Buyer Signaling Games in Auctions*

Weiran Shen
IIIS, Tsinghua University
Beijing, China
emerson@gmail.com

Pingzhong Tang
IIIS, Tsinghua University
Beijing, China
kenshinping@gmail.com

Yulong Zeng
IIIS, Tsinghua University
Beijing, China
zylhh123@gmail.com

ABSTRACT

We consider an auction setting where a seller sells one item to several buyers. Before a buyer's type is realized, he can commit himself to a so-called signal scheme. Mathematically, a signal scheme can be regarded as a linear decomposition of his prior type distribution into a probability distribution over a set of posterior distributions, each of which the seller can use a revenue optimal auction tailored for that distribution. It is known, from the literature of Bayes persuasion, that such signal schemes can lead to utility increase for both the seller and the buyers.

Our goal, is to analyze how a buyer should signal his distribution, given that other buyers may also signal their distributions. In other words, we want to find an equilibrium profile of signal schemes.

We obtain the closed-form solution for the single buyer case with regular distributions, and the multiple buyers case with symmetric type distributions under certain conditions. To prove our technique results, we also obtain some interesting intermediate results. In particular, we show that, if each buyer's signal scheme is to decompose his prior distribution into a set of posteriors that has the same virtual value function (in the exact sense of Myerson's virtual value function), his expected utility is equal to his utility in a first price auction game where his bidding function is always his virtual value function. Furthermore, perhaps surprisingly, we show that, certain distributions, including the uniform distribution, satisfy the property that every buyer's optimal signal scheme is indeed to decompose the prior into a set of posteriors that has the same virtual value function. As a result, we give the closed-form of an equilibrium profile of signal schemes for these cases.

CCS CONCEPTS

• **Information systems → Online auctions**; • **Theory of computation → Algorithmic game theory and mechanism design**;

KEYWORDS

Signaling game; Auction; Equilibrium

ACM Reference Format:
Weiran Shen, Pingzhong Tang, and Yulong Zeng. 2019. Buyer Signaling Games in Auctions. In *Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), Montreal, Canada, May 13–17, 2019,* IFAAMAS, 9 pages.

*This paper is supported in part by the National Natural Science Foundation of China Grant 61561146398, a China Youth 1000-talent Program, and an Alibaba Innovative Research Program. The authors thank all the anonymous reviewers for their thoughtful comments and Haifeng Xu for helpful discussions.

Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), N. Agmon, M. E. Taylor, E. Elkind, M. Veloso (eds.), May 13–17, 2019, Montreal, Canada.

1 INTRODUCTION

1.1 Problem description

Consider a monopoly pricing setting where a seller sells an item to a buyer to maximize revenue. They share the common knowledge that the buyer's type is drawn from the uniform distribution among $[0, 1]$. It is known from Myerson's theory [18] that the seller's strategy is to set a posted price at 0.5.

Now suppose, before the buyer's type is reported to the seller, he can reveal extra type information to the seller by connecting his type distribution to a set of publicly observable random variable called *signals* (E.g., this can be a test on the quality of the good): in particular, his type is lowered to a uniform distribution on $[0, 0.5]$ if the signal is observed to be *low*, while a uniform distribution on $[0.5, 1]$ if the signal is observed to be *high*. Furthermore, the chances of the signal being *low* or *high* are equally likely. Note that, the posterior type distributions are consistent with the prior common knowledge that the buyer's type distribution is from uniform $[0, 1]$.

Given the buyer's commitment on such information revelation strategy, the seller is now able to conditional the sale price on the signal realization: to maximize revenue, she will set price to be 0.25 when she sees low and set price to be 0.5 when high. Corempared to the previous case without signaling, both the seller and the buyer strictly benefit by $1/16$ respectively. It is known from recent literatures that the optimal signaling scheme for the buyer is to decompose his prior distribution into an infinite set of *equal revenue distributions*. Bergemann et al. [2] provide an existence proof to this result on the continuous distribution case. Later Shen et al. [23] provide a constructive proof to the general case.

In this paper, we extend the above analysis to the more general auction setting where a seller may sell an item to multiple buyers and each buyer may adopt a signal scheme as described early. Our goal, is to analyze how a buyer should signal his distribution, given that other buyers may also signal their distributions and the seller will use a revenue maximizing auction [18] on the posterior distributions.

1.2 Motivation and related works

The example described above is at the intersection of two important lines of economics research. The first investigates the power and limit of price discrimination [2, 19], where the buyer in the previous example is interpreted as a population, within which each individual has a deterministic type. The seller can segment the population into different markets (thus a sets of distributions) based on the characteristics of individuals in the population and price differently for each market (aka. *the third degree of price discrimination*). The impact of different segmentation strategies has been investigated and the set of (seller, buyer) utility profiles have been characterized under various scenarios. We refer readers to Bergemann et al. [2] for a comprehensive survey on price discrimination.

The second line of research concerns the power of signaling in the so-called persuasion model [8–11, 15]. This topic was initiated by the recent celebrated work of Kamenica and Gentzkow [15], where they study the general problem of a sender strategically revealing information based on external signals and give a method to find the optimal signaling scheme for the sender in a number of realistic scenarios. The basic model has been extended to a number of scenarios in the past five years: Gentzkow and Kamenica [11] consider the situation where sender's payoff also depends on the signal cost. Bhattacharya and Mukherjee [3], Chen and Olszewski [5], Gentzkow and Kamenica [13] study the simultaneous-move game where multiple senders simultaneously send signals. Dughmi [8] study the hardness of designing optimal information structures in zero-sum game, while Xu et al. [29] obtain hardness results of designing signal structures in Stackelberg Games. Gentzkow and Kamenica [12] proposes new approaches to the Bayesian persuasion problem.

Our problem, described in their terminology, is to find buyers' equilibrium profile of signaling schemes in the Myerson auction.

The signalling problem has also been studied in the auction scenario by Daskalakis et al. [7] and Bro Miltersen and Sheffet [4]. Both works consider the case where the seller has additional information than the buyer and how the seller can strategically reveal this additional information (together with designing the auction format itself in Daskalakis et al. [7]) to maximize revenue. In contrast, in our model, both parties share the same information and the buyer designs the signal.

Prior distributions play a key role in auctions with incomplete information. In the same spirit, Roesler and Szentes [20] analyze how to optimally spread a prior distribution with mean value reserved. Tang and Zeng [28] study the manipulation of prior distributions. Condorelli and Szentes [6], Shen et al. [22] analyze the optimal prior distribution for the buyer. In contrast, our model study the decomposition of prior distributions.

1.3 Our contributions

To best describe our contributions, let us start by reviewing the work of Bergemann et al. [2] and Shen et al. [23], which aim to understand the impact of signaling in the one buyer case, exactly the same as the setting described at the beginning of the paper. Bergemann et al. [2] characterize, for any discrete distribution, the set of (seller, buyer) utility profiles achievable by some buyer signaling scheme. It is not hard to see that, for any signaling scheme, the utility profile must necessarily satisfy the following three bounds: 1) the buyer's utility must be nonnegative, following from the individual rationality constraint; 2) the seller's utility must be no less than the case where she does not receive any signal at all; and 3) the sum of both parties utilities must be no higher than the value of the item. The main effort and result of these papers is to show that these three bounds are actually sufficient, in that they completely characterize all possible profiles achievable by any signaling scheme. Shen et al. [23] further extend this result by giving a constructive proof that applies for both discrete and continuous type distributions.

In this paper, we extend the analysis of the above problem to the case of multiple buyers. Our analysis is enabled by a series of innovative findings of the problem.

- We show that a buyer's utility can be written as a linear combination of the virtual values of posterior distributions. We further characterize the conditions that a set of such virtual values can be implemented by a signal scheme.
- Base on this property, we give an alternative proof for results in Bergemann et al. [2] for the single buyer case, the maximum utility is the expected social welfare minus the seller's optimal revenue on the prior distribution.
- We show that, if each buyer's signaling scheme is to decompose his prior distribution into a set of posteriors that has the same virtual value function (in the exact sense of Myerson's virtual value function), his expected utility is equal to his utility in a first price auction game where his bidding function is always his virtual value function.
- We further show that, certain distributions, including the uniform distribution, satisfy the property that every buyer's optimal signal scheme is indeed to decompose the prior into a set of posteriors that has the same virtual value function.

2 SETTING

Suppose the seller has a single item for sale to n buyers. Each buyer i's value v_i is drawn independently from a distribution F_i, called the *prior distribution*, with support $Supp(i)$ and density function f_i.[1] A signal scheme $\Omega_i = (T_i, \pi_i)$ for each buyer i consists of:

- a set (finite or infinite) of signals T_i;
- a signal distribution $\pi_i : Supp(i) \mapsto \Delta(T_i)$, where $\Delta(T_i)$ denotes the probability space of T_i.

Define $T = T_1 \times \ldots \times T_n$. For each signal $t_i \in T_i$ of buyer i, define $F_i(\cdot|t_i)$ to be the distribution, called *posterior distribution*, of random variable v_i given signal t_i. We use $Supp(t_i)$ and $f_i(\cdot|t_i)$ to denote the support and the density function of $F_i(\cdot|t_i)$ respectively. By Bayes rule,

$$f_i(v_i|t_i) = \frac{f_i(v_i)\pi_i(t_i|v_i)}{P(t_i)} = \frac{f_i(v_i)\pi_i(t_i|v_i)}{\int_{v_i' \in Supp(i)} f_i(v_i')\pi_i(t_i|v_i')\,\mathrm{d}v_i'},$$

where $P(t_i)$ denotes the probability of signal t_i.

Upon a signal profile $\mathbf{t} = (t_1, \ldots, t_n)$ for n buyers is realized by the nature, the seller runs an auction, which consists of an allocation rule $x_i : \mathbb{R}_+^n \mapsto [0, 1]$ and a payment rule $p_i : \mathbb{R}_+^n \mapsto \mathbb{R}$, based on the posterior distributions $F_i(\cdot|t_i)$. We use $x_i(\mathbf{b}|\mathbf{t})$ and $p_i(\mathbf{b}|\mathbf{t})$ to denote the allocation rule and the payment rule given signal profile $\mathbf{t}$ and a bid profile $\mathbf{b}$. We assume that the seller always optimizes his revenue, and thus runs Myerson auction based on posterior distributions. Due to the truthfulness of Myerson auction, when the value profile is $\mathbf{v} = (v_1, \ldots, v_n)$, buyer i's utility is $x_i(\mathbf{v}|\mathbf{t})v_i - p_i(\mathbf{v}|\mathbf{t})$.

Define $\phi_i(v_i|t_i)$ to be the *virtual value* of v_i with respect to the posterior distribution $F_i(\cdot|t_i)$, i.e.

$$\phi_i(v_i|t_i) = v_i - \frac{1 - F_i(v_i|t_i)}{f_i(v_i|t_i)}$$

[1] Throughout the paper, we consider continuous distributions that do not contain a point mass.

Similarly, define $\phi_i(v_i)$ to be the virtual value of v_i with respect to the prior distribution F_i. We say a distribution is *regular* if its virtual value is increasing.

In this paper, we consider the problem of designing optimal signal schemes $\Omega_i = (T_i, \pi_i)$ to maximize the buyer's expected utility.

Note that for any $v_i \in Supp(i)$,

$$\sum_{t_i \in T_i} f_i(v_i|t_i)P(t_i) = \sum_{t_i \in T_i} f_i(v_i)\pi_i(t_i|v_i) = f_i(v_i). \quad (1)$$

Equivalently,

$$\sum_{t_i \in T_i} F_i(v_i|t_i)P(t_i) = \sum_{t_i \in T_i} F_i(v_i)\pi_i(t_i|v_i) = F_i(v_i). \quad (2)$$

When T_i is an infinite set, $P(t_i)$ becomes probability density and satisfies $\int_{t_i \in T_i} P(t_i)\,\mathrm{d}t_i = 1$, and the above two equations becomes

$$\int_{t_i \in T_i} f_i(v_i|t_i)P(t_i)\,\mathrm{d}t_i = f_i(v_i) \quad (3)$$

$$\int_{t_i \in T_i} F_i(v_i|t_i)P(t_i)\,\mathrm{d}t_i = F_i(v_i) \quad (4)$$

So for each buyer, it is equivalent to design $P(t_i)$ and posterior distributions $F_i(\cdot|t_i)$ subject to (1) and (2). This is also equivalent to decompose the prior distribution $F_i(v_i)$ into posterior distributions $F_i(v_i|t_i), t_i \in T_i$. This equivalence is also investigated in the literature [2, 7]. We will use both the terms signaling scheme and decomposition, which is defined formally below, interchangeably from now on. We say a decomposition is regular if all posterior distributions are regular distributions.

Definition 2.1 (decomposition). A decomposition $\mathcal{P}_i$ of prior distribution F_i consists of:

- a set of probabilities, $\{P(t_i) \mid t_i \in T_i\}$;
- a set of posterior distributions, $\{F_i(\cdot|t_i) \mid t_i \in T_i\}$.

such that (2) is satisfied.

We focus on buyers' actions and optimize their utilities, in contrast to the rich literature that focuses on the seller's revenue[17, 21, 24–27].

3 TECHNICAL PRELIMINARIES

In this section, we establish our technical framework through a set of lemmas.

For each buyer i, the expected utility u_i equals the average utility among all possible realizations of signal profile $\mathbf{t}$. We use $f(\mathbf{v}|\mathbf{t})$ to denote $\prod_i f_i(v_i|t_i)$ and similarly $f_{-i}(\mathbf{v}_{-i}|\mathbf{t}_{-i}) = \prod_{j\neq i} f_j(v_j|t_j)$. By Myerson's Lemma [18], when all $F_i(v_i|t_i)$ are *regular* distributions, the utility of buyer i is

$$u_i = \int_{\mathbf{t}\in T} P(\mathbf{t}) \int_{\mathbf{v}} f(\mathbf{v}|\mathbf{t}) x_i(\mathbf{v}|\mathbf{t})(v_i - \phi_i(v_i|t_i))\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{t}.$$

Define

$$x_i^*(v_i|t_i) = \int_{t_{-i}\in T_{-i}} \left(P(\mathbf{t}_{-i}) \int_{\mathbf{v}_{-i}} x_i(\mathbf{v}|\mathbf{t}) f_{-i}(\mathbf{v}_{-i}|\mathbf{t}_{-i})\,\mathrm{d}\mathbf{v}_{-i} \right) \mathrm{d}\mathbf{t}_{-i}$$

Here $x_i^*(v_i|t_i)$ is known as the *interim allocation*, the expected allocation probability when buyer i's value is v_i and his signal is t_i, over the randomness of all other buyers' signals and values.

Note that in the Myerson auction, the interim allocation rule only depends on the virtual value $\phi_i(v_i|t_i)$ when $F_i(v_i|t_i)$ is regular. So we write it as a function of the virtual value, denoted by $y_i(\cdot)$, i.e. $y_i(\phi_i(v_i|t_i)) = x_i^*(v_i|t_i), \forall v_i, t_i$.

Lemma 3.1. *If all posterior distributions $F_i(\cdot|t_i)$ are regular, the expected utility for buyer i is*

$$u_i = \int_{t_i\in T_i} P(t_i) \int_{v_i} f_i(v_i|t_i) y_i(\phi_i(v_i|t_i))(v_i - \phi_i(v_i|t_i))\,\mathrm{d}v_i\,\mathrm{d}t_i.$$

Lemma 3.2. *For buyer i with value v_i,*

$$\int_{t_i\in T_i, f(v_i|t_i)\neq 0} P(t_i) f_i(v_i|t_i)\phi_i(v_i|t_i) \geq f_i(v_i)\phi_i(v_i)$$

Proof. For each $v_i, t_i \in T_i$ such that $f(v_i|t_i) \neq 0$, by definition $\phi_i(v_i|t_i) = v_i - \frac{1-F_i(v_i|t_i)}{f_i(v_i|t_i)}$, we have

$$P(t_i) f_i(v_i|t_i)\phi_i(v_i|t_i) = P(t_i) f_i(v_i|t_i) v_i - P(t_i)(1 - F_i(v_i|t_i))$$

Summing over all t_i's with $f(v_i|t_i) \neq 0$, and using Equation (3) and (4), we get

$$\begin{aligned}
&\int_{t_i\in T_i, f(v_i|t_i)\neq 0} P(t_i) f_i(v_i|t_i)\phi_i(v_i|t_i)\,\mathrm{d}t_i \\
=& v_i f_i(v_i) - \int_{t_i, f(v_i|t_i)\neq 0} P(t_i)(1 - F_i(v_i))\,\mathrm{d}t_i \\
\geq& v_i f_i(v_i) - (1 - F_i(v_i)) \\
=& f_i(v_i)\phi_i(v_i)
\end{aligned}$$

□

Note that by Lemma 3.2 it is straight forward that, the (Myerson's) revenue on the prior distributions is no more than the expected revenue among all posterior distributions.

Corollary 3.3. *The sellers revenue on the prior distributions is no more than the expected revenue among all posterior distributions for regular decompositions.*

Proof. For any value profile $\mathbf{v} = (v_1, ..., v_n)$,

$$\begin{aligned}
&\int_{\mathbf{t}\in T_i} \frac{\prod_i P(t_i) \prod_i f_i(v_i|t_i)}{\prod_i f_i(v_i)} \max_i\{\phi_i(v_i|t_i), 0\}\,\mathrm{d}t_i \\
\geq& \max_i \left\{ \frac{\int_{t_i\in T_i} P(t_i) f_i(v_i|t_i)\phi_i(v_i|t_i)\,\mathrm{d}t_i}{f_i(v)}, 0 \right\} \\
\geq& \max_i\{\phi_i(v_i), 0\}.
\end{aligned}$$

Integrating over $\mathbf{v}$ implies the corollary. □

If $\phi_i(v_i|t_i)$ are the same regardless of $t_i \in T_i$, then it is independent of t_i and can be written as $\psi_i(v_i)$. It can be directly obtained from Lemma 3.2 that $\psi_i(v_i) \geq \phi_i(v_i)$.

Definition 3.4. A decomposition of F_i is virtually identical with function $\psi_i(\cdot)$ if for all $v_i, t_i \in T_i$,

$$\phi_i(v_i|t_i) = \psi_i(v_i), \ \forall f_i(v_i|t_i) \neq 0$$

One of the main insights of this paper is to show that, for many cases, the buyer's optimal signaling scheme is a virtually identical decomposition. It follows from the Jensen's inequality and Lemma 3.2. We show the details in Section 5.

If one can prove that the a buyer's best choice is to choose a virtually identical decomposition, then does there exist a virtually identical decomposition of F_i with a given function $\psi_i(\cdot)$? It is obvious that $\phi(v) \le \psi(v) < v$ by definition and Lemma 3.2.

Lemma 3.5. *Assume $f(v)$ has support $[\underline{v}, \bar{v}]$. Given a continuous and increasing function $\psi(v)$ satisfying $\phi(v) \le \psi(v) < v, \forall \underline{v} < v < \bar{v}$ and $\psi(\bar{v}) = \bar{v}$, there exists a regular decomposition $(T, P(t), F(\cdot|t))$ such that*

$$\phi(v|t) = \psi(v), \forall t \in T, v \in Supp(t),$$

if the function $f(v)(v - \psi(v)) + F(v)$ is increasing.

Moreover, the closed-form of decomposition can be characterized as follows:

- $T = \{t|\underline{v} \le t < \bar{v}\}$;
- $P(t) = f(t) + \frac{\mathrm{d}}{\mathrm{d}t}(f(t)(t - \psi(t))), t \in T$;
- $F(v|t) = 1 - e^{-Q(v|t)}, \forall v \in (\underline{v}(t), \bar{v}(t)), t \in T$,

where

$$Q(v|t) = \int_{\underline{v}(t)}^{v} \frac{\mathrm{d}s}{s - \psi(s)}.$$

Proof. The proof is constructive and we focus on signals t such that the support of the corresponding distribution $f(v|t)$ is a single closed interval. Assume $Supp(t) = [\underline{v}(t), \bar{v}(t))$.

For any $t \in T$, $F(v|t)$ must satisfy

$$v - \frac{1 - F(v|t)}{f(v|t)} = \psi(v), \forall v \in (\underline{v}(t), \bar{v}(t)).$$

So $F(v|t) < 1, \forall v \in (\underline{v}(t), \bar{v}(t))$. We must have $\bar{v}(t) = \bar{v}, \forall t \in T$.

Also,

$$\frac{\mathrm{d}v}{v - \psi(v)} = \frac{\mathrm{d}F(v|t)}{1 - F(v|t)}, \forall v \in (\underline{v}(t), \bar{v}(t)).$$

Integrate on both sides (for ease of presentation, we change the integration variable to s), we have

$$\int_{\underline{v}(t)}^{v} \frac{\mathrm{d}s}{s - \psi(s)} = \int_{\underline{v}(t)}^{v} \frac{\mathrm{d}F(s|t)}{1 - F(s|t)}.$$

Since $F(\underline{v}(t)|t) = 0, \forall t \in T$, we have

$$Q(v|t) = -\ln(1 - F(v|t)), \forall v \in (\underline{v}(t), \bar{v}(t)),$$

where $Q(v|t) = \int_{\underline{v}(t)}^{v} \frac{\mathrm{d}s}{s - \psi(s)}$.

Therefore,

$$F(v|t) = 1 - e^{-Q(v|t)}, \forall v \in (\underline{v}(t), \bar{v}(t)), \tag{5}$$

$$f(v|t) = Q'(v|t)e^{-Q(v|t)} = \frac{e^{-Q(v|t)}}{v - \psi(v)} > 0, \forall v \in (\underline{v}(t), \bar{v}(t)).$$

From Equation (5) we know that when the minimum value $\underline{v}(t)$ is given, the whole posterior distribution $F(v|t)$ is determined. So without loss of generality we use signal t to represent the minimum value $\underline{v}(t)$.

Now we construct $P(t)$. Also note that for all $v \in [\underline{v}, \bar{v}]$, the following equation must hold:

$$\int_{t \in T} f(v|t)P(t)\,\mathrm{d}t = \int_{\underline{v}}^{v} f(v|t)P(t)\,\mathrm{d}t = f(v),$$

where $f(v|t) = 0$ if $v \notin Supp(t)$. Replacing $f(v|t)$, we have

$$\int_{\underline{v}}^{v} P(t)e^{-Q(v|t)}\mathrm{d}t = f(v)(v - \psi(v)).$$

Take derivative on both sides:

$$P(v) - \int_{\underline{v}}^{v} \frac{P(t)e^{-Q(v|t)}}{v - \phi(v)}\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}v}(f(v)(v - \psi(v))),$$

$$P(v) - \int_{\underline{v}}^{v} P(t)f(v|t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}v}(f(v)(v - \psi(v))),$$

$$P(v) - f(v) = \frac{\mathrm{d}}{\mathrm{d}v}(f(v)(v - \psi(v))).$$

In order for $P(t)$ to be a probability density function, we need $P(t) \ge 0, \forall t \in T$, or equivalently, we need $f(v)(v - \psi(v)) + F(v)$ to be an increasing function. Thus the lemma is proved. □

4 THE SINGLE BUYER CASE

In this section, we omit the subscript i since there is only one buyer. To warm up, we give an alternative proof for results in [2] and [23] which would be helpful for later arguments. We focus on the cases where only regular decompositions are allowed.

Suppose the buyer has cumulative valuation function $F(\cdot)$, with support $[\underline{v}, \bar{v}]$. The seller's action is to post a price r_t for each posterior distribution $F(\cdot|t)$, such that the seller's revenue $r_t(1 - F(r_t|t))$ is maximized.

Define $R(v) = v(1 - F(v))$ to be the revenue function and r^* to be its maximizer, which is the optimal reserve price for the prior distribution. Note that $R(v)$ is not exactly the same as the *revenue curve* well known in the literature [1, 14], since revenue curve is normally represented in *quantile* $q = 1 - F(v)$.

Now we use the function $R(v)$ to analyze the optimal signal scheme of the buyer. Formally, we have the following theorem.

Theorem 4.1 ([2, 23]). *If $F(\cdot)$ is regular and $R(v)$ is concave in the interval $(\underline{v}, r^*)$, the buyer's maximum utility is $\mathbf{E}[v] - R(r^*)$ and there exists a regular decomposition that achieves the maximum utility, where $\mathbf{E}[v]$ denotes the expected value of v.*

Proof. Since the posterior distributions are regular, the buyer gets the item if and only if the virtual value with respect to the posterior distribution is non-negative, i.e., for all v, t,

$$x^*(v|t) = y(\phi(v|t)) = \begin{cases} 0 & \text{if } \phi(v|t) < 0 \\ 1 & \text{if } \phi(v|t) \ge 0 \end{cases}.$$

Then the buyer's utility

$$\begin{aligned} u &= \int_v \int_{t \in T} P(t)f(v|t)y(\phi(v|t))(v - \phi(v|t))\,\mathrm{d}t\,\mathrm{d}v \\ &\le \int_{\phi(v)<0} \int_{t \in T} P(t)f(v|t)v\,\mathrm{d}t\,\mathrm{d}v + \\ &\quad \int_{\phi(v)\ge 0} \int_{t \in T} P(t)f(v|t)(v - \phi(v|t))\,\mathrm{d}t\,\mathrm{d}v \\ &\le \int_v vf(v)\,\mathrm{d}v - \int_{\phi(v)\ge 0} \phi(v)f(v)\,\mathrm{d}v \\ &= \mathbf{E}[v] - R(r^*). \end{aligned}$$

The last inequality follows from Lemma 3.2. The last equation holds because $\phi(v)f(v) = -R'(v)$. All inequalities hold in equality when choosing the virtually identical decomposition with $\psi(v) = \max\{0, \phi(v)\}$.

It is obvious that $\psi(v)$ satisfies the conditions in Lemma 3.5. The closed-form of optimal decomposition can be directly obtained by Lemma 3.5: For any $\underline{v} \le r \le r^*$, define signal t such that $\underline{v}(t) = r$ and

$$F(v|t) = \begin{cases} 0 & v \le \underline{v}(t) \\ 1 - \frac{\underline{v}(t)}{v} & \underline{v}(t) < v \le r^* \\ 1 - \frac{R(v)}{R(r^*)}\frac{\underline{v}(t)}{v} & r^* < v \le \bar{v} \end{cases}.$$

and the corresponding density for signal t is $P(t) = -R''(\underline{v}(t))$.

Since $R(v)$ is concave in the interval $(\underline{v}, r^*)$, we have that $P(t) \ge 0, \forall t$. The regularity of $\psi(v)$ implies the regularity of the decomposition. □

5 MULTIPLE BUYERS CASE

In this section, we assume that the prior distributions are regular and restrict the posterior distributions to be regular. We say decomposition is a best response if the buyer has no incentive to deviate from it. We say a profile of decompositions is an equilibrium if all buyers' decompositions are all best responses.

THEOREM 5.1. *Assume there are two buyers and they have identical and independent prior distribution $F(\cdot)$ with support $[\underline{v}, \bar{v}]$, if*

- *$F(v)$ and $\phi(v)$ are twice differentiable;*
- *$f'(v) \le 0$ and $\phi''(v) \ge 0$;*
- *$f(v)(v - \mathbf{E}_t[t < v]) + F(v)$ is increasing.*

then the virtually identical decomposition with the following $\psi(v)$ is an equilibrium.

$$\psi(v) = \max\{\mathbf{E}_t[t \le v], \phi(v)\},$$

where $\mathbf{E}_t[t \le v]$ is the expected value under the condition $t \le v$:

$$\mathbf{E}_t[t \le v] = \frac{\int_{\underline{v}}^{v} t f(t)\,\mathrm{d}t}{\int_{\underline{v}}^{v} f(t)\,\mathrm{d}t}.$$

It is not difficult to verify that all liner distributions and equal-revenue distributions satisfy the above three conditions. Before proving Theorem 5.1, we first consider some simple cases for a better understanding.

5.1 Two buyers with one buyer's value constant

Suppose buyer 1 has a deterministic value c and buyer 2's prior distribution is $F(\cdot)$, with support $[\underline{v}, \bar{v}]$. We compute the best response of buyer 2, our target buyer. We assume the tie breaking rule always maximizes the utility of our target buyer (We omit the subscript 2). Clearly, buyer 2 cannot win when his value is smaller than c. Thus we can move all values smaller than c to a single signal and only consider values greater than c.

LEMMA 5.2. *If $(v - c)(1 - F(v))$ is concave of v, the virtually identical decomposition with the following $\psi(v)$ is a best response:*

$$\psi(v) = \max\{c, \phi(v)\}, \forall v \in [c, \bar{v}].$$

PROOF. Let $R(v) = (v - c)(1 - F(v))$, and suppose r^* maximizes $R(v)$. Note that for any $v \ge c$, $\psi(v) \ge c$ and $y(\phi(v|t)) = x^*(v|t) = 1$.

Define

$$G_v(\Phi) = y(\Phi)(v - \Phi), 0 < \Phi < v,$$

which is maximized at $\Phi^* = c$.

It is notable that the function $G_v(\Phi)$ is defined here as "a part of" the utility function: the total utility is computed by integrating weighted $G_v(\phi(v|t)$ over all t's and all v's. So the concavity of function G_v and Lemma 3.2 enable us to use the Jensen's inequality (shown in the next subsection).

Suppose $\phi(d) = c$, then similar to the single buyer case, for $c < v < d$, we have

$$\begin{aligned}&\int_{t\in T} P(t)G(\phi(v|t))f(v|t)\,\mathrm{d}t \\ \le&\int_{t\in T} P(t)G(c)f(v|t)\,\mathrm{d}t \\ =&(v - c)f(v).\end{aligned}$$

And for $d < v < \bar{v}$, we have

$$\begin{aligned}&\int_{t\in T} P(t)G(\phi(v|t))f(v|t)\,\mathrm{d}t \\ \le&\int_{t\in T} P(t)f(v|t)(v - \phi(v|t))\,\mathrm{d}t \\ \le& vf(v) - f(v)\phi(v).\end{aligned}$$

So

$$\begin{aligned}u &= \int_v \int_{t\in T} P(t)G(\phi(v|t))f(v|t)\,\mathrm{d}t\,\mathrm{d}v \\ &\le \int_c^d (v - c)f(v)\,\mathrm{d}v + \int_d^{\bar{v}} (v - \phi(v))f(v)\,\mathrm{d}v.\end{aligned}$$

All equalities hold when choosing the decomposition described in this Lemma and $\psi(v)$ satisfies the conditions in Lemma 3.5. □

5.2 Best response

Supposes there are two buyers, each with $[0, 1]$ uniform prior distribution. Buyer 1 does not do any decomposition, i.e., T_1 is a singleton t_1 and $F(\cdot|t_1)$ is $[0, 1]$ uniform distribution. We compute the best response of buyer 2, our target buyer. We also omit the subscript 2 for simplicity.

LEMMA 5.3. *The virtually identical decomposition with the following $\psi(v)$ is a best response:*

$$\psi(v) = \max\{0, 2v - 1\}.$$

PROOF. Note that buyer one's virtual value $\phi_1(v_1) = 2v_1 - 1$, which is less than 0 for $v_1 \le \frac{1}{2}$.

Now the interim allocation rule is:

$$y(\phi(v|t)) = x^*(v|t) = \frac{\phi(v|t) + 1}{2},\ 0 \le \phi(v|t) \le 1.$$

Define $G_v(\Phi) = \frac{1}{2}(\Phi + 1)(v - \Phi), \forall 0 \le \Phi \le v$.

Note that $G'_v(\Phi) = -\Phi + \frac{v-1}{2} < 0$, so $G_v(\Phi)$ is decreasing for $0 \le \Phi \le v$, and is maximized at $\Phi^* = 0$.

So for $0 \le v \le \frac{1}{2}$, which means $\phi(v) \le 0$, we have

$$\int_{t\in T} P(t)G_v(\phi(v|t))f(v|t)\,\mathrm{d}t \le \int_{t\in T} P(t)G_v(0)f(v|t)\,\mathrm{d}t = \frac{v}{2}.$$

For $\frac{1}{2} < v \le 1$, $G_v''(\Phi) = -\frac{1}{2} < 0$, we have

$$\begin{aligned}&\int_{t\in T} P(t)G_v(\phi(v|t))f(v|t)\,\mathrm{d}t\\ \le& G_v\left(\int_{t\in T} P(t)f(v|t)\phi(v|t)\,\mathrm{d}t\right)\\ \le& G_v(\phi(v))\\ =& G_v(2v-1)\\ =& v(1-v),\end{aligned}$$

where the first inequality follows from Jensen's inequality and the second inequality follows from Lemma 3.2 and the decreasing monotonicity of G.

All equalities hold when choosing the decomposition described in this Lemma thus it is a best response. Note that, this $\psi(v)$ is defined identically to the singer bidder case with $[0, 1]$ uniform prior distribution, so the existence of the decomposition is proved and the closed-form decomposition is shown in Section 4.

The utility in this case is

$$\begin{aligned}u =& \int_{t\in T}\int_0^1 P(t)f(v|t)G(\phi(v|t))\,\mathrm{d}v\,\mathrm{d}t\\ \le& \int_0^{\frac{1}{2}} \frac{v}{2}\,\mathrm{d}v + \int_{\frac{1}{2}}^1 v(1-v)\,\mathrm{d}v = \frac{7}{48}.\end{aligned}$$

Compared to the utility in prior distribution, the utility of buyer 2 significantly increases.

□

5.3 Relation to first-price auctions

Lemma 5.4. *Consider n buyers with a prior distribution profile $(F_1(v_1), \dots, F_n(v_n))$. If for each buyer i, his decomposition is a regular virtually identical decomposition with $\psi_i(v_i)$, then the utility profile $(u_1, ..., u_n)$ is equivalent to the utility profile of the first-price auction where the bidders have a prior distribution profile $(F_1(v_1), \dots, F_n(v_n))$ and bidder i's bidding strategy is $b_i(v_i) = \psi_i(v_i)$.*

Proof. From Lemma 3.1 and Equation (1), the utility u_i is

$$\int_{v_i} y_i(\psi(v_i))(v_i - \psi(v_i))\,\mathrm{d}v_i.$$

Note that the first-price auction allocates the item to the buyer with the highest bid, and the function $y_i(\psi(v_i))$ equals the interim allocation in the first-price auction when bidding strategy profile $b_i(v_i)$ equals to $\psi_i(v_i)$. So the utility profiles are equivalent. □

Definition 5.5. Suppose that the bidders' value distribution profile is $(F_1(v_1), \dots, F_n(v_n))$. We say an auction $\mathcal{A}$ is an n-bidders *first-price auction with bidding constraint*, if each bidder i can only bid $b_i \ge \phi_i(v_i)$.

Lemma 5.6. *If the Bayes Nash equilibrium $(b_1(\cdot), ..., b_n(\cdot))$ of $\mathcal{A}$ with value distribution profile $(F_1(v_1), \dots, F_n(v_n))$ satisfies:*

- *$f_i(v_i)(v_i - b_i(v_i)) + F_i(v_i)$ is increasing with v_i for each i;*
- *$x_i^*(b)(v_i - b)$ is concave with b for each i,*

then if the prior distribution profile is $(F_1(v_1), \dots, F_n(v_n))$, it is an equilibrium for each buyer i's decomposition to be the virtually identical decomposition with $\psi_i(v_i) = b_i(v_i)$.

The first constraint guarantees the feasibility of the decomposition. The second constraint guarantees the optimality to choose a virtually identical decomposition, by Jensen's inequality. Lemma 5.4 and the property of BNE guarantee that $\psi_i(v_i) = b_i(v_i)$ is the best choice.

5.4 Proof of Theorem 5.1

To prove Theorem 5.1, we first show that there exists a cut-off point s such that $\mathbf{E}_t[t \le v] \ge \phi(v),\ \forall v \le s$ and $\mathbf{E}_t[t \le v] < \phi(v),\ \forall v > s$. Then we prove the optimality of the first part, based on the fact that $\mathbf{E}_t[t \le v]$ is a BNE of a corresponding first-price auction. To prove the second part, we first prove the concavity and monotonicity of the utility function. Finally, we apply Jensen's inequality.

Suppose buyer 1's decomposition is the virtually identical decomposition with $\psi(v)$ (given in Theorem 5.1). We prove that the same decomposition is buyer 2's best response. We omit the subscript 2 in the proof.

Lemma 5.7 ([16]). *The symmetric* Bayes Nash Equilibrium *(BNE) of the first-price auction with i.i.d prior distribution $F(\cdot)$ is $b(v) = \mathbf{E}_t[t \le v]$.*

Moreover, if buyer 1's bidding strategy is the BNE strategy, then for any value v, the expected utility $x^*(b(v))(y - b(v))$ is increasing with $b(v)$ for $b(v) \le \mathbf{E}_t[t \le v]$, and is decreasing for $b(v) > \mathbf{E}_t[t \le v]$.

Define $G_v(\Phi) = y(\Phi)(v - \Phi), 0 < \Phi < v$. Note that if there is no constraint for $\psi(v)$, then by Lemma 5.4, it is an equilibrium if both buyers choose $\psi(v) = \mathbf{E}_t[t \le v]$ (Each $\psi(v)$ maximizes $G_v(\Phi)$). However, $\psi(v)$ has a feasibility constraint in Lemma 3.2.

Lemma 5.8. *For buyer 2 with value v such that $\mathbf{E}_t[t \le v] \ge \phi(v)$, $\Phi = \mathbf{E}_t[t \le v]$ maximizes $G_v(\Phi)$.*

Proof. Define $\hat{y}(\Phi)$ to be the value of $y(\Phi)$ if we change buyer 1's decomposition into the virtually identical decomposition with $\hat{\psi}_1(v) = \mathbf{E}_t[t \le v],\ \forall v$, then for any $b \in \mathbf{R}$,

$$\begin{aligned}y(b)(v-b) &\le \hat{y}(b)(v-b)\\ &\le \hat{y}(\mathbf{E}_t[t \le v])(v - \mathbf{E}_t[t \le v])\\ &= y(\mathbf{E}_t[t \le v])(v - \mathbf{E}_t[t \le v]).\end{aligned}$$

The first inequality holds since $\hat{\psi}_1(v) \le \psi(v)$. The second inequality is due to Lemma 5.7. The last inequality comes from the symmetry that $y(\mathbf{E}_t[t \le v]) = y(\psi(v)) = F(v) = \hat{y}(\mathbf{E}_t[t \le v])$. □

We then prove that there exists a cut-off point:

Lemma 5.9. *There exist $s \in [\underline{v}, \bar{v}]$ such that*

- *$\psi(v) = \mathbf{E}_t[t \le v],\ \forall v \le s$;*
- *$\psi(v) = \phi(v),\ \forall v > s$.*

Proof. The lemma is equivalent to $\mathbf{E}_t[t \le v] \ge \phi(v),\ \forall v \le s$ and $\mathbf{E}_t[t \le v] < \phi(v),\ \forall v > s$. Since $\lim_{v \to \underline{v}} E[t \le \underline{v}] = \underline{v} > \phi(\underline{v})$, we can assume on the contrary that there exists s, m such that $\mathbf{E}_t[t \le s] = \phi(s)$, $\mathbf{E}_t[t \le m] = \phi(m)$ and $\mathbf{E}_t[t \le v] > \phi(v),\ \forall v \in (s, m)$.

For any $v \in [s, m]$, let $\Phi = \phi(w)$, $\check{y}'(\Phi) = \frac{\mathrm{d}F(w)}{\mathrm{d}\Phi} = \frac{f(w)}{\phi'(w)}$. So

$$G_v'(\Phi) = \frac{(v - \phi(w))f(w)}{\phi'(w)} - F(w). \tag{6}$$

Especially

$$G_v'(\phi(v)) = \frac{1 - F(v)}{\phi'(v)} - F(v). \tag{7}$$

By Lemma 5.8 , we have $G_s'(\phi(s)) = 0$ and $G_m'(\phi(m)) = 0$, thus

$$\phi'(s) = \frac{1 - F(s)}{F(s)} > \frac{1 - F(m)}{F(m)} = \phi'(m).$$

A contradiction with the second condition in Theorem 5.1. □

To make the use of Lemma 5.6, we need to prove the optimality of biding $\phi(v), v > s$ in auction $\mathcal{A}$, as well as the concavity of the utility function. Actually we only need to prove part of it by following lemmas:

Lemma 5.10. *Define $\check{y}(\Phi)$ to be the value of $y(\Phi)$ if we change buyer 1's decomposition to the virtually identical decomposition with $\check{\psi}_1(v) = \phi(v), \forall v$, then $\check{G}_v(\Phi) = \check{y}(\Phi)(v - \Phi), 0 < \Phi < v$ is concave.*

Proof. Let $\Phi = \phi(w)$, so $\mathrm{d}\Phi = \phi'(w)\,\mathrm{d}w$. By symmetry, $\check{y}(\Phi) = F(\phi^{-1}(\Phi)) = F(w)$. So

$$\check{y}'(\Phi) = \frac{\mathrm{d}F(w)}{\mathrm{d}\Phi} = \frac{f(w)}{\phi'(w)} > 0.$$

And

$$\check{y}''(\Phi) = \frac{\phi'(w)f'(w) - \phi''(w)f(w)}{\phi'(w)^3} < 0.$$

So $\check{G}_v''(\Phi) = (v - \Phi)\check{y}''(\Phi) - 2\check{y}'(\Phi) < 0$, the concavity is proved. □

Lemma 5.11. *For any $v > s$, $G_v(\Phi) = y(\Phi)(v - \Phi)$ is decreasing for $\Phi > \phi(v)$.*

Proof. Note that for any $v > s$,

$$\phi'(v) \ge \phi'(s) = \frac{1 - F(s)}{F(s)} > \frac{1 - F(v)}{F(v)}.$$

Plug this into Equation (7) and we have $G_v'(\phi(v)) \le 0$.

Also note that $G_v(\Phi) = \check{G}_v(\Phi)$ for all $\Phi \ge \phi(s)$ (By Lemma 5.9), so when $v > s$ and $\Phi > \phi(v)$, we have $\Phi > \phi(v) \ge \phi(s)$. Thus $G_v(\Phi)$ is concave according to Lemma 5.10. Therefore $G_v'(\Phi) \le G_v'(\phi(v)) \le 0$ for all $\Phi > \phi(v)$ □

Proof of Theorem 5.1. For $v < s$, by Lemma 5.8,

$$\begin{aligned}&\int_{t \in T} P(t)f(v|t)G(\phi(v|t))\,\mathrm{d}t\\ \le&\int_{t \in T} P(t)f(v|t)G\left(\mathbf{E}_t[t < v]\right)\,\mathrm{d}t\\ =&f(v)G(\mathbf{E}_t[t < v]).\end{aligned}$$

For $v \ge s$, note that $\mathbf{E}_t[t < v] \ge \mathbf{E}_t[t < s] = \phi(s)$. So we know that $G_v(\Phi), \Phi < \phi(s)$ is increasing according to Lemma 5.7. So it is never optimal for any $\phi(v|t)$ to have value less than $\phi(s)$. As $G_v(\Phi) = \check{G}_v(\Phi)$ is concave for all $\Phi \ge \phi(s)$, apply Jensen's inequality and we have

$$\begin{aligned}&\int_{t \in T} \frac{P(t)f(v|t)}{f(v)} G(\phi(v|t))\,\mathrm{d}t\\ \le&G\left(\int_{t \in T} \frac{P(t)f(v|t)}{f(v)}\phi(v|t)\,\mathrm{d}t\right)\\ \le&G(\phi(v)).\end{aligned}$$

All equalities hold when choosing the decomposition described in Theorem 5.1, thus it is an equilibrium. □

Example 5.12. If both buyers' value distribution is uniform [0, 1]. Then the virtually identical decomposition with the following $\psi(v)$ is an equilibrium:

$$\psi(v) = \begin{cases} \frac{v}{2} & v \in [0, \frac{2}{3}] \\ 2v - 1 & \frac{2}{3} < v < 1 \end{cases}.$$

For any $0 \le r < \frac{2}{3}$, define signal t such that $\underline{v}(t) = r$, with probability density $P(t) = \frac{3}{2}$ and distribution

$$F(v|t) = \begin{cases} 1 - \frac{t^2}{v^2} & t \le v < \frac{2}{3} \\ 1 - \frac{27t^2}{4}(1 - v) & v \ge \frac{2}{3} \end{cases}.$$

5.5 Extension to the n symmetric buyers case

Note that the BNE of the first-price auction for n symmetric buyer also has a closed-form

$$b(v) = \frac{\int_{\underline{v}}^{v} (n - 1)tF^{n-2}(t)f(t)\,\mathrm{d}t}{F(v)}.$$

Thus we have the following theorem:

Theorem 5.13. *Assume there are n buyers with i.i.d. prior distribution $F(v)$. Then the virtually identical decomposition with $\psi(v) = \max[b(v), \phi(v)]$ is an equilibrium if the following conditions hold:*

- *$F^{n-1}(\phi^{-1}(\Phi))$ is concave with respect to Φ;*
- *$\phi''(v) \ge 0$;*
- *$f(v)[v - b(v)] + F(v)$ is increasing with v.*

All the conditions holds for any uniform distribution. The proof is similar to the two symmetric buyers case.

6 BEYOND OPTIMALITY

For the single buyer case, Bergemann et al. [2] prove that as long as

- the seller revenue is no less than the revenue;
- the buyer utility is non-negative;
- the sum of the above two is no less than the maximum social welfare,

then the (revenue, utility) pair can be implemented by some decomposition. A natural extension is to consider the implementable (revenue, utility) pair for the multiple buyers case, where each buyer can arbitrary choose his signal scheme. However, we show that for the two buyers case, even if the above three conditions are satisfied, there exists some (revenue, utility) pairs that can not be implemented by any decomposition.

To analyze the problem, we first introduce a kind of decomposition, as shown in [23], which corresponds to the extreme point with the maximum welfare and 0 utility.

Definition 6.1. A decomposition for the prior distribution $F(\cdot)$ is extremal if and only if for all $v, t \in T$,

$$\phi(v|t) = 0 \text{ or } v, \text{ for all } f(v|t) \neq 0.$$

Shen et al. [23] also prove the following lemma:

Lemma 6.2. *There exists a close form extremal decomposition for any prior distribution F.*

Based on the extremal decomposition, we are able to show that the feasible range is not a triangle for the two buyers case by the following lemmas:

Lemma 6.3. *When two buyers has i.i.d. continuous distributions, if the buyer utility is 0, then the seller's revenue is strictly larger than that of the Myerson auction.*

Proof. If the utility is 0, the decompositions for both buyers can only be extremal decompositions, i.e., the virtual values are v_1 (v_2) or 0. When the value profile is (v_1, v_2), $v_1 < v_2$ and $0 < \phi(v_1) < v_1, 0 < \phi(v_2) < v_2$, for the case where buyer 2 wins, the revenue he contributes is

$$\int_{t_2:\phi_2(v_2|t_2)=v_2} P(v_2) f(v_2|t_2) v_2 \, \mathrm{d}t_2.$$

By Lemma 3.2,

$$\int_{t_2:\phi_2(v_2|t_2)=v_2} P(v_2) f(v_2|t_2) v_2 \, \mathrm{d}t_2 \geq f(v_2)\phi(v_2),$$

since $\phi_2(v_2|t_2) = 0$ if it is equal to v_2. Note that $f(v_2)\phi(v_2)$ is already the revenue of the Myerson auction (of the prior distribution) that buyer 2 contributes. However, for the case when buyer 2 loses, buyer one wins when $\phi_1(v_1|t_1) = v_1$ and contributes strictly positive revenue, of which the probability is non-negligible. So the total revenue exceeds that of the Myerson auction. □

It is not difficult to verify that the revenue is minimized when each buyer takes the following extreme decomposition

- with probability $\frac{\phi_i(v_i)}{v_i}$, $\phi(v_i|t_i) = v_i$;
- with probability $1 - \frac{\phi_i(v_i)}{v_i}$, $\phi(v_i|t_i) = 0$.

That is, subject to Lemma 3.2, taking the extremal decomposition that maximizes the probability that the posterior virtual value is 0. We use P_1 to denote this point in the (utility, revenue) coordinate.

Lemma 6.4. *For the two buyers case with i.i.d. continuous distributions, if the revenue equals that of the Myerson revenue, then total utility is strictly larger than 0.*

Proof. In this case, we prove that both buyers can no do any decomposition, otherwise the revenue will exceed the revenue of the Myerson auction. If buyer 1 makes a decomposition such that for some signal t_1, $\phi_1(v_1|t_1) < \phi_1(v_1)$, then when such signals are realized, the wining probability for buyer 1 decreases (since the virtual values of buyer 2 is a continuous distribution). This means that with positive probability, the winner has changed compared to the prior distributions. By the concavity of the max function, the seller's revenue strictly increase. As for the utility of the prior distribution, it is clearly positive. □

We use P_2 to denote the point with the utility and revenue of prior distributions in the (utility, revenue) coordinate.

Therefore, given that the feasible range for the two buyers case is not a triangle, a naïve guess would be that the feasible range is a quadrilateral, i.e., the southwest boundary is the segment P_1P_2. However, our simulation shows that this is not the case (see Figure 1).

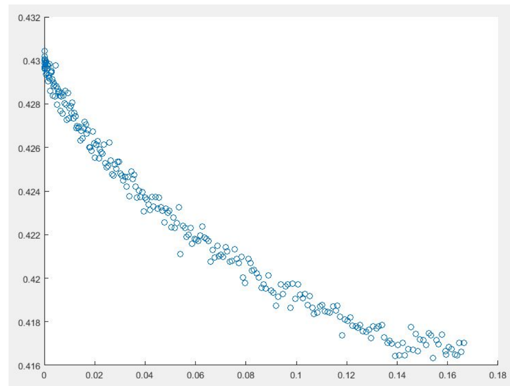

Figure 1: Simulation results of the (utility,revenue) pairs, where the x-axis is the buyers' utility and the y-axis is the seller's revenue

We choose the prior distribution of the two buyers to be $[0, 1]$ uniform distribution. We let each buyer use a linear combination of the signaling schemes of points P_1 and P_2. Since we put constraints on each buyer's decompositions, the curve shown in Figure 1 is not the boundary of the attainable area of (utility, revenue) pairs. However, it already shows that the boundary is not the straight line P_1P_2. So what is the closed-form of the boundary curve still remains a unknown.

7 CONCLUSION AND FURTURE WORK

We analyze the buyer singling game where each buyer chooses a signaling scheme that best responds to others. We study the virtually identical decomposition, where for any v, the virtual value corresponding to any posterior distribution is the same. We characterize the set of such decompositions that can be implemented. We relate the signaling game to the BNE of the first price auction, and show that under certain conditions, the equilibrium strategy $b(v)$ in the first price auction is exactly the virtual value of the virtually identical decomposition. In particular, for the n buyers symmetric case, we give closed-form solutions to the unique equilibrium under certain conditions.

One interesting future work is, of course, to find the solution to the game for the general multiple buyers case, and also for the case where the conditions in the symmetric case are relaxed.

It is known the (revenue, utility) pairs form a triangle. Therefore, another open problem is how to generalize this result to the multiple buyers case. We exclude some points on the boundary by Lemma 6.3 and Lemma 6.4 and do a simulation to show even for the simplest case the boundary is non-trivial.

REFERENCES

[1] Saeed Alaei, Jason Hartline, Rad Niazadeh, Emmanouil Pountourakis, and Yang Yuan. 2015. Optimal auctions vs. anonymous pricing. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*. IEEE, 1446–1463.

[2] Dirk Bergemann, Benjamin Brooks, and Stephen Morris. 2015. The Limits of Price Discrimination. *American Economic Review* 105, 3 (2015), 921–57.

[3] Sourav Bhattacharya and Arijit Mukherjee. 2013. Strategic information revelation when experts compete to influence. *The RAND Journal of Economics* 44, 3 (2013), 522–544.

[4] Peter Bro Miltersen and Or Sheffet. 2012. Send Mixed Signals: Earn More, Work Less. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC '12)*. ACM, New York, NY, USA, 234–247.

[5] Ying Chen and Wojciech Olszewski. 2014. Effective persuasion. *International Economic Review* 55, 2 (2014), 319–347.

[6] Daniele Condorelli and Balazs Szentes. 2016. Buyer-optimal demand and monopoly pricing. *Manuscript, Department of Economics, London School of Economics* (2016).

[7] Constantinos Daskalakis, Christos H. Papadimitriou, and Christos Tzamos. 2016. Does Information Revelation Improve Revenue?. In *EC*. ACM, 233–250.

[8] Shaddin Dughmi. 2014. On the hardness of signaling. In *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*. IEEE, 354–363.

[9] Shaddin Dughmi and Haifeng Xu. 2016. Algorithmic bayesian persuasion. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*. ACM, 412–425.

[10] Shaddin Dughmi and Haifeng Xu. 2017. Algorithmic persuasion with no externalities. In *Proceedings of the 2017 ACM Conference on Economics and Computation*. ACM, 351–368.

[11] Matthew Gentzkow and Emir Kamenica. 2014. Costly persuasion. *The American Economic Review* 104, 5 (2014), 457–462.

[12] Matthew Gentzkow and Emir Kamenica. 2016. A Rothschild-Stiglitz approach to Bayesian persuasion. *The American Economic Review* 106, 5 (2016), 597–601.

[13] Matthew Gentzkow and Emir Kamenica. 2017. Competition in persuasion. *The Review of Economic Studies* 84, 1 (2017), 300–322.

[14] Jason D Hartline. 2012. Approximation in economic design. *Book in preparation* (2012).

[15] Emir Kamenica and Matthew Gentzkow. 2011. Bayesian Persuasion. *American Economic Review* 101, 6 (Oct. 2011), 2590–2615.

[16] Vijay Krishna. 2009. *Auction theory*. Academic press.

[17] Vahab Mirrokni, Renato Paes Leme, Pingzhong Tang, and Song Zuo. 2017. Non-clairvoyant dynamic mechanism design. (2017).

[18] Roger B Myerson. 1981. Optimal auction design. *Mathematics of operations research* 6, 1 (1981), 58–73.

[19] A.C. Pigou. 2006. *The Economics of Welfare*.

[20] Anne-Katrin Roesler and Balázs Szentes. 2017. Buyer-optimal learning and monopoly pricing. *American Economic Review* 107, 7 (2017), 2072–80.

[21] Weiran Shen and Pingzhong Tang. 2017. Practical versus optimal mechanisms. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems*. International Foundation for Autonomous Agents and Multiagent Systems, 78–86.

[22] Weiran Shen, Pingzhong Tang, and Yulong Zeng. 2018. Buyer-Optimal Distribution. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems*. International Foundation for Autonomous Agents and Multiagent Systems, 1513–1521.

[23] Weiran Shen, Pingzhong Tang, and Yulong Zeng. 2018. A Closed-Form Characterization of Buyer Signaling Schemes in Monopoly Pricing. In *Proceedings of the 17th Conference on Autonomous Agents and MultiAgent Systems*. International Foundation for Autonomous Agents and Multiagent Systems.

[24] Pingzhong Tang and Tuomas Sandholm. 2012. Mixed-bundling auctions with reserve prices. In *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*. International Foundation for Autonomous Agents and Multiagent Systems, 729–736.

[25] Pingzhong Tang and Tuomas Sandholm. 2012. Optimal auctions for spiteful bidders. In *Twenty-Sixth AAAI Conference on Artificial Intelligence*.

[26] Pingzhong Tang and Zihe Wang. 2016. Optimal auctions for negatively correlated items. In *Proceedings of the 2016 ACM Conference on Economics and Computation*. ACM, 103–120.

[27] Pingzhong Tang and Zihe Wang. 2017. Optimal mechanisms with simple menus. *Journal of Mathematical Economics* 69 (2017), 54–70.

[28] Pingzhong Tang and Yulong Zeng. 2018. The price of prior dependence in auctions. In *Proceedings of the 2018 ACM Conference on Economics and Computation*. ACM, 485–502.

[29] Haifeng Xu, Rupert Freeman, Vincent Conitzer, Shaddin Dughmi, and Milind Tambe. 2016. Signaling in Bayesian Stackelberg Games. In *Proceedings of the 2016 International Conference on Autonomous Agents & Multiagent Systems*. International Foundation for Autonomous Agents and Multiagent Systems, 150–158.