Maxmin Share Fair Allocation of Indivisible Chores to Asymmetric Agents*

Extended Abstract

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ABSTRACT

We initiate the study of indivisible chore allocation for agents with asymmetric shares. The fairness concepts we focus on are natural generalizations of maxmin share: WMMS fairness and OWMMS fairness. We first highlight the fact that commonly-used algorithms that work well for allocation of goods to asymmetric agents, and even for chores to symmetric agents do not provide good approximations for allocation of chores to asymmetric agents under WMMS. As a consequence, we present a novel polynomial-time constant-approximation algorithm, via linear program, for OWMMS. For two special cases: binary valuation case and 2-agent case, we provide exact or better constant-approximation algorithms.

KEYWORDS

Fair division; maxmin fair share; approximation algorithms; chores

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1 INTRODUCTION

We consider fair allocation of indivisible chores when agents have asymmetric shares. In contrast to the case of goods for which agents have positive value, chores are disliked by agents and they have negative value for them. The fairness concept we focus on is maxmin share (MMS) fairness which was designed for allocation of indivisible items. MMS is based on the thought experiment that if the items are partitioned into bundles and an agent would always get the least preferred bundle of items, what is the best way she can partition the items. The value of such a bundle is the maxmin share of the agent. An allocation is deemed MMS fair if each agent gets her required share.

Maxmin share fairness was proposed by Budish [5] as a fairness concept for allocation of indivisible items. It is a relaxation of *proportionality* fairness that requires that each of the n agents should get value that is at least $1/n$ of the total value she has for the set of all items. When items are divisible, maxmin share fairness coincides with proportionality. Maxmin share fairness is a weaker concept when items are indivisible. Procaccia and Wang [12] identified a counter-example where maxmin fair allocations do not always exist. Since then, there has been several works on algorithms that find an approximate MMS allocation [1, 3, 4, 9]. All these works make the typical assumption that agents are symmetric and should be treated in a similar manner.

Farhadi et al. [6] were the first to consider MMS fairness for the case where indivisible goods are allocated and the agents are *not* symmetric because they may have different entitlement share of the goods. Ideally, an agent would expect to get a share of the total value that is proportional to her entitlement. However when items are indivisible, MMS fairness needs to be suitably generalized to the cater for asymmetric entitlement shares. Farhadi et al. generalized MMS fairness to that of the more general MMS concept as *weighted MMS (WMMS)* that caters for entitlements. Beyond the results for goods [6, 7], not much is known about chore allocation when the agents are asymmetric despite the recent active research in fair allocation of goods and chores. Furthermore, it is not clear whether the results for goods from one setting could carry over the other [2].

In this paper, we focus on fair allocation of *chores* for asymmetric agents. In the case of chores, agents do not have entitlements but relative shares. If an agent has a higher share, she is expected to take a higher load of the chores. Treating agents asymmetrically may be a requirement for several reasons. For example, countries with larger population and CO2 emission may be liable to undertake more responsibility to clean up the environment. The central research question we examine is: *When indivisible chores are to be allocated among agents with asymmetric shares, for what approximation factor do approximately WMMS fair allocations exist and how efficiently can they be computed?*

2 SETTING AND FAIRNESS CONCEPTS

2.1 Setting

Let $N = \{1, 2, \cdots, n\}$ be a set of n agents, and $M = \{1, 2, \cdots, m\}$ be a set of m indivisible items. Each agent has a valuation function $V_i : 2^M \to \mathbb{R}$. Denote by $V_{ij} = V_i(\{j\})$. We assume that items are chores to every agent, i.e., $V_{ij} \leq 0$ for all $j \in M$ and the valuations are additive, i.e., for any $S \subseteq M$, $V_i(S) = \sum_{j \in S} V_{ij}$. Without loss of generality and just for ease of presentation, it is assumed that all of the valuations are normalized, i.e. $V_i(\emptyset) = 0$ and $V_i(M) = -1$.

In this work, we consider the case when agents are asymmetric. Particularly, every agent has a share for the chores, namely $s_i \in (0, 1]$. The shares add up to 1, i.e., $\sum_{i \in N} s_i = 1$. Letting $\boldsymbol{V} = (V_1, \cdots, V_n)$ and $\boldsymbol{s} = (s_1, \cdots, s_n)$, we use $\mathcal{I} = (N, M, \boldsymbol{s}, \boldsymbol{V})$ to denote a chore allocation instance. Let $\Pi(M)$ be the set of all

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n-partitions of the items. A generic allocation will be denoted by $X = \langle X_1, X_2, \ldots, X_n \rangle$ where X_i is the bundle of agent i.

2.2 WMMS fairness

Before presenting the WMMS fairness concept that takes into account the shares of the agents, we first present the standard MMS fairness concept that assumes the shares of the agents are equal. For symmetric agents, the classical *maxmin share* (MMS) of an agent i with valuation V_i is defined as

$$\mathsf{MMS}_i = \max_{\langle X_i \rangle_{i \in N} \in \Pi(M)} \min_{j \in N} V_i(X_j).$$

Intuitively, when allocating items to n agents, each agent should get an allocation with value that is $1/n$ of the total value they have for all the items. Since the items are not divisible, this proportionality requirement may be not achievable for the agents. In view of this, MMS_i can be viewed as a relaxed lower bound on the value that agent i hopes for if she has the chance to partition the items into n bundles and every other agent adversarially choses a bundle before i. Next, we generalize the classical MMS notion to the setting with asymmetric agents.

Definition 2.1 (Weighted MMS). Given any chore allocation instance $\mathcal{I} = (N, M, \boldsymbol{s}, \boldsymbol{V})$, for every agent $i \in N$, the *weighted maxmin share (WMMS)* value of i is defined as:

$$\mathsf{WMMS}_i(\mathcal{I}) = \max_{\langle X_i \rangle_{i \in N} \in \Pi(M)} \min_{j \in N} V_i(X_j) \frac{s_i}{s_j}.$$

The definition above for WMMS fairness is exactly the same as that of WMMS as formalized by Farhadi et al. [6] for the case of goods except that the entitlement e_i of an agent i is replaced by her share s_i. As mentioned in the introduction, whereas a higher entitlement for goods is desirable for an agent, a higher share for chores is undesirable for the agent.

We call an allocation WMMS if the value of the allocation to each agent i is worth at least WMMS_i to her. Similarly, an allocation is called α-WMMS, if the total value of the share allocated to each agent i is at least $\alpha\mathsf{WMMS}_i$ to her for $\alpha \geq 1$.

Note that when all shares are equal, WMMS coincides with MMS fairness so it is a proper generalization of MMS. Secondly, we spell out an insight that also provides justification for the WMMS concept that was defined by Farhadi et al. [6]. We note that when the items are *divisible*, then $\mathsf{WMMS}_i = s_i V_i(M)$. Hence, for divisible chores, WMMS fairness also implies a natural generalization of proportionality that takes into account the shares of agents.

Next we show a simple algorithm, Naive, which returns an n-WMMS allocation. Algorithm Naive produces an allocation that allocates all of the items to a single agent who has the highest share (ties are broken arbitrarily).

Lemma 2.2. *Let $\mathcal{I} = (N, M, \boldsymbol{s}, \boldsymbol{V})$ be any chore allocation instance and $\langle X_i \rangle_{i \in N}$ be the output of Algorithm* Naive. *Then $V_i(X_i) \geq n\mathsf{WMMS}_i(\mathcal{I})$ for any $i \in N$.*

2.3 Optimal WMMS fairness

It is well known that for symmetric agents, no matter the items are goods or chores, an MMS allocation always exists for 2-agent case. But for asymmetric agents, we note that an exact WMMS allocation may not exist even when there are only two agents. Indeed, by the following lemma, we see that the lower bound of the problem is at least $\frac{4}{3}$, which means that there is no allocation that can guarantee each agent's value to be greater than $\frac{4}{3}\mathsf{WMMS}_i(\mathcal{I})$ for every $i \in N$.

Lemma 2.3. *In the chore allocation problem, any algorithm has an approximation ratio of at least $\frac{4}{3}$ for WMMS fairness.*

Since an exact WMMS allocation barely exists, it is natural to consider a relaxed version, *optimal WMMS (OWMMS) fairness*, which is similar to the one introduced in [3].

Definition 2.4 (Optimal WMMS). Let $\mathcal{I} = (N, M, \boldsymbol{s}, \boldsymbol{V})$ be any chore allocation instance. The *optimal WMMS (OWMMS) ratio* α^* is defined as the minimal $\alpha \in [1, \infty)$ for which an α-WMMS allocation always exists. Let $\mathsf{OWMMS}_i(\mathcal{I}) = \alpha^* \mathsf{WMMS}_i$ for any $i \in N$. A partition $X = \langle X_1, X_2, \ldots, X_n \rangle$ is called an OWMMS allocation, if $V_i(X_i) \geq \mathsf{OWMMS}_i(\mathcal{I})$ for all $i \in N$.

For any partition $X = \langle X_i \rangle_{i \in N}$, if $V_i(X_i) \geq c \cdot \mathsf{OWMMS}_i(\mathcal{I})$ for all $i \in N$, X is called c-approximation to the OWMMS allocation.

3 APPROXIMATION ALGORITHMS

For the case of goods allocation, the greedy *round robin* algorithm considered by Farhadi et al. [6] gives the best guarantee (of n-approximation for goods). Interestingly, the same algorithm was proved to provide a 2-approximation for MMS allocation of chores when agents are symmetric [3]. However, when agents have different shares, such an algorithm can be arbitrarily poor. We can show that some natural attempts to 'fix' the bad performance of the greedy sequential algorithm does not help.

For the general setting, we design a polynomial-time algorithm that is $(4 + \epsilon)$-approximation with respect to OWMMS fairness.

Theorem 3.1. *Given any chore allocation instance $\mathcal{I} = (N, M, \boldsymbol{s}, \boldsymbol{V})$ with α^* being the OWMMS ratio. For any $\epsilon > 0$, there is an algorithm that runs in polynomial time (for any number of agents) and returns an allocation $\langle X_i \rangle_{i \in N}$ such that for any agent i, $V_i(X_i) \geq (4 + \epsilon)\mathsf{OWMMS}_i(\mathcal{I})$.*

The algorithm used in Theorem 3.1 establishes the connection of our problem to the parallel processors scheduling problem [8, 10]. Through this connection, the computation of the OWMMS ratio is formulated as an integer program, and using the rounding technique in [11], we are able to round a fractional assignment of the relaxation of the integer program to an integer assignment.

Next, we consider two restricted cases. Given any instance $\mathcal{I} = (N, M, \boldsymbol{s}, \boldsymbol{V})$ with $N = \{1, 2\}$, we prove that it is always possible to guarantee each agent i's value to be at least $\frac{3}{2}\mathsf{WMMS}_i(\mathcal{I})$, via a divide-and-choose style algorithm. Thus, by Lemma 2.3, the OWMMS ratio α^* for 2-agent case is within $[\frac{4}{3}, \frac{3}{2}]$.

Theorem 3.2. *Let $\mathcal{I} = (N, M, \boldsymbol{s}, \boldsymbol{V})$ with $N = \{1, 2\}$. There is an algorithm that returns an allocation $\langle X_1, X_2 \rangle$ such that for any agent $i \in N$, $V_i(X_i) \geq \frac{3}{2}\mathsf{WMMS}_i(\mathcal{I})$.*

Finally, we study the case with any number of agents but all of them have (different) binary valuations: $V_{ij} \in \{0, c_i\}$, where $c_i \leq 0$, for all $i \in N$ and $j \in M$.

Theorem 3.3. *When the valuations of the agents are binary, a* WMMS *allocation exists and can be found in polynomial time.*

REFERENCES

[1] G. Amanatidis, E. Markakis, A. Nikzad, and A. Saberi. 2015. Approximation Algorithms for Computing Maximin Share Allocations. In *Proceedings of the 35th International Colloquium on Automata, Languages, and Programming (ICALP)*. 39–51.

[2] H. Aziz. 2016. Computational Social Choice: Some Current and New Directions. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*. 4054–4057.

[3] H. Aziz, G. Rauchecker, G. Schryen, and T. Walsh. 2017. Algorithms for Max-Min Share Fair Allocation of Indivisible Chores. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*. 335–341.

[4] S. Barman and S. Kumar Krishna Murthy. 2017. Approximation Algorithms for Maximin Fair Division. In *Proceedings of the 18th ACM Conference on Economics and Computation (ACM-EC)*. 647–664.

[5] E. Budish. 2011. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. *Journal of Political Economy* 119, 6 (2011), 1061–1103.

[6] A. Farhadi, M. Hajiaghayi, M. Ghodsi, S. Lahaie, D. M. Pennock, M. Seddighin, S. Seddighin, and H. Yami. 2017. Fair Allocation of Indivisible Goods to Asymmetric Agents. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems, AAMAS 2017, São Paulo, Brazil, May 8-12, 2017*. 1535–1537.

[7] A. Farhadi, M. Hajiaghayi, M. Ghodsi, S. Lahaie, D. M. Pennock, M. Seddighin, S. Seddighin, and H. Yami. 2019. Fair Allocation of Indivisible Goods to Asymmetric Agents. *Journal of Artifical Intelligence Research (JAIR)* (2019).

[8] D. Friesen. 1987. Tighter bounds for LPT scheduling on uniform processors. *SIAM J. Comput.* 16, 3 (1987), 554–560.

[9] M. Ghodsi, M. HajiAghayi, M. Seddighin, S. Seddighin, and H. Yami. 2018. Fair allocation of indivisible goods: Improvements and generalizations. In *Proceedings of the 19th ACM Conference on Economics and Computation (ACM-EC)*. ACM Press.

[10] T. Gonzalez, O. Ibarra, and S. Sahni. 1977. Bounds for LPT schedules on uniform processors. *SIAM J. Comput.* 6, 1 (1977), 155–166.

[11] J. Lenstra, D. Shmoys, and E. Tardos. 1990. Approximation algorithms for scheduling unrelated parallel machines. *Mathematical programming* 46, 1-3 (1990), 259–271.

[12] A. D. Procaccia and J. Wang. 2014. Fair Enough: Guaranteeing Approximate Maximin Shares. In *Proceedings of the 15th ACM Conference on Economics and Computation (ACM-EC)*. ACM Press, 675–692.