

	Existence guaranteed?			Computational complexity of testing		Computational complexity of computing		
	W	W+EX	L	W	L	W	W+EX	L
BJR	yes	yes	yes	P (Prop. 3.10)	P (Prop. 3.10)	P ($W = \emptyset$)	P (Prop. 3.9)	P (Prop. 3.9)
Strong-BJR	yes	no	no	P (Prop. 3.10)	P (Prop. 3.10)	P ($W = \emptyset$)	NP-h (Prop. 3.4)	NP-h (Prop. 3.4)
Local-BPJR	yes	yes	yes	Co-NP-c	Co-NP-c	P ($W = \emptyset$)	P (Prop. 3.13; GPseq)	P (Prop. 3.13; GPseq)
BPJR	yes	yes	yes	Co-NP-c	Co-NP-c	P ($W = \emptyset$)	Open	NP-h (Prop. 3.8)
Strong-BPJR	yes	no	no	Co-NP-c	Co-NP-c	P ($W = \emptyset$)	NP-h (Prop. 3.4)	NP-h (Prop. 3.4)

Table 1: Summary of our results. We consider two versions of proportionality axioms, the standard one respect to L (the given budget limit), and the other variant with respect to W (the budget actually used) that is discussed in Section 3.4. We also consider some axioms when additionally imposing exhaustiveness (+EX). For each proportionality axiom \mathcal{PA} we state (1) whether a budget satisfying \mathcal{PA} is guaranteed to exist; (2) what is the computational complexity of testing whether a specific budget satisfies \mathcal{PA} ; and (3) what is the computational complexity of computing a budget satisfying \mathcal{PA} . (NP-h denotes NP-hardness, Co-NP-c denotes Co-NP-completeness, EX denotes exhaustiveness, and GPseq denotes that our generalization of Phragmen’s sequential rule satisfies the axiom. The co-NP-c results follow from the fact that testing PJR is co-NP-complete in approval-based multi-winner voting [4].)

In a similar manner to the definition of BJR-L, below we introduce the axiom of BPJR-L. Here, however, our requirement allows for greater flexibility in the additional requirement of cohesive voters supporting a ‘sufficiently cheap’ item. In particular, we allow for bundles of items and let the ‘sufficiently cheap’ criteria grow proportionally with the size of the voters group.

Definition 3.6 (BPJR-L). A budget W satisfies BPJR-L if for all $\ell \in [1, L]$ there exists no set of voters $V' \subseteq V$ with $|V'| \geq \ell n/L$ such that $w(\cap_{i \in V'} A_i) \geq \ell$ and $w((\cup_{i \in V'} A_i) \cap W) < \max \{w(C') : C' \subseteq \cap_{i \in V'} A_i \text{ and } w(C') \leq |V'|L/n\}$.

The idea of the definition above is that every group of (at least) $\ell \cdot n/L$ voters should control (at least) ℓ units of the budget. Technically, however, for this to be satisfied we need a bundle of items with ‘low enough’ cost $\leq \ell$. We then require that at least the cost of the most expensive bundle which is also ‘low enough’ is used in representing this group of $\ell \cdot n/L$ voters. Note that indeed BPJR-L generalizes BJR-L; specifically, BPJR-L boils down to BJR-L when $\ell = 1$.

It turns out that exhaustive BPJR-L budgets always exist. (While the procedure described in the next proof is super-polynomial, afterwards we show that this is not a coincidence, by showing the problem of computing a budget satisfying BPJR-L is NP-hard.)

PROPOSITION 3.7. *For any given budget limit L there always exists a feasible budget W which is exhaustive and satisfies BPJR-L.*

PROOF. We describe an algorithm which produces feasible, exhaustive budgets, which satisfy BPJR-L. The algorithm proceeds as follows. We iterate over ℓ' , where initially $\ell' = L$ and ℓ' can only decrease, until $\ell' = 1$. Let $A' = V$ (initially, considering all voters; during the course of the algorithm, we will “discard” voters as we take care for them), and let $W = \emptyset$ (initially, no item is budgeted).

The overall idea is greedy, where we decrease the total cost ℓ' of the sets we greedily consider. Formally, in each iteration, first check whether $w(W) + \ell' \leq L$; if this is not the case, then decrease ℓ' by one and continue to the next iteration. Otherwise, let

$$C^* := \{C' \subseteq C : w(C) = \ell'\}.$$

If $C^* = \emptyset$, then decrease ℓ' by one and continue to the next iteration. Otherwise, for each $C' \in C^*$, let

$$A(C') = \{i \in A' : C' \subseteq A_i\}$$

denote the voters from A' which approve all items of C' . Select any $C' \in C^*$ with maximal size of $|A(C')|$ and check whether $|A(C')| \geq \ell' \cdot n/L$; If this is not the case, then decrease ℓ' by one and continue to the next iteration. Otherwise, set $W \mapsto W \cup C'$ and redefine $A' \mapsto A' \setminus A(C')$. Leave ℓ' as is and continue to the next iteration. The algorithm halts whenever $\ell' = 0$, in which case, if W is non-exhaustive, then we arbitrarily add items to it until it does, while keeping it feasible.

Next we prove the algorithm’s correctness. Let W be the output from the above algorithm and for the purpose of a contradiction suppose that BPJR-L is not satisfied. That is, there exists an $\ell \in [1, L]$ and a set of voters V' with $|V'| \geq \ell n/L$ and $w(\cap_{i \in V'} A_i) \geq \ell$ such that there exists $C' \subseteq \cap_{i \in V'} A_i$ with $w(C') \leq \ell$ such that

$$w((\cup_{i \in V'} A_i) \cap W) < w(C').$$

Furthermore, let ℓ be the smallest number such that the above holds.

Now suppose that $w(W) = L$. Then every unit spent in the budget represents at least n/L additional voters, and so the total number of adequately represented voters is at least

$$w(W) \cdot n/L = n,$$

thus, there cannot be a group of $|V'|$ voters which are not represented by at least $|V'|/(n/L) \geq \ell$ units of budget; thus, a contradiction.

Now suppose that $w(W) < L$. Then, every unit spent in the budget represents at least n/L additional voters, and so the total number of adequately represented voters is at least

$$w(W) \cdot n/L < n.$$

Denote this set of voters by $R \subseteq V$ and note that any subset $V'' \subseteq R$ is adequately represented in the budget with at least $|V''|/(n/L)$ units of expenditure. Noting that there exists a group V' of size $|V'| \geq \ell n/L$ who are inadequately represented – recalling that ℓ is the smallest number such that this holds – it must be that the group was not represented due to the budget limit and thus

$$w(W) + \ell > L.$$

Now if there exists a (smallest) group V' of inadequately represented voters it must be that

$$V' \subseteq V \setminus R$$

and so $|V'| \leq n - |R| \leq n - \frac{w(W)n}{L} = \frac{n}{L}(L - w(W)) < \ell \frac{n}{L} \leq |V'|$ (the strict inequality follows since $w(W) + \ell > L$), thus no such set V' can exist and we have derived a contradiction. We conclude that BPJR-L must be satisfied. \square

The next result explains why the algorithm presented in the proof above is not polynomial-time. Notice that there is some gap in our complexity analysis, as the next proposition only shows weak NP-hardness.

PROPOSITION 3.8. *Computing a BPJR-L budget is NP-hard.*

PROOF. We describe a Turing reduction from the NP-hard problem Partition [13] which, given integers x_1, \dots, x_m whose sum is $2B$, asks to find a subset of them which sums to B .

Given an instance of Partition with integers x_1, \dots, x_m of sum $2B$, we construct a PB scenario for which, for a yes-instance of Partition, a BPJR-L budget corresponds to a subset of those integers whose sum equals to B . Specifically, we construct items c_1, \dots, c_m , where item c_j ($j \in [m]$) costs x_j and have 1 voter approving all of them; we set the budget limit L to B . This finishes the construction.

Given a solution to the Partition instance, consisting of a subset $X' \subseteq X$ with $\sum_{x' \in X'} x' = B$, we construct a budget $W = X'$ which costs $B = L$ and thus is exhaustive and satisfies BPJR-L. For the other direction, notice that, according to BPJR-L, the single voter deserves the whole budget, thus a budget satisfying BPJR-L shall correspond to a solution to the Partition instance. \square

Computing BJR-L budgets, however, can be done in polynomial time. The proof of the next proposition is by a greedy algorithm, somehow resembling Approval-based greedy Chamberlin-Courant.

PROPOSITION 3.9. *There is a polynomial-time algorithm which computes an exhaustive budget satisfying BJR-L.*

PROOF. If $L = 0$, then $W = \emptyset$ satisfies the proposition. Let $L > 0$ be a budget limit and define the set of cheapest items:

$$C^* := \{c \in C : w(c) = 1\}.$$

If $|C^*| \leq L$, then any budget W such that $C^* \subseteq W$ is feasible and satisfies BJR-W. To satisfy the exhaustiveness property we add items $c \notin W$ to the budget until it is exhaustive.

If $|C^*| > L$, then we continue according to the following procedure. Let $A' = A$, $W = \emptyset$, and let $s(c, A') = |\{i \in A' : c \in A_i\}|$ denote the approval score of item c with respect to the ballot profile A' . Select any $c \in C^*$ with maximal score $s(c, A')$; then, set $W \mapsto W \cup \{c\}$, remove all approval ballots with $c \in A_i$ from A' , and redefine $C^* \mapsto C^* - \{c\}$. Repeat this process until $w(W) \geq \lfloor L \rfloor$ or until $C^* = \emptyset$.

Note that at each stage where an item $c \in C^*$ is added to W , a group of $s(c, A')$ unrepresented voters become represented. Also note that at each stage $s(c, A')$ is weakly decreasing; that is, we remove voters who have been represented and so the approval score of any item can never increase.

In the first case, the algorithm terminates with W such that $w(W) \geq \lfloor L \rfloor$, thus the exhaustiveness and feasibility properties

are satisfied. Now, for the purpose of a contradiction, suppose that BJR-W-L is not satisfied. Thus, there exists an item $c \in C^*$ with $s(c, A') \geq n/L$, for A' at the algorithm's termination. But since $s(c, A')$ is weakly decreasing at each stage and the item $\tilde{c} \in C^*$ with maximal score is added each time, but $c \in C^*$ was never elected, it must be that every one of the $|W| = w(W)$ items added to W represented at least $|W| \cdot n/L$ distinct voters.

Notice, however, that as

$$|W| \cdot n/L \geq \lfloor L \rfloor n/L > (L-1)n/L = n - n/L,$$

it would mean that strictly more than $n - n/L$ voters were represented – meaning that there can no exists a set V' of unrepresented voters with $|V'| \geq n/L$; this gives a contradiction.

For the second case, note that there is no cheapest item supported by any voter – let alone a cohesive group of $\geq n/L$ voters. Thus, the budget W satisfies BJR-L. To ensure that the exhaustiveness property is satisfied we (arbitrarily) add items $c \notin W$ until the budget is exhaustive (but still remains feasible). \square

We end this section by considering the complexity of testing whether a given budget satisfies BJR-L (and Strong-BJR-L).

PROPOSITION 3.10. *There is a polynomial-time algorithm to test whether a given budget satisfies BJR-L and Strong-BJR-L.*

PROOF. Given a PB scenario with items C , voters V , and budget limit L , and a budget W , the task is to decide whether W satisfies BJR-L or Strong-BJR-L. We proceed by describing a procedure for BJR-L and mention how it shall be modified for Strong-BJR-L.

First, we find all voters which are not represented at all; formally, let $V'' := \{v \in V : V \cap W = \emptyset\}$. Next, for each $c \in C$ (or, for Strong-BJR-L, for each $c \in \{c \in C : w(c) = 1\}$), consider the voters in V'' which approve c ; formally, let $V'_c = \{v'' \in V'' : c \in v''\}$. Then, if $|V'_c| \geq n/L$, reject. If the end is reached, then accept. \square

3.3 Local-BPJR

As we are interested in efficient budgeting methods which output budgets satisfying certain forms of proportional representation, the computational hardness result of the last section is somewhat disappointing, as it presumably rules out the possibility of efficient methods which compute exhaustive BPJR-L budgets. Here we consider weaker versions of these concepts and then we describe an efficient budgeting method which computes budgets satisfying it.

Definition 3.11 (Local-BPJR-L). A budget W satisfies *Local-BPJR-L* if for all $\ell \in [1, L]$ there exists no set of voters $V' \subseteq V$ such that $W' = (\cup_{i \in V'} A_i) \cap W$, $|V'| \geq \ell n/L$ and there exists some $W'' \supset W'$ such that

$$W'' \in \arg \max \{w(C') : C' \subseteq \cap_{i \in V'} A_i \text{ and } w(C') \leq \ell\}.$$

Janson [16] reports on several interesting rules, developed by Phragmen, which were designed to achieve proportionality axioms in multi-winner voting. Brill et al. [7] proved that one of these rule, which they referred to as *Phragmen's sequential rule*, computes a committee that satisfies PJR (this refers to the proportionality axiom for multi-winner voting). Here, we generalize Phragmen's sequential rule [7, 16] to the case of PB.

Our generalized rule is referred to as GPseq (Generalized Phragmen's sequential rule) and proceeds as follows. Items are iteratively

```

Input:  $(N, C, w, L)$  % resp.: voters, items, cost function, limit
Output:  $W$  % budget
1  $W \leftarrow \emptyset$ 
2 while  $C'' = \{c \notin W : w(W) + w(c) \leq L \wedge \exists i \in N : c \in A_i\} \neq \emptyset$ 
   do
3   Let  $C^* = \{c' : c' \in \arg \min_{c' \in C''} s_{c'}\}$  where: % argmin set
4    $x_{c,i} \geq 0$  ( $\forall c \in C, \forall i \in N$ )
5    $x_{c,i} = 0$  ( $\forall c \in C, \forall i \in N$  such that  $c \notin A_i$ )
6    $\sum_{i \in N} x_{c,i} = w(c)$  ( $\forall c \in W \cup \{c'\}$ )
7    $\sum_{i \in N} x_{c,i} = 0$  ( $\forall c \notin W \cup \{c'\}$ )
8    $x_i = \sum_{c \in C} x_{ci}$  ( $\forall i \in N$ )
9    $s_{c'} \geq x_i$  ( $\forall i \in N$ )
10  Let  $c^* \in C^*$  % break ties arbitrarily
11   $W \leftarrow W \cup \{c^*\}$ 
12 return  $W$ 

```

Algorithm 1: Generalized Phragmen’s sequential rule for PB (GPseq).

added until no item can be added without exceeding the budget limit. An item that is added is required to spread its cost among voters who approve it. When an item is considered to be added to the set of selected items, we check what will be the maximum cost received by a voter. We select the item that minimizes the maximum cost received by voters (we discuss tie breaking later). The rule is also shown in Algorithm 1.

REMARK 3. *When distributing the cost of the current item, we are allowed to redistribute the cost of the already-chosen items; this can be directly implemented using a linear program (as in Algorithm 1), but also shown to be polynomial-time solvable using a combinatorial argument, as shown by Brill et al. [7].*

REMARK 4. *Currently, Line 2 in Algorithm 1 does not consider items c which are not approved by any voter. As can be seen from the proof of Proposition 3.13, leaving the algorithm as is still results in it satisfying Local-BPJR. Perhaps unsatisfactory, however, the resulting budgets might be non-exhaustive due to some items c which are not approved by any voter (but fits within the budget limit). One possible fix is by including a post-processing phase, specifically looking for such items at the end of the algorithm and adding them exhaustively.*

Before we prove that GPseq satisfies Local-BPJR-L, the next example shows that, indeed, it does not satisfy BPJR-L.

Example 3.12. Consider the following instance with $C = \{a, b, c, d\}$ and $w(a) = 2, w(b) = w(c) = 1.5$ and $w(d) = 1$. Let there be 6 voters with approvals

$$A_1 = \dots = A_4 = \{a, b\} \quad A_5 = A_6 = \{c\}.$$

If GPseq is run with $L = 3$, then in the first iteration $W_1 = \{b\}$, and in the second we add item c . Thus the computed budget is:

$$W = \{b, c\} \quad \text{and} \quad w(W) = 3.$$

This budget does not satisfy BPJR-L since the group of voters $V' = \{1, \dots, 4\}$ is of size $\geq 2 \cdot n/w(W) = 4$ and unanimously support a bundle $\{a\}$ with cost 2 but they were only represented by a bundle of cost 1.5 (i.e., item b).

PROPOSITION 3.13. *GPseq satisfies Local-BPJR-L.*

PROOF. Assume, towards a contradiction, that there is a PB scenario where GPseq outputs a budget W which violates Local-BPJR-L. By definition, this means that there is a number $\ell \in [1, L]$, a set of voters $|V'| \geq \ell n/L$, and some $W'' \supset W' := (\cup_{i \in V'} A_i) \cap W$ such that

$$W'' \in \max \left\{ w(C') : C' \subseteq \cap_{i \in V'} A_i \text{ and } w(C') \leq \ell \right\}.$$

First, observe that it must be the case that $w(W') < \ell$ (since otherwise no such W'' could be feasible). As $W' \subset W''$, there must be some $c^* \in W'' \setminus W'$, and, as $W' = (\cup_{i \in V'} A_i) \cap W$, it must be that $c^* \notin W$. Further, since $W' \cup \{c^*\} \subseteq W''$, the following holds:

$$w(W') + w(c^*) \leq w(W'') \leq \ell. \quad (1)$$

Recall that GPseq works in iterations, where in each iteration another item is added to the intermediate budget W . Further, before the first iteration it is possible to add c^* to the partial budget, since the partial budget is empty; while after the last iteration it is surely not possible anymore, as if it was so, then GPseq would not terminate. The proof now follows by considering the iteration at which the corresponding ‘switch’ occurred, and will then show that the maximum voter spread could have been smaller if c^* were chosen instead of the other item which was chosen; this then contradicts the way by which GPseq works.

To be more formal, the following notation is helpful. Denote the intermediate budget at the completion of the j th iteration of GPseq by W^j ; and by x_i^j and s_j^* , the spread of voter i and the maximum voter spread (respectively; specifically, $s_j^* = \max_{i \in V} x_i^j$) at the completion of that iteration.

Supplied with the above notation, let j be the index of the first iteration for which the following hold:

$$w(W^{j-1}) + w(c^*) \leq L, \quad (2)$$

$$w(W^j) + w(c^*) > L. \quad (3)$$

Recall that such j must exist as otherwise GPseq would not have terminated.

Next we compute a lower bound on the maximum voter spread in W^j . As $W' = (\cup_{i \in V'} A_i) \cap W$ it follows that the total weight of the items in $W_j \setminus (W' \cap W^j)$ is spread over at most $n - |V'|$ voters (those corresponding to voters not in V'). Then, from averaging, it follows that there must be at least one voter $k \in V \setminus V'$ for which the following hold:

$$\begin{aligned}
 x_k^j &\geq \frac{w(W^j) - w(W' \cap W^j)}{n - |V'|} \\
 &> \frac{L - w(c^*) - w(W' \cap W^j)}{n - |V'|} && \text{by (3)} \\
 &\geq \frac{L - \ell}{n - |V'|} && \text{by (1)} \\
 &\geq \frac{L - \ell}{\frac{n}{L}(L - \ell)} && \text{since } |V'| \geq \ell \frac{n}{L}, \\
 &= \frac{L}{n}.
 \end{aligned}$$

As $s_j^* = \max_{i \in V} x_i^j$, it follows that $s_j^* > \frac{L}{n}$.

Next we compute an upper bound on the maximum voter spread if, instead of adding the item added in the j th iteration, we would

add c^* at that iteration. Thus, the budget at the completion of the j th iteration will be $W^{j'} = W^{j-1} \cup \{c^*\}$. Let us denote the alternative values of x_i^j and s_j^* by $x_i^{j'}$ and $s_j^{*'}$, respectively.

Recall that GPseq considers minimizing the maximum voter spread, thus to compute an upper bound it is sufficient to consider one way of spreading c^* 's weight; we will consider spreading it evenly among the voters in V' . That way, the spread of voters $k \in V \setminus V'$ remains unchanged to the previous iteration $j - 1$; i.e., for $k \in V \setminus V'$ it holds that:

$$x_k^{j'} = x_k^{j-1}. \quad (4)$$

Let

$$\tilde{s}_{j-1}^* = \max_{k \in V'} x_k^{j-1} = \max_{k \in V'} x_k^{j'}, \quad (5)$$

this denotes the maximum spread among voters in $V \setminus V'$ at the $(j - 1)$ th iteration (or equivalently the alternative maximum spread of such voters in the j th iteration). Some weight of items in $W' \cap W^{j-1}$ may be distributed to voters in $V \setminus V'$ at this $(j - 1)$ th iteration.

Further, by redistributing the weight at the $(j - 1)$ th iteration, $\sum_{i \in V'} x_i^{j-1} \leq w(W' \cap W^{j-1})$, evenly among voters in V' we can maintain equality in (4). We then construct the alternative value of spreads for voters in V' by evenly distributed the weight of item c^* as well. Thus, for all $i \in V'$

$$\begin{aligned} x_i^{j'} &= \frac{\sum_{i \in V'} x_i^{j-1}}{|V'|} + \frac{w(c^*)}{|V'|} \leq \frac{w(W' \cap W^{j-1})}{|V'|} + \frac{w(c^*)}{|V'|} \\ &\leq \frac{\ell}{|V'|} \\ &\leq \frac{L}{n}. \end{aligned} \quad \text{by (1)}$$

Recall (5); it follows that under this spread,

$$s_j^{*' } = \max \left(\tilde{s}_{j-1}^*, \max_{i \in V'} (x_i^{j'}) \right).$$

Next we consider two cases. First, if $\tilde{s}_{j-1}^* \leq \frac{L}{n}$, then we are done as this would imply that $s_j^{*' } \leq \frac{L}{n} < s_j^*$, which contradicts the fact the GPseq minimizes s_j^* at each iteration.

For the second case, suppose that $\tilde{s}_{j-1}^* > \frac{L}{n}$ and so $s_{j-1}^* > \frac{L}{n}$. This would imply that at some earlier iteration, say $t < j$, we have $s_{t-1}^* \leq \frac{L}{n}$ and $s_t^* > \frac{L}{n}$ (as initially no weight is spread on the voters). Further, as $c^* \notin W$ and $W^{t-1} \subseteq W$ and $W^t \subseteq W$, for the intermediate budgets W^{t-1} and W^t it holds that $w(W^{t-1}) < w(W^t) \leq w(W^{j-1}) \leq L - w(c^*)$. This means that adding the item c^* does not cause us to exceed the budget limit L since $t < j$ and j is the earliest stage such that (2) and (3) are satisfied. Thus, if at stage t instead item c^* was added to the intermediate budget W^t then by spreading the additional weight $w(c^*)$ among agents in V' we can attain the following alternative maximum voter spread:

$$s_t^{*' } \leq \frac{L}{n} < s_t^*.$$

This contradicts the fact that GPseq minimizes s_j^* at each iteration.

Overall, we conclude that no such set V' can exist and thus Local-BPJR-L is satisfied. \square

Tie breaking. Notice that both Theorem 3.13 and Example 3.12 are oblivious to the tie breaking used by GPseq. Arguably, its tie-breaking can result in some non-intuitive behavior, as the following example shows. Let $C = \{c_1, c_2\}$ with $w(c_1) = 1$ and $w(c_2) = 2$ and let there be voters $V = \{1, \dots, 6\}$ with the following ballots:

$$A_1 = \dots = A_4 = \{c_2\} \quad A_5 = A_6 = \{c_1\}.$$

Then, if we have a budget limit $L = 2$, then the items to be chosen for Phragmen-budget will be either c_1 or c_2 . Now, if we favor cheaper items, or simply choose c_1 arbitrarily, then we will 'satisfy' only 2 voters; whilst, perhaps a more intuitive outcome would be $W' = \{c_2\}$ which is of a cost $w(W) = L = 2$ and satisfies 4 voters. Both budgets, however, satisfy BPJR-L in this case.

3.4 W-variants and Relations between Axioms

So far our axioms of proportionality depended on an external budget limit L . Indeed, being based on L , such definitions are easily communicated, as usually L is known before hand; further, each group of voters can easily compute and appreciate the fraction of L which they can claim for themselves.

There is, however, some merit in being oblivious to L , by considering proportionality axioms which are oblivious to L , and are properties of the budget itself (with respect to the electorate, of course). This is possible through what we refer to as W -variants. Specifically, instead of considering groups of voters of $\ell \cdot n/L$ (which, as our definitions above state, deserve ℓ units of the budget), in our W -variants we concentrate not on the external budget limit L but on the actual total cost of the budget $w(W)$, and consider groups of voters of $\ell \cdot n/w(W)$, which deserve ℓ units of the budget. We feel that these definitions, which are based on $w(W)$ are, mathematically speaking, more elegant, as they are properties of the budget itself and are oblivious to the externally-imposed budget limit. Formally, we suggest the following definitions (which are analogous to their L -variants described in the previous sections).

Definition 3.14 (Strong-BJR-W). A budget W satisfies *Strong-BJR-W* if there exists no set of voters $V' \subseteq V$ with $|V'| \geq n/w(W)$, such that $w(\cap_{i \in V'} A_i) \geq 1$ but $w((\cup_{i \in V'} A_i) \cap W) = 0$.

Definition 3.15 (Strong-BPJR-W). A budget W satisfies *Strong-BPJR-W* if for all $\ell \in [1, w(W)]$ there does not exist a set of voters $V' \subseteq V$ with $|V'| \geq \ell n/w(W)$, such that $w(\cap_{i \in V'} A_i) \geq \ell$ but $w((\cup_{i \in V'} A_i) \cap W) < \ell$.

Definition 3.16 (BJR-W). A budget W satisfies *BJR-W* if there exists no set of voters $V' \subseteq V$ with $|V'| \geq n/w(W)$ such that $w(\cap_{i \in V'} A_i) \geq 1$, $w((\cup_{i \in V'} A_i) \cap W) = 0$, and there exists some $c \in \cap_{i \in V'} A_i$ with $w(c) = 1$.

Definition 3.17 (BPJR-W). A budget W satisfies *BPJR-W* if for all $\ell \in [1, w(W)]$ there exists no set of voters $V' \subseteq V$ with $|V'| \geq \ell n/w(W)$ such that $w(\cap_{i \in V'} A_i) \geq \ell$ and

$$w((\cup_{i \in V'} A_i) \cap W) < \max \left\{ w(C') : C' \subseteq \cap_{i \in V'} A_i \text{ and } w(C') \leq \ell \right\}.$$

Definition 3.18 (Local-BPJR-W). A budget W satisfies *Local-BPJR-W* if for all $\ell \in [1, L]$ there exists no set of voters $V' \subseteq V$ such that $W' = (\cup_{i \in V'} A_i) \cap W$, $|V'| \geq \ell n/w(W)$ and there exists some

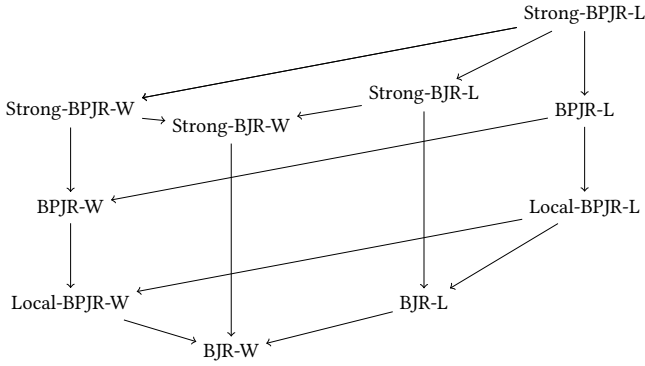


Figure 1: Logical relations between proportionality concepts. An arrow from (A) to (B) denotes that concept (A) implies concept (B).

$W'' \supset W'$ such that

$$W'' \in \arg \max \left\{ w(C') : C' \subseteq \cap_{i \in V'} A_i \text{ and } w(C') \leq \ell \right\}.$$

Notice that each L-variant implies its corresponding W-variant.

PROPOSITION 3.19. *Any L-variant implies its W-variant.*

PROOF. In the L-variants $\ell \in [1, L]$ while in the W-variants $\ell \in [1, w(W)]$, and $w(W) \leq L$ always holds. Further, the sets V' considered in the L-variants satisfy $|V'| \geq \ell n/L$ while those in the W-variants satisfy $|V'| \geq \ell n/w(W)$; again, $w(W) \leq L$ holds, it follows that $n/L \leq n/w(W)$. Therefore, each L-variant considers all sets considered by its corresponding W-variant. \square

Notice that W-variants always exist, as, for example, the empty budget $B = \emptyset$ satisfies them (as our voter set is always finite). Thus, it makes more sense to require also exhaustiveness from such budgets; for example, considering the complexity of computing exhaustive Local-BPJR-W budgets. Indeed, tractability of computing an L-variant implies tractability of computing an exhaustive W-variant. On the other hand, exhaustive W-variants might be computationally easier to compute than their corresponding L-variants; while we know that computing exhaustive Strong-BJR-W budgets and exhaustive Strong-BPJR-W budgets is NP-hard, we could not modify the proof of hardness of computing BPJR-L budgets (Theorem 3.8) to apply to exhaustive BPJR-W budgets as well. We conjecture, however, that indeed computing exhaustive BPJR-W is NP-hard as well.

We conclude the section by further discussing the logical relations between the axioms presented and discussed in the sections above. These relations are also pictorially represented in Figure 1.

PROPOSITION 3.20. *Strong-BPJR-(W/L) implies BPJR-(W/L) which implies Local-BPJR-(W/L) which implies BJR-(W/L).*

PROOF. The definition of BPJR-(W/L) includes a check on V' which is not present in the definition of Strong-BPJR-(W/L), and thus is implied by it. The definition of Local-BPJR-(W/L) considers specific sets W'' to represent certain groups, therefore is implied by BPJR-(W/L) which consider more sets.

To show that Local-BPJR-(W/L) implies BJR-(W/L) we use the contrapositive by assuming that BJR-(W/L) is not satisfied. This implies that there exists a group of voters V' such that $|V'| \geq n/w(W)$ (or $\geq n/L$), $w(\cup_{i \in V'} A_i) \geq 1$, and some $c \in \cap_{i \in V'} A_i$ with $w(c) = 1$ exists, but nevertheless $w(\cup_{i \in V'} A_i \cap W) = 0$. Recalling the definition of Local-BPJR-(W/L), this implies that $W'' = \emptyset$ and so Local-BPJR-(W/L) is not satisfied if there exists *any* budget W'' (since every budget contains the empty set) containing some subset of items $C' \subseteq \cap_{i \in V'} A_i$ and $w(C') \leq \ell$ such that $w(W'') > w(W')$. Such a budget always exists since a BJR-(W/L) budget always exists (Proposition 3.7), and a budget satisfying BJR-(W/L) satisfies the conditions for W'' to fail W as a Local BPJR-(W/L) budget. \square

4 CONCLUSIONS

Participatory budgeting is an interesting and widely applicable setting, gaining growing attention from the research community and being more extensively deployed. The axiomatic, normative study of methods of participatory budgeting is still lacking, and issues of proportionality and representation are currently not well understood; this is especially unfortunate as many times it is desirable to spend funds in a proportional way, taking into account issues of representativeness, and not letting the majority control all the available budget. Thus, in this paper, we proposed several new, proportional representation axioms as well as efficient corresponding algorithms. Many of our results are summarized in Table 1.

As we framed participatory budgeting as a generalization of multi-winner voting, our axioms and rules can also be viewed as interesting generalizations of work on multi-winner voting. Some of the interesting insights include the following: Whereas both PAV and Phragmen's sequential rule are considered compelling rules for approval-based multi-winner voting, the latter is more suitable in being extended to more general settings such as PB. Recalling our research motivation from Section 1, where we stated our aim at finding the right axiom and corresponding rule for PB, as a conclusion we can say that Local-BPJR-L appears to be a compelling axiom for proportional representation of PB and GPseq seems to be a particularly useful and desirable rule in this context.

We envisage further work on axiomatic and computational aspects of participatory budgeting. It will be interesting to explore the trade-offs between the axioms we proposed and other axioms that might be important to consider in these applications. It may also be useful to, theoretically and empirically, compare how different rules in the literature fare in terms of axiomatic properties.

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