The Complexity of the Possible Winner Problem with Partitioned Preferences

Extended Abstract

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ABSTRACT

The Possible-Winner problem asks, given an election where the voters' preferences over the set of candidates is partially specified, whether a distinguished candidate can become a winner. In this work, we consider the computational complexity of the Possible-Winner problem under the assumption that the voter preferences are *partitioned*. That is, we assume that every voter provides a complete order over sets of incomparable candidates (e.g., candidates are ranked by their level of education). We consider elections with partitioned profiles over positional scoring rules. Our first result is a polynomial time algorithm for voting rules with two distinct values, which include the common k-approval voting rule. We then go on to prove NP-hardness for the class of voting rules that produce scoring vectors with at least four distinct values, and a large class of voting rules that produce scoring vectors with three distinct values.

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1 INTRODUCTION

In political elections, web site rankings, and multiagent systems, preferences of different parties (*voters*) have to be aggregated to form a joint decision. A general solution to this problem is to have the agents vote over the alternatives. The voting process is conducted as follows: each agent provides a ranking of the possible alternatives (*candidates*). Then, a *voting rule* takes these rankings as input and produces a set of chosen alternatives (*winners*) as output. However, in many real-life settings one has to deal with *partial votes*: Some voters may have preferences over only a subset of the candidates. The Possible-Winner problem, introduced by Konczak and Lang [2005], is defined as follows: Given a partial order for each of the voters, can a distinguished candidate c win for at least one extension of the partial orders?

The answer to the Possible-Winner problem depends on the voting rule that is used. In this work we consider *positional scoring rules.* A positional scoring rule provides a score value for every position that a candidate may take within a linear order, given as a scoring vector of length m in the case of m candidates. The scores of the candidates are added over all votes and the candidates with the maximal score win. For example, the k-approval voting rule, typically used in political elections, defined by $(1, \dots, 1, 0, \dots, 0)$ starting with k ones, enables voters to express their preference for k candidates. A popular special case of k-approval is *plurality*, defined by $(1, 0, \dots, 0)$.

The Possible-Winner problem has been investigated for various voting systems [5, 12, 15, 18]. For positional scoring rules, Betzler and Dorn [2010] proved a result that was just one step away from a full dichotomy for the Possible-Winner problem with positional scoring rules, and any number of candidates. In particular, they showed NP-completeness for all but three scoring rules, namely plurality, veto, and the rule with the scoring vector $(2, 1, \dots, 1, 0)$. For plurality and veto, they showed that the problem is solvable in polynomial time, but the complexity of Possible-Winner remained open for the scoring rule $(2, 1, \dots, 0)$ until it was shown to be NP-complete as well by Baumeister and Rothe [2012].

Partitioned preferences provide a good compromise between complete orders and arbitrary partial orders. Intuitively, the user provides a complete order over *sets* of incomparable items. In the machine learning community, partitioned preferences were shown to be common in many real-life datasets, and have been used for learning statistical models on full and partial rankings [10, 13, 14]. In recommender systems, the items are often partitioned according to their numerical level of desirability [16] (e.g., the common star-rating system, where the scores range between 1 and 5 stars). In such a scenario, all items with identical scores are incomparable. Partitioned profiles are also used to represent agent preferences in resource allocation problems [1].

In this work we investigate the computational complexity of the Possible-Winner problem with partitioned preference profiles. Our first result is that deciding whether a candidate is a possible winner can be performed in polynomial time for 2-valued voting rules (i.e., that produce scoring vectors with 2 distinct values), which include the k-approval, and k-veto voting rules. We then show that our algorithm also solves the possible winner problem for the $(2, 1, \dots, 1, 0)$ voting rule. In general, when the partitioned assumption is dropped, the problem is NP-complete for both of these rules [3, 4]. For other voting rules the Possible-Winner problem remains hard, despite the assumption of partitioned profiles. We prove hardness for the class of voting rules that produce scoring vectors containing at least 4 distinct values, and a large class of voting rules with 3 distinct values.

Prior Work. Previous work has considered the complexity of Possible-Winner under various restrictions to the voter profile [5, 8, 17, 18]. In particular, Dey and Misra [2017] studied the computational complexity of Possible-Winner with respect to the

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number of undetermined candidate pairs in every vote. Xia and Conitzer [2011] proved that k-approval and k-veto are NP-complete even when every vote contains just 4 undetermined pairs. The scoring rule $(2, 1, \ldots, 1, 0)$ remains NP-complete even when every vote contains just $m-1$ undetermined pairs, where m is the number of candidates [8]. In this work, we focus on partitioned profiles, showing that Possible-Winner can be solved in polynomial time for these three rules, regardless of the number of undetermined pairs. Our hardness results make use of techniques introduced by Betzler and Dorn [2010], and Dey and Misra [2017], demonstrating their applicability to elections with various restrictions.

2 PRELIMINARIES

Elections and Voting Rules

A *preference* is a binary relation $\succ$ over a set of *alternatives*, or *candidates* C that satisfies transitivity ($a \succ b$ and $b \succ c$ implies $a \succ c$) and irreflexivity ($a \succ a$ never holds).

A *voting profile* $\mathcal{V}$ is a sequence $(v_1, \ldots, v_n)$ of total (linear) orders over a set C of candidates $\{c_1, \ldots, c_m\}$. Each v_i in the profile $\mathcal{V}$ stands for the complete ranking of the candidates by the ith voter. A *voting rule* r is a function that maps a given voting profile $\mathcal{V}$ into a nonempty set $r(\mathcal{V})$ of *co-winners*. A candidate c is a *co-winner* if $c \in r(\mathcal{V})$, and a *winner* if $r(\mathcal{V}) = \{c\}$.

A *scoring vector* over a set of m candidates is a sequence $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_m)$ of m natural numbers such that $\alpha_1 \geq \cdots \geq \alpha_m$. Whenever candidate c is ranked in place j in the ranking v_i, it contributes $s_{v_i}(c) = \alpha_j$ points to the total score of c. The co-winners are the candidates with a maximum total score $\sum_{i=1}^{n} s_{v_i}(c)$. A *positional scoring rule* $r = (\overrightarrow{\alpha}_m)_{m \in \mathbb{N}^+}$ is a function that associates a scoring vector $\overrightarrow{\alpha}_m$ with each number $m \in \mathbb{N}^+$ of candidates. The value in the jth index of $\overrightarrow{\alpha}_m$ is denoted by $\overrightarrow{\alpha}_m(j)$.

Let r be a positional scoring rule. We assume that for all $m > 0$, the scoring vector $\overrightarrow{\alpha}_m$ contains at least one nonzero element. We also assume that r is *normalized*: for every $m > 0$, the greatest common divisor of the numbers in $\overrightarrow{\alpha}_m$ is one, and there exists a j such that $\overrightarrow{\alpha}_m(k) = 0$ for all $k > j$. These assumptions do not restrict the generality of the class of positional scoring rules [4, 9]. Therefore, we assume that all positional scoring rules are normalized. The rule r is *pure* if for every $m \geq 2$, the scoring vector $\overrightarrow{\alpha}_m$ is obtained from $\overrightarrow{\alpha}_{m-1}$ by inserting a score value at any position subject to satisfying the monotonicity constraint $\alpha_1 \geq \cdots \geq \alpha_m$. Essentially, *all* studied positional scoring rules are pure [2, 4, 9]. An election is denoted by $\mathcal{I}(C, \mathcal{V}, r)$, where C is the set of candidates, $\mathcal{V}$ the set of complete votes, and r is the positional scoring rule.

Partial Voting Profiles

A *partial voting profile* O is a sequence $(o_1, \ldots, o_n)$ of partial orders over C. We denote by $lin(o_i)$ the set of all linear extensions of o_i, where a *linear extension* of o_i is a total order v_i such that $c \succ_{v_i} c'$ whenever $c \succ_{o_i} c'$. An *extension* of O is a member of $lin(o_1) \times \cdots \times lin(o_n)$, that is, a voting profile $\mathcal{V} = (v_1, \ldots, v_n)$ where each v_i is a linear extension of o_i. Let r be a voting rule. A candidate c is a *possible winner* if there is an extension $\mathcal{V}$ of O such that c is a winner (or co-winner) under $\mathcal{V}$. The following is known about the complexity of determining the possible winners under positional scoring rules. Note that the rule is fixed, and the input consists of the candidates and partial profile.

Theorem 2.1. [3, 18] *Assume that r is pure. The possible winners can be found in polynomial time if r is either the plurality rule or the veto rule; otherwise, it is NP-complete to determine whether a given candidate is a possible winner.*

In this paper we consider a known type of partial preference known as *partitioned preferences.*

Definition 2.2 (Partitioned profile). A partial voting profile $O = (o_1, \ldots, o_n)$ is *partitioned* if every preference $o_j \in O$ consists of a partition of the candidates C into disjoint sets $A_1, \ldots, A_q$ such that: (1) for all $i < l$, if $c \in A_i$ and $c' \in A_l$ then $c \succ c'$ in o_j; and (2) for each $i \leq q$, candidates in A_i are incomparable in o_j (i.e., $a \not\succ b$ and $b \not\succ a$ in o_j, for every $a, b \in A_i$).

3 SUMMARY OF RESULTS

In the full version of this paper we detail and prove the results of this section.

Definition 3.1 (K-valued voting rule). We say that a positional scoring rule $r = (\overrightarrow{\alpha}_m)_{m \in \mathbb{N}^+}$ is *K-valued* if there exists a number $n_0 \in \mathbb{N}^+$ such that for all $m \geq n_0$, the score vector $\overrightarrow{\alpha}_m$ contains exactly K distinct values.

By this definition, the k-approval, veto, and plurality voting rules are 2-valued, while Borda has an unbounded number of different score values.

Definition 3.2 (unbounded-value voting rule). A positional scoring rule $r = (\overrightarrow{\alpha}_m)_{m \in \mathbb{N}^+}$ has an *unbounded number of positions with equal score values* if, for every $l \in \mathbb{N}^+$, there exists a number $n_0 \in \mathbb{N}^+$ such that for all $m \geq n_0$, the score vector $\overrightarrow{\alpha}_m$ contains at least l consecutive positions with the same value.

Let $\mathcal{I} = (C, O, r)$ denote an election where O is a partitioned profile (i.e., all of the partial votes are partitioned). The main results are summarized in Theorem 3.3. A scoring rule is called *differentiating* [8] if it produces a scoring vector containing two distinct nonzero differences between consecutive positions in the score vector. Theorem 3.3 covers all positional scoring rules except those producing vectors of the form $(\underbrace{2, \ldots, 2}_{k_2}, 1, \ldots, 1, \underbrace{0, \ldots, 0}_{k_0})$, where k_0 and k_2 are fixed constants such that $k_0 + k_2 > 2$, for which the complexity remains open.

Theorem 3.3. *Let $r = (\overrightarrow{\alpha}_m)_{m \in \mathbb{N}^+}$ be a pure positional scoring rule. Then we have the following when the preference profile is partitioned.*

1. *If r is 2-valued or if r is $(2, 1, \ldots, 1, 0)$, then Possible-Winner over r is solvable in polynomial time.*
2. *If r is K-valued, where $K \geq 4$, then Possible-Winner is NP-complete for r.*
3. *If r is 3-valued, and r produces a size-m scoring vector that is differentiating, or where the number of positions occupied by either α_m or α_1 is unbounded, then Possible-Winner is NP-complete for r.*

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