and not only its winner(s)), giving rise to four additional problems: CCPC-TE, CCPC-TP, DCPC-TE, and DCPC-TP.[3]

In addition to the *unique-winner model* that was used in the definition of CCPVG (and thus, implicitly, in that of the other problems as well) by requiring p to be a sole winner of the final run-off, we will also consider the *nonunique-winner model* where the question is either whether p can be made a winner (possibly among several winners) by the control action at hand in its constructive variant or whether p can be precluded from being a winner by the control action at hand in its destructive variant. In Section 3 we will consider both winner models. In Section 4, while we will focus on the unique-winner model that Hemaspaandra et al. [19] use in their definition of control by partition of voter groups, we will also briefly discuss the nonunique-winner model.

Let $\mathfrak{C}$ be a control type, such as constructive control by partition of voters in model TP. We say an election system $\mathscr{E}$ (e.g., a scoring protocol σ) is *immune to* $\mathfrak{C}$ if it is impossible for the chair to achieve her control goal via exerting control of type $\mathfrak{C}$ (e.g., for the constructive case in the unique-winner model, to turn the designated candidate c into a unique winner via the control action at hand; or, for the destructive case in the nonunique-winner model, to ensure via the control action that c will not be a winner). Otherwise, we say $\mathscr{E}$ is *susceptible to* $\mathfrak{C}$. For example, it is easy to see that veto is susceptible to every type of control defined above (in both winner models); due to space constraints, we omit giving detailed examples to verify these claims.

If an election system $\mathscr{E}$ is susceptible to some control type $\mathfrak{C}$, one commonly is interested in the computational complexity of the corresponding control problem: $\mathscr{E}$ is said to be *vulnerable to* $\mathfrak{C}$ if the control problem corresponding to $\mathfrak{C}$ can be solved in polynomial time, and $\mathscr{E}$ is said to be *resistant to* $\mathfrak{C}$ if $\mathfrak{C}$ is NP-hard.

3. CONTROL BY PARTITION WHEN TIES PROMOTE IN VETO

We now turn to the cases of control by partition in veto elections that Maushagen and Rothe [25] left open. All these cases concern the ties-promote rule, and we will show that the related problems each are NP-complete, using (essentially[4]) just one construction. This construction will be provided in Section 3.1, we will be concerned with the open questions regarding control by partition of voters in Section 3.2, while we handle the open cases of control by partition of candidates in Section 3.3.

[3]Hemaspaandra et al. [18, Thm. 8 on p. 386] have shown that for all voting systems, DCRPC-TP = DCPC-TP in the nonunique-winner model and DCRPC-TE = DCPC-TE in both the unique-winner and the nonunique-winner model; the difference between these two winner models is explained in the next paragraph.

[4]Specifically, except for the proof of Theorem 4 that uses a different construction, and up to some minor modifications in some of the other proofs in this section.

3.1 The Construction

To prove NP-hardness of the problems in Sections 3.2 and 3.3, we reduce from the problem ONE-IN-THREE-POSITIVE-3SAT, which is a variant of the well-known NP-complete problem ONE-IN-THREE-3SAT where the input is a boolean formula whose clauses have only unnegated variables [16, p. 259]:

ONE-IN-THREE-POSITIVE-3SAT	
Given:	A set X of boolean variables and a set S of clauses over X, each with exactly three unnegated literals.
Question:	Is there a truth assignment to the variables in X satisfying that in each clause of S exactly one literal is set to true?

Given an instance (X,S) of ONE-IN-THREE-POSITIVE-3SAT, where $X = \{x_1,\ldots,x_m\}$ and $S = \{S_1,\ldots,S_n\}$, $n > 1$, with $S_j \subseteq X$ and $|S_j| = 3$ for each j, $1 \leq j \leq n$, construct an election (C,V) with candidate set $C = \{c,w\} \cup X$ and the following votes in V:[5]

- For each i, $1 \leq i \leq m$, there are $3n^2+1+i(2n^2+4n)$ votes of the form $w\ c\ \cdots\ x_i$.
- There are $n-1$ votes of the form $w\ \cdots\ c$.
- For each j, $1 \leq j \leq n$, and for each $x_i \in S_j$, we have $c\ \cdots\ w\ S_j \setminus \{x_i\}$ (representing three votes for each j as noted in Footnote 5).
- For each j, $1 \leq j \leq n$, there are $2n$ votes of the form $w\ \cdots\ c\ S_j$.

In total, we have $m+2$ candidates and

$$\begin{aligned} & 2n^2+4n-1+\sum_{i=1}^{m}(3n^2+1+i(2n^2+4n)) \\ = \quad & 2n^2+4n-1+m((m+4)n^2+2(m+1)n+1) \end{aligned}$$

votes. As we will use the same reduction for several proofs, there will be more voters than actually needed in some proofs. It is obvious that this transformation can be computed in polynomial time.

3.2 Partition of Voters

We now solve the two cases of partition of voters with the ties-promote rule for veto that Maushagen and Rothe [25] left open.

THEOREM 1. *In the unique-winner model, veto-*CCPV-TP *is* NP-*complete.*

Proof. Membership of veto-CCPV-TP in NP is easy to see. For proving NP-hardness, we reduce from ONE-IN-THREE-POSITIVE-3SAT by using the construction in Section 3.1 that transforms a given ONE-IN-THREE-POSITIVE-3SAT instance (X,S) with $X = \{x_1,\ldots,x_m\}$ and $S = \{S_1,\ldots,S_n\}$, where $n > 1$, $S_j \subseteq X$, and $|S_j| = 3$, $1 \leq j \leq n$, into an election (C,V). Let c be the distinguished candidate.

We will now show that (X,S) is in ONE-IN-THREE-POSITIVE-3SAT if and only if (C,V,c) is in veto-CCPV-TP.

[5]The dots in these votes represent all remaining candidates in an arbitrary, fixed order (e.g., in lexicographic order). Furthermore, "$c\ \cdots\ w\ S_j \setminus \{x_i\}$ for each $x_i \in S_j$" represents three votes: For example, if $S_3 = \{x_1,x_4,x_6\}$, it stands for $c\ x_1\ x_2\ x_3\ x_5\ x_7\ \cdots\ x_n\ w\ x_4\ x_6$, $c\ x_2\ x_3\ x_4\ x_5\ x_7\ \cdots\ x_n\ w\ x_1\ x_6$, and $c\ x_2\ x_3\ x_5\ x_6\ x_7\ \cdots\ x_n\ w\ x_1\ x_4$. On the other hand, whenever a subset S_j of the candidates is given in $w\ \cdots\ c\ S_j$, this represents just one vote: For example, if $S_3 = \{x_1,x_4,x_6\}$, it stands for $w\ \cdots\ c\ x_1\ x_4\ x_6$.

From left to right, let (X,S) be a yes-instance of ONE-IN-THREE-POSITIVE-3SAT. Then there is a subset $U \subseteq X$ such that for each clause S_j we have $|U \cap S_j| = 1$. Partition the voters into V_1 and V_2, where V_1 contains exactly one vote of the form $w\ c\ \cdots\ x_i$ for each candidate $x_i \in X \setminus U$ and V_2 contains all remaining votes. In subelection (C,V_1), none of the candidates c, w, and $x_i \in U$ is vetoed and thus they all have the maximum score, whereas all candidates $x_i \in X \setminus U$ score exactly one point fewer. Therefore, candidates c, w, and all $x_i \in U$ proceed to the final run-off from this subelection. In the other subelection, (C,V_2), each candidate except w is vetoed at least once, so w moves forward to the run-off alone. Thus (C',V') is the run-off election with V' being V restricted to the candidates in $C' = \{c,w\} \cup U$.

In the run-off, candidate w is vetoed in each vote in V' that results from a vote of the form $c\ \cdots\ w\ S_j \setminus \{x_i\}$; these are n vetoes in total. However, c is vetoed only by the $n-1$ votes in V' that result from votes of the form $w\ \cdots\ c$.[6] Each candidate x_i, on the other hand, gets so many vetoes (namely, at least $3n^2 + 1 + i(2n^2 + 4n)$) from the votes in V' that result from votes of the form $w\ c\ \cdots\ x_i$ that there is no need to discuss them in detail. It follows that, having the fewest vetoes, c alone scores the most points in the run-off election and thus is the unique winner, i.e., (C,V,c) is in veto-CCPV-TP in the unique-winner model.

From right to left, let (X,S) be a no-instance of ONE-IN-THREE-POSITIVE-3SAT. We have to show that (C,V,c) is a no-instance of veto-CCPV-TP as well. Since w is vetoed by no voter in election (C,V), w will always (regardless of which voter partition is chosen) be a winner of either first-round subelection and will move to the final run-off. Whoever joins w in the final run-off, w will prevail as a run-off winner (in more detail, one can argue along the lines of the proof of Theorem 3 when (X,S) is a no-instance of ONE-IN-THREE-POSITIVE-3SAT). This implies that c cannot win alone, i.e., (C,V,c) is not in veto-CCPV-TP in the unique-winner model. ❑

As corollary to the proof of Theorem 1, we obtain the following by changing the distinguished candidate in this proof from c to w.

COROLLARY 1. *In the nonunique-winner model, veto-*DCPV-TP *is* NP*-complete.*

By a slight modification in the construction from Section 3.1, we can prove the following result.

THEOREM 2. *In the nonunique-winner model, veto-*CCPV-TP *is* NP*-complete.*

Proof. NP membership again is clear. For proving NP-hardness, use the construction from Section 3.1 that maps a given ONE-IN-THREE-POSITIVE-3SAT instance (X,S) to the election (C,V), except that now we have n instead of $n-1$ votes of the form $w\ \cdots\ c$. Again, let c be the distinguished candidate.

For proving the left-to-right direction of the equivalence $(X,S) \in$ ONE-IN-THREE-POSITIVE-3SAT $\iff$ $(C,V,c) \in$ veto-CCPV-TP, we again have that c, w and all candidates from $U \subseteq X$ with $|U \cap S_j| = 1$ for each clause S_j take part in the final run-off. However, c and w now have the same number n of vetoes, so they both are winners.

For proving the right-to-left direction of this equivalence, note that the additional veto for c ensures that w now is the only winner of the run-off election, no matter how the voters are partitioned. ❑

[6]In the $2n^2$ votes in V' that result from votes of the form $w\ \cdots\ c\ S_j$, c always takes the second-to-last position and so only narrowly evades being vetoed.

Again, as a corollary to the proof of Theorem 2, we obtain the following by changing the distinguished candidate in this proof from c to w.

COROLLARY 2. *In the unique-winner model, veto-*DCPV-TP *is* NP*-complete.*

3.3 Partition of Candidates

Maushagen and Rothe [25] stated without proof that veto-CCRPC-TP is NP-complete in the unique-winner model, noting that the omitted proof uses a suitable modification of their proof that veto-CCRPC-TE is NP-complete. Using the construction from Section 3.1, which builds on this latter proof, we provide the missing result for veto-CCRPC-TP in the *nonunique*-winner model.

THEOREM 3. *In the nonunique-winner model, veto-*CCRPC-TP *is* NP*-complete.*

Proof. Membership of veto-CCRPC-TP in NP is obvious. To prove its NP-hardness, we use the construction from Section 3.1 that maps a given ONE-IN-THREE-POSITIVE-3SAT instance (X,S) to the election (C,V) stated there, with distinguished candidate c.

We claim that (X,S) is in ONE-IN-THREE-POSITIVE-3SAT if and only if (C,V,c) is in veto-CCRPC-TP.

From left to right, let (X,S) be a yes-instance of ONE-IN-THREE-POSITIVE-3SAT. Then there is a subset $U \subseteq X$ such that for each clause S_j we have $|U \cap S_j| = 1$. Partition the candidates into $C_1 = \{c,w\} \cup U$ and $C_2 = C \setminus C_1$. As in the proof of Theorem 1,[7] it follows that c alone wins in subelection (C_1,V') with V' being V restricted to the candidates in C_1.

The unique winner of the other subelection, (C_2,V), is the candidate from $X \setminus U$ with smallest subscript, say x_i, because this candidate has at most $3n^2 + 1 + i(2n^2 + 4n) + 3n + 2n^2$ vetoes while every other candidate in C_2 has at least $3n^2 + 1 + (i+1)(2n^2 + 4n)$ vetoes, i.e., at least n vetoes more.

The final run-off, therefore, is $(\{c,x_i\},V'')$ with V'' being V restricted to $\{c,x_i\}$. Candidate c can get at most $2n^2 + n - 1$ vetoes from the second and fourth voter groups, which is smaller than the number of vetoes x_i must get from the first and third voter groups. It follows that c alone wins the run-off. Thus (C,V,c) is a yes-instance of veto-CCRPC-TP (even in the unique-winner model).

From right to left, assuming that (X,S) is a no-instance of ONE-IN-THREE-POSITIVE-3SAT, we will now show that for each partition of the candidates, c is not a winner of the final run-off. To this end, we consider all possible candidate partitions below.

Case 1: Candidates c and w belong to distinct first-round subelections. Since c and w take the first two positions in the first voter group, which outnumbers the other three voter groups by far, c and w win alone their subelections and proceed to the final run-off. In direct comparison between them, w precedes c in all but the third voter groups and thus wins, while c is not a winner.

Case 2: C is partitioned into $C_1 = \{c,w\}$ and $C_2 = X$. As discussed for the run-off in Case 1, w wins the subelection (C_1,V') with V' being V restricted to C_1. Since c does not move forward to the run-off, c does not win.

Case 3: C is partitioned into $C_1 = \{c,w\} \cup U$ with $\emptyset \neq U \subseteq X$ and $C_2 = C \setminus C_1$. We consider three subcases that differ in terms of the

[7]Specifically, in the left-to-right direction of the proof of Theorem 1, recall the argument about the run-off election (C',V') with $C' = \{c,w\} \cup U$ and V' being V restricted to the candidates in C'; note that C' in that proof is C_1 in this proof.

number of clauses S_j for which $S_j \cap U = \emptyset$. In what follows, let $Q = m((m+4)n^2 + 2(m+1)n + 1)$ denote the number of votes in the first voter group.

Case 3.1: There is exactly one clause S_j with $S_j \cap U = \emptyset$. Then we have the following bounds on the scores of the candidates in their subelections:

$$\begin{aligned} score(c) &\leq Q + 3n + 2n^2 - 2n = Q + 2n^2 + n, \\ score(w) &\geq Q + n - 1 + 2n - 2 + 2n^2 = Q + 2n^2 + 3n - 3, \\ score(x_i) &\leq Q - (5n^2 + 4n + 1) + n - 1 + 3n + 2n^2. \end{aligned}$$

Since $n > 1$, w scores more points than c in the first subelection, so c cannot move forward to the run-off and thus cannot win. (w also defeats each other candidate in the first subelection, moves on to the final run-off to face the candidate who won the other subelection—namely, the candidate from X with smallest subscript in this other subelection—whom w defeats as well.)

Case 3.2: There are at least two clauses S_j with $S_j \cap U = \emptyset$. Then we have the following bounds on the scores of the candidates in their subelections:

$$\begin{aligned} score(c) &\leq Q + 3n + 2n^2 - 4n = Q + 2n^2 - n, \\ score(w) &\geq Q + n - 1 + 2n^2 = Q + 2n^2 + n - 1, \\ score(x_i) &\leq Q - (5n^2 + 4n + 1) + n - 1 + 3n + 2n^2. \end{aligned}$$

It is easy to see that w scores at least $2n - 1$ points more than c. Again, c doesn't reach the final run-off and cannot win. (Furthermore, w scores at least $5n^2 - n - 1$ points more than every other candidate in the first subelection and then wins alone the run-off.)

Case 3.3: $S_j \cap U \neq \emptyset$ for each clause S_j. Since we have started from a no-instance (X,S) of ONE-IN-THREE-POSITIVE-3SAT, there is at least one clause S_j with $|S_j \cap U| \geq 2$. Then we have the following bounds on the scores of the candidates in their subelections:

$$\begin{aligned} score(c) &\leq Q + 3n + 2n^2, \\ score(w) &\geq Q + n - 1 + 2n + 1 + 2n^2 = Q + 3n + 2n^2, \\ score(x_i) &\leq Q - (5n^2 + 4n + 1) + n - 1 + 3n + 2n^2. \end{aligned}$$

Note that we have $score(c) = score(w)$ in the first subelection if and only if there is exactly one clause S_j with $|S_j \cap U| \geq 2$. In this case, both candidates win their subelection and move on to the final run-off, while in the other subelection, (C_2, V), the candidate from X with smallest subscript in this subelection wins alone, say x_i, so c, w, and x_i face each other in the final run-off. Since (X,S) is a no-instance of ONE-IN-THREE-POSITIVE-3SAT, we have $S_j \cap \{x_i\} = \emptyset$ for at least one clause S_j, which gives the same election outcome as in Cases 3.1 and 3.2: w is the only winner and, therefore, c cannot win.

If $|S_j \cap U| \geq 2$ holds true for more than one clause S_j, however, w scores more points than c in the first subelection. Hence, c does not win this subelection and does not proceed to the final run-off.

This completes the case distinction. We have shown that, regardless of how the candidates are partitioned, c cannot win. It follows that (C,V,c) is a no-instance of veto-CCRPC-TP (even in the nonunique-winner model). ❑

By changing the distinguished candidate in this proof from c to w, we obtain the following corollary.

COROLLARY 3. *In the unique-winner model, veto-*DCRPC-TP *is* NP-*complete.*

For veto-DCPC-TP in the unique-winner model, we need to use a different construction when we reduce again from ONE-IN-THREE-POSITIVE-3SAT to show its NP-hardness. The construction presented by Maushagen and Rothe [25, Thm. 5] for showing NP-completeness of veto-DCRPC-TE will do; however, we need to argue here with a different partition of the candidates.

THEOREM 4. *In the unique-winner model, veto-*DCPC-TP *is* NP-*complete.*

Proof. It is obvious that veto-DCPC-TP is in NP. To prove NP-hardness, we again reduce from ONE-IN-THREE-POSITIVE-3SAT. Let (X,S) be a given ONE-IN-THREE-POSITIVE-3SAT instance, where $X = \{x_1, \ldots, x_m\}$ and $S = \{S_1, \ldots, S_n\}$, $n > 1$, with $S_j \subseteq X$ and $|S_j| = 3$ for each j, $1 \leq j \leq n$. Construct an election (C,V) with $m+2$ candidates, $C = \{c, w\} \cup X$, and $m(3n+1) + n(2n+6)$ votes as follows:

- For each i, $1 \leq i \leq m$, there are $3n+1$ votes of the form $w\ c\ \cdots\ x_i$.
- There are n votes of the form $w\ \cdots\ c$.
- For each j, $1 \leq j \leq n$, and for each $x_i \in S_j$, we have $c\ \cdots\ w\ S_j \setminus \{x_i\}$ (representing three votes for each j as noted in Footnote 5).
- For each j, $1 \leq j \leq n$, there are $2n+2$ votes of the form $w\ \cdots\ c\ S_j$.

Let w be the distinguished candidate. In election (C,V), c is vetoed by n voters, w by no voter at all, and each candidate x_i by at least $3n+1$ voters. Therefore, w is the only veto winner of this election.

It remains to show that (X,S) is a yes-instance of ONE-IN-THREE-POSITIVE-3SAT if and only if (C,V,w) is a yes-instance of veto-DCPC-TP.

From left to right, assume that (X,S) is in ONE-IN-THREE-POSITIVE-3SAT. Then there exists a subset $U \subseteq X$ such that $|S_j \cap U| = 1$ for each clause S_j. Partition the candidates into $C_1 = \{w\} \cup (X \setminus U)$ and $C_2 = C \setminus C_1 = \{c\} \cup U$. In subelection (C_1, V), w is again the only veto winner, with no more than $3n$ vetoes, whereas every other candidate in C_1 has at least $3n+1$ vetoes. Thus w moves on to face all candidates from C_2 in the final run-off. In this run-off, we have the following scores:

$$\begin{aligned} score(c) &= (3n+1)m + (2n+2)n + 3n, \\ score(w) &= (3n+1)m + (2n+2)n + n + 2n, \\ score(x_i) &\leq (3n+1)(m-1) + (2n+2) + 4n. \end{aligned}$$

Since c and w tie for winning the run-off, w is not a unique winner. It follows that (C,V,w) is a yes-instance of veto-DCPC-TP in the unique-winner model.

From right to left, let (X,S) be a no-instance of ONE-IN-THREE-POSITIVE-3SAT. We will show that in every partition of the candidates only w emerges victorious. To this end, it is enough to show that w is the only winner of every subelection. In what follows, we consider every possible subelection (C', V'), where $C' \subseteq C$ is a subset of the candidates and V' is V restricted to the candidates in C'. Let $U \subseteq X$ denote an arbitrary nonempty subset of X.

Case 1: $C' = \{c, w\}$. In this subelection (C', V'), w is the only winner, since w precedes c in all but $3n$ votes.

Case 2: $C' = \{w\} \cup U$. In this subelection (C', V'), w is the only winner, since w can be vetoed only by the $3n$ voters from the third voter group, whereas every x_i is vetoed by at least $3n+1$ voters in the first voter group alone.

Case 3: $C' = \{c,w\} \cup U$. Suppose there are k clauses S_j such that $S_j \cap U = \emptyset$. Then the candidates in C' have the following scores in subelection (C',V'):

$$\begin{aligned} score(c) &= (3n+1)m + (2n+2)(n-k) + 3n, \\ score(w) &\geq (3n+1)m + (2n+2)n + n + 2(n-k), \\ score(x_i) &\leq (3n+1)(m-1) + (2n+2)n + 4n. \end{aligned}$$

By the argument from Case 2, we know that $score(w) > score(x_i)$ for all $x_i \in U$. To show that $score(w) > score(c)$, we note that:

$$\begin{aligned} score(w) &> score(c) \\ (2n+2)n + n + 2(n-k) &> (2n+2)(n-k) + 3n \\ 2n(1+k) &> 2n, \end{aligned}$$

which is always true for $k > 0$. For $k = 0$, however, we can use a different argument to show that w defeats c. Note that $k = 0$ means that there is no clause S_j with $S_j \cap U = \emptyset$. But since (X,S) is a no-instance of ONE-IN-THREE-POSITIVE-3SAT, there must exist at least one clause S_j such that $|S_j \cap U| \geq 2$. But then c and w have the following scores in subelection (C',V'):

$$\begin{aligned} score(c) &= (3n+1)m + (2n+2)n + 3n, \\ score(w) &\geq (3n+1)m + (2n+2)n + 3n + 1. \end{aligned}$$

Thus, scoring at least one point more than c in this case, w wins (C',V') alone. It follows that (C,V,w) is a no-instance of veto-DCPC-TP in the unique-winner model. ❑

4. CONTROL BY PARTITION OF VOTER GROUPS FOR SCORING PROTOCOLS

We now turn to control by partition of voter groups, a very interesting model recently introduced by Erdélyi et al. [10]. Since this model generalizes the original model of control by partition of voters [2], it is clear that NP-hardness lower bounds in the latter model are inherited by the former model. For example, since it is known that plurality is resistant to constructive and destructive control by partition of voters with the ties-promote rule [19], the same immediately follows for constructive and destructive control by partition of voter groups. On the other hand, when we use the ties-eliminate rule then plurality is vulnerable to this problem [19], i.e., both plurality-CCPV-TE and plurality-DCPV-TE are in P. By contrast, Erdélyi et al. [10] proved that the voter group variants of the problem plurality-CCPVG-TE is NP-complete in the unique-winner model. In this section, we will extend their result to the class of nontrivial pure scoring protocols in both the constructive and the destructive variant and to both the unique-winner and the nonunique-winner model (although we will focus here on the former due to space limitations).

In the proof of Theorem 5 below, we reduce from the following problem that is well known to be NP-complete [16]:

EXACT-COVER-BY-3-SETS (X3C)	
Given:	A set $B = \{b_1, \ldots, b_{3m}\}$ and a family of subsets of B, $\mathscr{S} = \{S_1, \ldots, S_n\}$, each S_i having three elements.
Question:	Does there exist an exact cover of B, i.e., a subfamily $\mathscr{S}' \subseteq \mathscr{S}$ with $|\mathscr{S}'| = m$ such that each element $b_j \in B$ occurs in exactly one subset $S_i \in \mathscr{S}'$?

THEOREM 5. *For each nontrivial pure scoring protocol* σ, σ-CCPVG-TE *is* NP-*complete in the unique-winner model.*

Proof. For any nontrivial scoring protocol σ, it is obvious that σ-CCPVG-TE is in NP. To show NP-hardness, we reduce from X3C. Let k be the minimal number with the property that $\sigma_1 > \sigma_k$; we will allow only instances of X3C satisfying $3m > k$. Our reduction is based on the reduction of Erdélyi et al. [10] for plurality-CCPVG-TE. Let $(B,\mathscr{S})$ be an instance of X3C with $B = \{b_1, \ldots, b_{3m}\}$, $\mathscr{S} = \{S_1, \ldots, S_n\}$, and $S_i = \{b_{i,1}, b_{i,2}, b_{i,3}\} \subseteq B$ for each i, $1 \leq i \leq n$, where we may assume that $m > 1$ and $n > m+1$ because this restriction of X3C is still NP-complete. We now construct an election (C,V) with candidates $C = B \cup \{c,d,p\}$, where p is the distinguished candidate the chair wants to make a unique winner.

We now describe V along with its partition into voter groups, using the following notation. If a vote over C is divided into, say, two blocks $v = v_1\ v_2$, where v_i is a ranking of the candidates in C_i, $i \in \{1,2\}$, with $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = C$, then we let $v_1\ [v_2]$ denote the list of $|C_2|$ votes that each start with v_1, followed by one of the cyclic shifts of v_2. For example, if $v = v_1\ v_2 = p\ c\ d\ b_1\ b_2 \cdots b_6$ for $v_1 = p\ c\ d$ and $v_2 = b_1\ b_2 \cdots b_6$, then $v_1\ [v_2]$ stands for the six votes

$$\begin{array}{lll} p\ c\ d\ b_1\ b_2 \cdots b_6, & p\ c\ d\ b_2\ b_3 \cdots b_1, & p\ c\ d\ b_3\ b_4 \cdots b_2, \\ p\ c\ d\ b_4\ b_5 \cdots b_3, & p\ c\ d\ b_5\ b_6 \cdots b_4, & p\ c\ d\ b_6\ b_1 \cdots b_5. \end{array}$$

We define the following $n+3$ voter groups:

- For each i, $1 \leq i \leq n$, there is one group G_i with $7(3m+2)$ votes. Specifically, G_i contains
  - two copies of each of the $3m+2$ votes $p\ [b_1\ b_2 \cdots b_{3m}\ c\ d]$,
  - the $3m+2$ votes $b_{i,1}\ [B \setminus \{b_{i,1}\}\ c\ d\ p]$,
  - the $3m+2$ votes $b_{i,2}\ [B \setminus \{b_{i,2}\}\ c\ d\ p]$,
  - the $3m+2$ votes $b_{i,3}\ [B \setminus \{b_{i,3}\}\ c\ d\ p]$, and
  - two copies of each of the $3m+2$ votes $d\ [b_1\ b_2 \cdots b_{3m}\ c\ p]$.

  The above construction with cyclic shifts has the effect that p and d score

  $$2(3m+2) \cdot \sigma_1 + 5 \sum_{i=2}^{3m+3} \sigma_i$$

  points from the votes in G_i, each member of S_i scores

  $$(3m+2) \cdot \sigma_1 + 6 \sum_{i=2}^{3m+3} \sigma_i$$

  points, and each other candidate scores fewer points from the votes in G_i, namely only

  $$7 \sum_{i=2}^{3m+3} \sigma_i.$$

  Since we consider only nontrivial scoring protocols $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{3m+3})$, we know that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{3m+3}$ and $\sigma_1 > \sigma_{3m+3}$. Thus, setting $X = (3m+2) \cdot \sigma_1$ and $Y = \sum_{i=2}^{3m+3} \sigma_i$, we can conclude that $X > Y$.

- There is one group G_B with $(3m+2)(6mn - 3n + 2m)$ votes: For each j, $1 \leq j \leq 3m$, letting $\ell_j = |\{S_i \in \mathscr{S} \mid b_j \in S_i\}|$, G_B contains
  - $2n - \ell_j$ copies of the $3m+2$ votes $b_j\ [B \setminus \{b_j\}\ c\ d\ p]$ and

– $2m$ copies of the $3m+2$ votes $p\ [b_1\ b_2\ \cdots\ b_{3m}\ c\ d]$.

- There is one group G_c with $(3m+2)(2(n+m)+1)$ votes, namely $2(n+m)+1$ copies of the $3m+2$ votes $c\ [b_1\ \cdots\ b_{3m}\ d\ p]$.
- There is one group G_d with $(3m+2)(2n+1)$ votes, namely $2n+1$ copies of the $3m+2$ votes $d\ [b_1\ \cdots\ b_{3m}\ c\ p]$.

Clearly, this σ-CCPVG-TE instance with the $3m+3$ candidates and $(3m+2)(6mn+8n+4m+2)$ voters can be constructed in polynomial time. Note that in (C,V), with X and Y as defined above, each $b_j \in B$ scores $2nX+(6mn+6n+4m+2)Y$ points, c scores $(2(n+m)+1)X+(6mn+6n+2m+1)Y$ points, d scores $(4n+1)X+(6mn+4n+4m+1)Y$ points, and p scores $2(n+m)X+(6mn+6n+2m+2)Y$ points, so d is the unique σ-winner in this election because $X > Y$.

We will now show that there exists an exact cover for $\mathscr{S}$ if and only if p can be made a unique σ-winner of the election via the CCPVG-TE control action.

From left to right, suppose there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ of B. Partition V into V_1 and V_2 as follows: V_2 consists of the m groups G_i corresponding to $\mathscr{S}'$ and of the groups G_c and G_d, while $V_1 = V \setminus V_2$ consists of the remaining $n-m$ groups G_i corresponding to the complement of $\mathscr{S}'$ and of group G_B. It is easy to see that p is the unique σ-winner of the first subelection, (C,V_1): p scores $2(n-m)X+(6mn+2n-5m)Y+2mX = 2nX+5(n-m)Y$ points, c scores $0X+(6mn+4n-5m)Y$ points, d scores $2(n-m)X+(6mn+2n-3m)Y$ points, and each $b_j \in B$ scores $(2n-1)X+(6mn+2n-5m+1)Y$ points. In subelection (C,V_2), however, c and d tie for winning, both with $(2(n+m)+1)X+(7m+2n+1)Y$ points, while p scores only $2mX+(5m+2(n+m)+1+2n+1)Y$ points and each $b_j \in B$ scores $X+(9m+4n+1)Y$ points. Therefore, c and d eliminate each other according to the TE rule, so p is the only one in the final run-off and wins alone.

From right to left, suppose that p can be made the sole σ-winner respectively a σ-winner of the election according to the CCPVG-TE control action. Since p participates in the final run-off, the TE rule implies that p must be a unique winner of one of the two first-round subelections, say of (C,V_1). This enforces voter groups G_c and G_d to be contained in V_2, since—no matter which other groups are added to V_1—c and d are ranked higher than p more often and so would spoil p's victory in (C,V_1). However, since both c and d would defeat p in the final run-off as well, neither of them can participate in it; but since they have a higher score than any other candidate, they have to eliminate each other via the TE rule in (C,V_2). To make this happen, exactly m of the voter groups G_i must participate in subelection (C,V_2) because if fewer than m voters groups G_i were in V_2 then c would win in (C,V_2), and if more than m voters groups G_i were in V_2 then d would win in (C,V_2).

With only the $n-m$ remaining voter groups G_i participating in (C,V_1) (regardless of which ones), p and d would tie for winning in (C,V_1). However, since p emerges victorious alone in this subelection, voter group G_B must be contained in it as well. Now, p scores $2(n-m)X+5(n-m)Y+(6mn-3n)Y+2mX = 2nX+(2n-5m+6mn)Y$ points in (C,V_1) and the only way to prevent that some $b_j \in B$ scores exactly as many points in (C,V_1) (and so would spoil p being the unique winner in this subelection) is to have the m groups G_i in (C,V_2) correspond to an exact cover of B. ❑

We remark that σ-CCPVG-TE remains NP-complete in the nonunique-winner model, via essentially a similar reduction, even though some effort must be made to discuss some technical difficulties arising there.

For the destructive variant of the previous problem, no result is known even for a scoring protocol as simple as plurality. We provide the following result that shows NP-completeness of DCPVG-TE for all nontrivial pure scoring protocols, focusing on the unique-winner model but again remarking that this problem remains NP-complete in the nonunique-winner model, via essentially the same proof. We will use a reduction from the following problem that is well known to be NP-complete [16, p. 259]:

PARTITION	
Given:	A set $A = \{1,\dots,n\}$ and a list $s = (s_1,\dots,s_n)$ of positive integers such that $\sum_{i=1}^{n} s_i$ is even.
Question:	Does there exist a subset $A' \subset A$ such that $\sum_{i\in A'} s_i = \sum_{i\in A\setminus A'} s_i$?

THEOREM 6. *For each nontrivial pure scoring protocol σ with at least five candidates, σ-DCPVG-TE is NP-complete in the unique-winner model.*

Proof. Membership of σ-DCPVG-TE in NP is obvious. We show its NP-hardness by a reduction from PARTITION. Let $m \geq 5$ be the number of candidates for $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$. Since σ is assumed to be nontrivial, we know that $\sigma_1 > \sigma_m$. Given an instance (A,s) of PARTITION, where $A = \{1,\dots,n\}$ and $s = (s_1,\dots,s_n)$ is a list of positive integers such that $\sum_{i=1}^{n} s_i = 2K$ for some positive integer K, we construct the election (C,V) with candidates $C = \{c_1,c_2,d_1,d_2,p\} \cup Y$ with $Y = \bigcup_{j=6}^{m} y_j$, where p is the distinguished candidate, and with the following voter groups:

- For each i, $1 \leq i \leq n$, there is a group G_i with $(m-1)s_i$ votes: s_i copies of each of the $m-1$ votes $p[c_1\ c_2\ d_1\ d_2\ Y]$.
- There is a group G_c with $2(m-1)K$ votes:
  - K copies of each of the $m-1$ votes $c_1[c_2\ d_1\ d_2\ p\ Y]$, and
  - K copies of each of the $m-1$ votes $c_2[c_1\ d_1\ d_2\ p\ Y]$.
- There is a group G_d with $2(m-1)K$ votes:
  - K copies of each of the $m-1$ votes $d_1[c_1\ c_2\ d_2\ p\ Y]$, and
  - K copies of each of the $m-1$ votes $d_2[c_1\ c_2\ d_1\ p\ Y]$.

Let $Q = \sum_{i=2}^{m} \sigma_i$. In election (C,V), we have the following scores:

$$\begin{aligned} score(c_1) &= (m-1)K\sigma_1 + 5KQ, \\ score(c_2) &= (m-1)K\sigma_1 + 5KQ, \\ score(d_1) &= (m-1)K\sigma_1 + 5KQ, \\ score(d_2) &= (m-1)K\sigma_1 + 5KQ, \\ score(p) &= 2(m-1)K\sigma_1 + 4KQ, \\ score(y_j) &= 6KQ. \end{aligned}$$

Candidate p has the highest score and wins alone since

$$(2(m-1)K - (m-1)K)\sigma_1 + (4K-5K)Q > 0,$$

which is equivalent to

$$(m-1)K\sigma_1 > K(\sigma_2 + \sigma_3 + \cdots + \sigma_m),$$

which in turn is true because we have $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ and $\sigma_1 > \sigma_m$.

We claim that (A,s) is a yes-instance of PARTITION if and only if (C,V,p) is a yes-instance of σ-DCPVG-TE.

From left to right, suppose (A,s) is a yes-instance of PARTITION. Then there is a subset $A' \subseteq A$ such that $\sum_{i\in A'} s_i = \sum_{i\in A\setminus A'} s_i$.

Now, partitioning V into $V_1 = \bigcup_{i\in A'} G_i \cup G_c$ and $V_2 = V \setminus V_1 = \bigcup_{i\in A\setminus A'} G_i \cup G_d$, we will show that p does not win. In the first subelection, (C,V_1), we have the following scores:

$$\begin{aligned}
score(c_1) &= (m-1)K\sigma_1 + 2KQ,\\
score(c_2) &= (m-1)K\sigma_1 + 2KQ,\\
score(d_1) &= 3KQ,\\
score(d_2) &= 3KQ,\\
score(p) &= (m-1)K\sigma_1 + 2KQ,\\
score(y_j) &= 3KQ.
\end{aligned}$$

In the second subelection, (C,V_2), we have the following scores:

$$\begin{aligned}
score(c_1) &= 3KQ,\\
score(c_2) &= 3KQ,\\
score(d_1) &= (m-1)K\sigma_1 + 2KQ,\\
score(d_2) &= (m-1)K\sigma_1 + 2KQ,\\
score(p) &= (m-1)K\sigma_1 + 2KQ,\\
score(y_j) &= 3KQ.
\end{aligned}$$

Since no unique winner emerges in any of the two subelections (p tieing with c_1 and c_2 in (C,V_1) and with d_1 and d_2 in (C,V_2)), no candidate proceeds to the final run-off. In particular, p does not win the run-off. Hence, (C,V,p) is a yes-instance of σ-DCPVG-TE.

From right to left, let (A,s) be a no-instance of PARTITION. We will show that the constructed (C,V,p) is a no-instance of σ-DCPVG-TE as well, i.e., we will show that p is a unique winner for every possible partition of voter groups. To this end, we consider the two cases where G_c and G_d belong or do not belong to the same first-round subelection.

Case 1: G_c and G_d are in the same first-round subelection, say in (C,V_1), i.e., $G_c \subseteq V_1$ and $G_d \subseteq V_1$. If V_2 is empty, $(C,V_1) = (C,V)$, so p is the only winner of this subelection and thus—being the only candidate participating in the final run-off—the unique overall winner. If V_2 is not empty, V_2 contains only some of the voter groups G_i. In each group G_i, p scores the most points and, therefore, also the most points in (C,V_2), and thus proceeds to the final run-off. In (C,V_1), either p wins alone or c_1, c_2, d_1 and d_2 tie for winning in this subelection. It follows that p again is the only participant of the final run-off and thus the sole overall winner.

Case 2: G_c and G_d are in different first-round subelections. Without loss of generality, we may assume that $G_c \subseteq V_1$ and $G_d \subseteq V_2$. Let $A_1 \subseteq A$ be any subset of A and $A_2 = A \setminus A_1$ its complement. Then we have either $\sum_{i\in A_1} s_i > \sum_{i\in A_2} s_i$ or $\sum_{i\in A_1} s_i < \sum_{i\in A_2} s_i$. We may assume the former inequality to hold (otherwise, we just rename A_1 and A_2 accordingly). It follows that $\sum_{i\in A_1} s_i > K$ and $\sum_{i\in A_2} s_i < K$. Let $\hat{K} = \sum_{i\in A_1} s_i$. Due to symmetry it is enough to consider the case that all voter groups G_i with $i \in A_1$ are in V_1 and all voter groups G_i with $i \in A_2$ are in V_2. Then we have the following scores in subelection (C,V_1):

$$\begin{aligned}
score(c_1) &= (m-1)K\sigma_1 + (\hat{K} + K)Q,\\
score(c_2) &= (m-1)K\sigma_1 + (\hat{K} + K)Q,\\
score(d_1) &= (\hat{K} + 2K)Q,\\
score(d_2) &= (\hat{K} + 2K)Q,\\
score(p) &= (m-1)\hat{K}\sigma_1 + 2K Q,\\
score(y_j) &= (\hat{K} + K)Q.
\end{aligned}$$

In subelection (C,V_1), p is the unique winner since

$$((m-1)\hat{K} - (m-1)K)\sigma_1 + (K-\hat{K})(\sigma_2 + \sigma_3 + \ldots + \sigma_m) > 0,$$

which is equivalent to

$$(\hat{K} - K)(m-1)\sigma_1 > (\hat{K} - K)(\sigma_2 + \sigma_3 + \ldots + \sigma_m),$$

which in turn is true because we have $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ and $\sigma_1 > \sigma_m$ because the scoring vector is nontrivial. This also implies that

$$\begin{aligned}
(m-1)K\sigma_1 - K\sigma_2 - K\sigma_3 - \cdots - K\sigma_m &> 0\\
(m-1)K\sigma_1 &> K(\sigma_2 + \sigma_3 + \cdots + \sigma_m).
\end{aligned}$$

In the second subelection, (C,V_2), we have the following scores:

$$\begin{aligned}
score(c_1) &= (4K - \hat{K})Q,\\
score(c_2) &= (4K - \hat{K})Q,\\
score(d_1) &= (m-1) \quad K \quad \sigma_1 + (3K - \hat{K})Q,\\
score(d_2) &= (m-1) \quad K \quad \sigma_1 + (3K - \hat{K})Q,\\
score(p) &= (m-1)(2K - \hat{K})\sigma_1 + 2KQ,\\
score(y_j) &= (3K - \hat{K})Q.
\end{aligned}$$

Since d_1 is tieing with d_2, no further candidate can be in the final round and thus p is the only candidate and winner in the run-off. Hence, (C,V,p) is a no-instance of σ-DCPVG-TE. ❑

5. CONCLUSIONS

We have studied control by partition of either voters or candidates for veto elections in terms of their computational complexity, settling all cases (for the standard control scenarios) that have been left open by Maushagen and Rothe [25]. As they suspected, we have now confirmed that (other than for constructive coalitional weighted manipulation where veto and plurality behave quite differently [5]), veto and plurality here display the same behavior: The problems of control by partition are exactly the same for them, namely, control by partition of candidates is hard, whereas control by partition of voters is easy in the ties-eliminate model [25] and is hard in the ties-promote model (Theorems 1 and 2), even in the destructive case (Corollaries 1 and 2).

We have also considered the interesting, more natural model of control by partition of voter groups due to Erdélyi et al. [10]. In particular, we have generalized their result for plurality in the constructive case and in the unique-winner model to the class of all nontrivial pure scoring protocols in both the constructive and the destructive case and in both the unique-winner and the nonunique-winner model.

Interesting questions remain open, for example, investigating control by partition of voter groups for other voting systems and obtaining results not only in terms of classical complexity but also in terms of parameterized complexity or, even more demanding, in terms of typical-case complexity.

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