Local Core Stability in Simple Symmetric Fractional Hedonic Games

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ABSTRACT

We initiate the study of *local core* stability in *simple symmetric fractional hedonic games*. The input is an unweighted undirected graph G where vertices are the agents and edges model social connection (i.e., acquaintance) among agents. We assume that if there is an edge between two agents then they value 1 each other otherwise they value 0 each other, i.e., we consider the simple setting where an agent values 1 all and only her acquaintances. A coalition structure is a partition of the agents into coalitions where the utility of an agent is equal to the number of agents inside her coalition that are valued 1 divided by the size of the coalition. A coalition structure is in the *core* if no subset of agents can strictly improve all their utility by forming a new coalition together. In [7] it is shown that simple symmetric fractional hedonic games may not admit a core stable coalition structure. However, the fact that the core is required to be resilient to deviations by any groups of agents could be sometimes unrealistic, especially in systems with large populations. In fact, it may be difficult that agents are able to coordinate each other in order to understand whether there is the possibility of deviating together.

Motivated by the above considerations, we define a relaxation of the core, called *local core*. A coalition structure is in the local core if there is no subset of agents which (1) induces a clique in the graph G and (2) such that all agents can improve their utility by forming a new coalition together. We first show that any local core dynamics converges, which implies that a local core stable coalition structure always exists. We then study its performance with respect to the classic utilitarian social welfare and provide tight and almost tight bounds on the local core price of anarchy and stability, respectively.

KEYWORDS

Coalition Formation Games; Hedonic Games; Local Core; Price of Anarchy; Price of Stability

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1 INTRODUCTION

Hedonic games, introduced in [10], are a game-theoretic approach to the study of coalition formation problems. The outcomes of these games are coalition structures, which are partitions of the agents into coalitions, over which the agents have valuations, or preferences. One of the main properties of hedonic games is the non-externality, that is agents are interested only in the coalition they belong to and they do not care about how agents outside their coalition are allocated. A significant amount of research considered the study of many classes of hedonic games and characterized various solutions concepts like Nash stability, core stability and individual stability (see [3] for a survey on the topic).

This work is about symmetric fractional hedonic games, a subclass of hedonic games, where agents have mutual, cardinal valuations over each other. Given a coalition structure, the utility of an agent is equal to the sum of her valuations over the agents in her coalition divided by the size of the coalition. We focus on the special case in which all the valuations are either 0 or 1. These games are known in the literature as *simple symmetric fractional hedonic games* (SS-FHG) [1]. It is worth mentioning that SS-FHG have a succinct representation of each agent's valuations over coalitions. This is different from most of the classes of hedonic games, for which it takes exponential space to represent agents' valuations over coalitions.

SS-FHG suitably model a basic economic scenario, referred to in [1] as Bakers and Millers, where each agent can be considered as a buyer or a seller. There are only edges connecting buyers and sellers and every agent sees the others of the same type as market competitors. Each agent prefers to be situated in a group (market) with a small number of competitors, that is, each buyer wants to be in a group with many sellers and few other buyers, thus maximizing their ratio, in order to decrease the price of the good. On the other hand, a seller wants to be situated in a group maximizing the number of buyers against the number of sellers, in order to be able to increase the price of the good and gain a higher profit. SS-FHG can also model other realistic scenarios: (i) politicians may want to be in a party that maximizes the fraction of like-minded members; (ii) people may want to be with an as large as possible fraction of people of the same ethnic or social group. Other examples of real-life scenarios modelled by SS-FHG can be found in [1, 6, 7, 16].

One of the most well-known stability concept in the literature is the *core*. A coalition structure is said to be core stable, or that it is in the core, if no subset of agents can strictly improve all their utility by forming a new coalition together. This concept has been applied to many classes of hedonic games (see for example [1, 7, 9, 21]). In particular, in [7], it is given an instance of SS-FHG that does not admit a core stable coalition structure. However, the fact that the core is required to be resilient to deviations of any group of agents could be sometimes unrealistic, especially in systems with large populations. In fact, in this scenario, it may be difficult for

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agents to coordinate each other in order to understand whether there is the possibility of deviating together. Motivated by this concern, we incorporate social connection aspects into SS-FHG and consider only deviations of groups composed by agents who know each other. In particular, we model SS-FHG by an unweighted undirected graphs G, where vertices are the agents and edges model social connection (i.e., acquaintance) among agents. We assume that if there is an edge between two agents then they value 1 each other, otherwise they value 0 each other (i.e., an agent values 1 all and only her acquaintances). We define a relaxation of the core stability, called *local core*. A coalition structure is in the local core if there is no subset of agents, inducing a clique on the graph G (we assume that all agents must know each other to coordinate in order to form a new coalition together), that can all improve their utility by forming a new coalition together.

1.1 Our results

We start by showing in Theorem 3.1 that any local core dynamics converges, which implies that a local core stable coalition structure always exists. We then turn our attention to the study of its performance by considering the classic utilitarian social welfare, which is defined as the sum of all the agents' utilities. In particular, we study the local core price of anarchy and the local core price of stability, that is the ratio of the optimal social welfare divided by the social welfare of the worst local core and the best local core, respectively. We prove in Theorem 4.2 and Proposition 4.4 that the local core price of anarchy is at most 4 and that this bound is tight, while we provide a lower bound equal to 2 (in Theorem 5.1) and an upper bound equal to 8/3 (in Theorem 5.4) for the local core price of stability. The latter one is our main technical result.

Due to space constraints, the proof of Lemma 5.3 is only sketched.

1.2 Related work

Hedonic games, where each agent has a complete and transitive preference relation over all possible coalitions she can belong to without any form of externality, have been first formalized by Dréze and Greenberg [10], who analyze them under a cooperative perspective. A significant amount of research considered the study of many classes of hedonic games and characterized various solutions concepts like Nash stability, core stability and individual stability (see [3] for a survey on the topic).

Fractional hedonic games, a natural and succinctly representable class of hedonic games, have been introduced by Aziz et al. [1]. They mainly focus on core stable outcomes and prove that for general graphs the core can be empty, and even if the core is not empty, computing and verifying a core stable partition is NP-hard and coNP-complete, respectively. However, they also show that the core is not empty for restricted undirected graph topologies like graphs with degree at most 2, multipartite complete graphs, bipartite graphs admitting a perfect matching and regular bipartite graphs. Brandl et al. [7] study the existence of both core and individually stable coalition structures and the computational complexity of the related existence decision problems. In particular, they show that also in SS-FHG the core can be empty. Bilò et al. [6] consider Nash stable outcomes in fractional hedonic games. They show that a Nash equilibrium is not guaranteed to exist in symmetric fractional hedonic games with negative weights. However, they notice that it always exists when weights are non-negative. Furthermore, they give bounds on the (Nash) price of anarchy and stability. In [5, 16] the authors provide improved bounds on the (Nash) price of stability for SS-FHG. In [6] the authors show an instance of SS-FHG that does not admit a 2-strong stable coalition structure. Elkind et al. [11] study Pareto optimal coalitions in fractional hedonic games and provide bounds on the price of Pareto optimality. Other stability concepts applied to fractional hedonic games are discussed in [7, 20]. Aziz et al. [2] consider the computational complexity of computing welfare maximizing partitions (not necessarily Nash stable) for fractional hedonic games, while in [12] the authors consider the online scenario. Finally, Strategyproof mechanisms for fractional hedonic games have been proposed in [13].

Olsen [19] considers a slight variant of the symmetric fractional hedonic games called modified fractional hedonic games, where the utility of each agent in a coalition structure is equal to the sum of the weights of the incident edges in the coalition she belongs to, divided by the size of the coalition minus 1, that is, the agent herself is not accounted to the population of the coalition. It is worth mentioning that in this setting the core is always non-empty. Monaco et al. [18] consider Nash and core stable outcomes for modified fractional hedonic games and provide bounds on the core price of anarchy and stability.

As far as concerns social connection aspects in SS-FHG, Igarashi and Elkind [15] study hedonic games in which a subset of agents can form a coalition if and only if they are connected in a given input graph. They investigate the complexity of finding stable outcomes in such games for several notions of stability, mainly focusing on acyclic graphs. Our work is different than [15] for the following aspects: i) we admit stable outcomes in which coalitions may not be isomorphic to cliques, because we require that they are cliques only when they are formed by improving deviations; ii) we consider SS-FHG (which are not considered in [15]) defined over any graph; iii) we also provide an analysis on the performance of the core. In [14] the authors consider social connection aspects in hedonic games and analyze the effects of network-based visibility and structure on the convergence of coalition formation processes to stable states. However, they consider games with correlated preferences, i.e., each coalition has a weight that agents share equally. In particular, their results do not apply to the case where each agent has preference order over coalitions.

The concept of local stable outcomes close in the spirit to our local core is studied also in other important contexts like network design games [17], network creation games [4] and max k-cut games [8].

2 PRELIMINARIES

For any $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. Given an undirected unweighted graph $G = (N, E)$, a *Simple Symmetric Fractional Hedonic Game* (*SS-FHG*) $\mathcal{G}(G)$ induced by G is a game in which (1) each node $u \in N$, where $N = [n]$, is a selfish *agent*, (2) if edge $\{u, v\} \in E$, then u and v value each other 1, while if $\{u, v\} \notin E$, then u values v 0 and vice versa. Each agent u has to choose a coalition to join among a set of n available ones. Specifically, the strategy set of each agent u is an integer $i \in [n]$. Thus, a strategy profile of the game is a *coalition structure* $C = \{C_1, \dots C_n\}$, where $C(u)$ denotes the coalition agent u belongs to, and C_i contains all the agents u such that $C(u) = i$. Observe that C is a partition of the nodes into n coalitions, that is, $\bigcup_{i \in [n]} C_i = N$ and $C_i \cap C_j = \emptyset\ \forall i, j \in [n]$, with $i \neq j$. Since for a strategy profile some coalitions may be empty, we define $|C|$ to be the number of non-empty coalitions. Let $\delta_G(u)$ be the number of neighbors of agent u in G, and let $\delta_C(u)$ be the number of neighbors of u that belong to coalition C. Given coalition $C_i \in C$, let $G(C_i)$ be the subgraph induced by the nodes belonging to coalition C_i. Given a partition C, the *utility* or *payoff* $\mu_u(C)$ of agent u is equal to the number of neighbors in her own coalition divided by the total number of agents in the coalition, that is

$$\mu_u(C) = \frac{\delta_{C(u)}(u)}{|C(u)|}.$$

The *(utilitarian) social welfare* of a strategy profile $C = \{C_1, \dots, C_n\}$ is given by the sum of agents' utilities. Analogously, the social welfare of a coalition C_i is given by the sum of the utilities of all agents belonging to C_i. If, for any coalition C, we define $E(C)$ to be the set of edges whose nodes are both in coalition C, then we obtain, for every $i \in [n]$,

$$SW(C_i) = \sum_{u \in C_i} \mu_u(C) = \frac{2E(C_i)}{|C_i|}$$

and

$$SW(C) = \sum_{u \in N} \mu_u(C) = \sum_{i=1}^{n} SW(C_i).$$

A coalition structure C is *q-core stable* (*q-CS*), $1 \leq q \leq n$, if there exists no subset $T \subseteq N$ of size at most q such that every agent $u \in T$ strictly prefers T to her coalition $C(u)$, that is, $\forall T \subseteq N$, $\exists u \in T$ such that $\frac{\delta_{C(u)}(u)}{|C(u)|} \geq \frac{\delta_T(u)}{|T|}$. When $q = n$, then a n-CS coalition is said to be *core stable* (*CS*) and it is said to be in the *core*. Intuitively, for any $q > 1$, a coalition structure that is q-CS, is also $(q-1)$-CS.

A coalition structure C is ***local core stable*** (*LCS*) if there exists no subset $T \subseteq N$ such that (1) the subgraph $G(T)$ of G induced by T is a clique, and (2) $\mu_u(C) < \frac{|T|-1}{|T|}\ \forall u \in T$.

Considering local core stability, given a coalition structure C, a strategy $i \in [n]$ is a *local core improving deviation* in C for agents belonging to set $T = u_1, \dots, u_k$ (with $k \geq 2$) if $G(T)$ is a clique and agents in T can improve their utilities by forming together a new coalition C_i. Notice that in an LCS coalition structure there exists no set of agents possessing a local core improving deviation. A *local core dynamics* is a sequence of coalition structures such that for any two consecutive coalition structures C and C', C' is obtained by performing an improving local core deviation in C. A game has the *finite improvement path property* under the local core stability if it does not admit a local core dynamics of infinite length. Notice that such a game converges to an LCS coalition structure and therefore, clearly, it always admits an LCS coalition structure.

Given a game $\mathcal{G}(G)$, let $C^*(\mathcal{G}(G))$ be the solution maximizing the utilitarian social welfare, and let *q-CS*$(\mathcal{G}(G))$ and *LCS*$(\mathcal{G}(G))$ be the set of coalition structures that are q-core stable and local core stable, respectively. The *q-core price of anarchy* (resp. *local core price of anarchy*) of a simple symmetric fractional hedonic game $\mathcal{G}(G)$ is defined as the ratio between the social welfare of the optimal outcome C^* and that of the worst q-core stable (resp. local core stable) outcome. Formally, $q-CPoA(\mathcal{G}(G)) = max_{C \in \text{q-CS}(\mathcal{G}(G))} \frac{SW(C^*(\mathcal{G}(G)))}{SW(C)}$ (resp. $LCPoA(\mathcal{G}(G)) = max_{C \in \text{LCS}(\mathcal{G}(G))} \frac{SW(C^*(\mathcal{G}(G)))}{SW(C)}$). Analogously, the *q-core price of stability* (resp. *local core price of stability*) is the ratio between the social welfare of the optimal outcome C^* and that of the best q-core stable (resp. local core stable) outcome. Formally, q-$CPoS(\mathcal{G}(G)) = min_{C \in \text{q-CS}(\mathcal{G}(G))} \frac{SW(C^*(\mathcal{G}(G)))}{SW(C)}$ (resp. $LCPoS(\mathcal{G}(G)) = min_{C \in \text{LCS}(\mathcal{G}(G))} \frac{SW(C^*(\mathcal{G}(G)))}{SW(C)}$). Clearly, for any game $\mathcal{G}(G)$ it holds that $1 \leq$q-CPoS$(\mathcal{G}(G)) \leq$ q-CPoA$(\mathcal{G}(G))$ (resp. $1 \leq$ LCPoS$(\mathcal{G}(G)) \leq$ LCPoA$(\mathcal{G}(G))$).

3 EXISTENCE OF AND CONVERGENCE TO LOCAL CORE STABLE COALITIONS

It is known that an instance of SS-FGH may admit no core stable coalition structure [7]. Here we show instead that a local core stable coalition structure always exists and also that it is guaranteed that any local core dynamics converges.

Theorem 3.1. *Any instance of SS-FHG has the finite improvement path property.*

Proof. Consider any dynamics $\mathcal{D}$ starting from any coalition structure C^0. We show that $\mathcal{D} = \langle C^0, C^1 \dots \rangle$ has finite length, i.e. that an LCS coalition structure is eventually reached. For any $i \geq 1$, let T^i be the set of agents (forming a clique in G) that performs a local core improving deviation leading from coalition structure C^{i-1} to C^i, and let $N^i = \bigcup_{j=1}^{i} T^i$ be the set of all agents involved in some of the first i improvement moves of $\mathcal{D}$. Notice that all agents in T^i will belong to a coalition isomorphic to a clique in any coalition structure C^j with $j \geq i$. For any $i \geq 0$ and any agent $u \in N$, let

$$\eta_u^i = \begin{cases} 0 & \text{if } u \notin N^i \\ \mu_u(C^i) & \text{otherwise.} \end{cases}$$

Moreover, for any $i \geq 0$, let $\vec{x}^i$ be the vector obtained by listing η_u^i (for all $u \in N$) in non-increasing order. As usual, given two n-dimensional vectors $\vec{y}$ and $\vec{y}'$, we say that the first one is smaller than the second one for the lexicographical order (and we write $\vec{y} \prec \vec{y}'$) if $y_i < y_i'$ for the first component i for which y_i and y_i' differ.

It holds, for any $i \geq 0$, that $\vec{x}^{i-1} \prec \vec{x}^i$, i.e., vectors $\vec{x}^i$ always lexicographically increase after each improving deviation. In order to prove this property, we have to consider all agents $u \in N$ such that $\eta_u^i \neq \eta_u^{i-1}$. In fact, consider for any $i \geq 1$ any agent $u \in T^i$: clearly, $\eta_u^i > \eta_u^{i-1}$ because either $u \notin N^{i-1}$, and in this case it trivially holds that $\eta_u^i = \mu_u(C^i) > 0 = \eta_u^{i-1}$, or $u \in N^{i-1}$ and also in this case it holds $\eta_u^i = \mu_u(C^i) > \mu_u(C^{i-1}) = \eta_u^{i-1}$ given that every agent in T^i improves her utility. It remains to deal with any $u \in N \setminus T^i$ such that $\eta_u^i \neq \eta_u^{i-1}$: first of all, notice that $u \in N^{i-1}$, because otherwise $\eta_u^i = \eta_u^{i-1} = 0$; furthermore, it is easy to see that u is lowering her utility, i.e., $\mu_u(C^i) < \mu_u(C^{i-1})$, because u has to belong, in C^{i-1}, to a clique having at least a node in T^i, say $u' \in T^i$. Even if $\eta_u^i = \mu_u(C^i) < \mu_u(C^{i-1}) = \eta_u^{i-1}$, it still holds that $\vec{x}^{i-1} \prec \vec{x}^i$ because there exists node u' such that $\eta_{u'}^i > \eta_{u'}^{i-1} = \eta_u^{i-1}$.

Since the set of possible vectors is finite, the claim directly follows. □

4 LOCAL CORE PRICE OF ANARCHY

In order to prove the bounds on the local core price of anarchy, we first show in Proposition 4.1 an interesting relation between local core stability and 2-core stability: LCS ⊆ 2-CS, that is a local core stable coalition is always 2-core stable.

Proposition 4.1. *Given an instance of SS-FHG, a local core stable coalition is always 2-core stable.*

Given Proposition 4.1, an upper bound to the 2-core price of anarchy is also an upper bound to the local core price of anarchy. The following theorem shows that the 2-core price of anarchy of any SS-FHG is at most 4.

Theorem 4.2. *Given any graph G, $2 - CPoA(\mathcal{G}(G)) \leq 4$.*

Proof. Let $\mathcal{G}(G)$ be the instance of SS-FHG induced by graph G, and let C be a 2-CS coalition structure of the nodes in G. We partition the nodes in two sets A, B such that $A = \{u \in N \mid \mu_u(C) \geq \frac{1}{2}\}$ and $B = N \backslash A$. Let $C^* = \{C_1^*, \dots C_{|C^*|}^*, \emptyset, \dots, \emptyset\}$ be the coalition structure maximizing the social welfare. Intuitively, there can be no edge $\{u, v\}$ whose endpoints are both in B, otherwise u and v could join together to form a coalition and earn $\frac{1}{2}$ each, violating the 2-core stability. Thus, if $A_i = C_i^* \cap A$ and $B_i = C_i^* \cap B$ denote the agents of coalition C_i^* that are in A and B, respectively, we can upper bound the sum of the utilities of agents in C_i^* as follows:

$$\begin{aligned} \sum_{u \in C_i^*} \mu_u(C^*) &= \frac{2E(C_i^*)}{|C_i^*|} \leq \\ &\leq \frac{2\binom{|A_i|}{2} + 2|A_i||B_i|}{A_i + B_i} = \\ &= \frac{|A_i|\,(|A_i| - 1 + 2|B_i|)}{A_i + B_i} \leq 2|A_i| \end{aligned} \tag{1}$$

Inequality 1 holds because the only edges in G are the ones between either nodes both in A_i, or a node in A_i and a node in B_i. By summing all the agents' utilities in C^*, we get $SW(C^*) \leq 2|A|$. On the other hand, since $\mu_u(C) \geq \frac{1}{2}$ $\forall u \in A$, then $SW(C) \geq \frac{|A|}{2}$. Hence, 2-CPoA ≤ 4. □

Next corollary directly follows from Proposition 4.1 and Theorem 4.2.

Corollary 4.3. *Given any graph G, $LCPoA(\mathcal{G}(G)) \leq 4$.*

In the following proposition we provide a class of instances showing that the 2-core price of anarchy and local price of anarchy cannot be less then 4, thus providing a matching lower bound to the 2-CS and local core price of anarchy.

Proposition 4.4. *For any $\epsilon > 0$, there exists a graph G such that $2 - CPoA(\mathcal{G}(G)) \geq 4 - \epsilon$ and $LCPoA(\mathcal{G}(G)) \geq 4 - \epsilon$.*

Proof. Consider the game $\mathcal{G}(G)$ induced by the graph G depicted in Figure 1. There is a cycle $C_4 = [4]$ of 4 nodes, and each node $i \in C_4$ is connected with k additional nodes $4 + k(i-1) + j$, with $j = 1, \dots, k$. Thus, the total number of edges in G is $4(k+1)$.

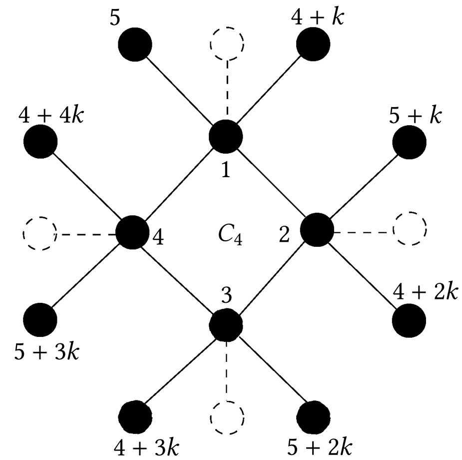

Figure 1: The graph inducing a game with $2-CPoA$ and $LCPoA$ tending to 4 as k tends to ∞.

On the one hand, there exists a solution $\bar{C}$ that puts each agent in the cycle and her leaves together, that is, $\bar{C} = \{\bar{C}_1, \dots, \bar{C}_4, \emptyset, \dots, \emptyset\}$ where, for any $i = 1, \dots, 4$, $\bar{C}_i = \{i, 4+k(i-1)+1, \dots, 4+k(i-1)+k\}$. Therefore, the optimal social welfare is $SW(C^*) \geq SW(\bar{C}) = \frac{8k}{k+1}$.

On the other hand, consider coalition structure $C = \{\{1, 2\}, \{3, 4\}, \{5\}, \{6\}, \dots, \{4 + 4k\}, \emptyset, \emptyset\}$ that puts pair of neighboring agents of C_4 together and leaves all the remaining ones alone. C is both $2 - CS$ and LCS, because the maximum clique in G is of size 2 and none of the agents in the cycle can strictly improve her utility by deviating together with only one neighbor. The social welfare of C is $SW(C) = 2$, thus the ratio between $SW(C^*)$ and $SW(C)$ is at least $4 - \epsilon$ for k sufficiently large. □

5 LOCAL CORE PRICE OF STABILITY

By turning our attention to the local core price of stability, we first wonder whether a coalition structure maximizing the utilitarian social welfare is always local core stable. We answer negatively to this question by providing an instance for which the unique local core stable coalition structure is not optimal. More specifically, in the following theorem we show that the local core price of stability is at least 2.

Theorem 5.1. *For any $\epsilon > 0$, there exists a graph G such that $LCPoS(\mathcal{G}(G)) \geq 2 - \epsilon$.*

Proof. Consider the game $\mathcal{G}(G)$ induced by the graph G depicted in Figure 2. The graph is similar to the one used in the proof of Proposition 4.4, but instead of a cycle here we have a clique K_x of $x = k+2$ nodes $1, \dots, x = k+2$, and each node $i \in K_x$ is connected to $k \geq 1$ leaf nodes $x + k(i-1) + j$, with $j = 1, \dots, k$. $k \geq 4$ is an even number that will be determined later. The total number of edges is $x\left(\frac{x-1}{2} + k\right) = \frac{(k+2)(3k+1)}{2}$.

We first show that the only local core stable coalition structure contains coalition $C_1 = [x]$, i.e., in any LCS coalition structure C, $C_1 \in C$ and all agents corresponding to leaf nodes have utility 0; notice that $SW(C) = x - 1 = k + 1$. To this aim, we need to prove a useful property.

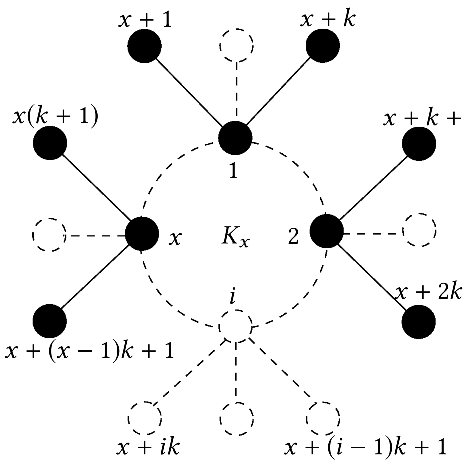

Figure 2: The graph, with $x = k+2$, inducing a game with *LCPoS* tending to 2 as k tends to ∞.

Property 5.1. *In any LCS coalition structure C, for any node $u = 1, \ldots, x$ it holds that $\mu_u(C) \geq \frac{k}{k+2}$.*

In order to prove Property 5.1, consider any node $u = 1, \ldots, x$ and let B_u be the set containing all and only the leaf nodes v connected in G to u and such that $C(v) = C(u)$; similarly, let $\bar{B}_u$ be the set containing all and only the leaf nodes v connected in G to u and such that $C(v) \neq C(u)$; moreover, let $\beta_u = |B_u|$ and $k - \beta_u = |\bar{B}_u|$. If $\mu_u(C) \geq \frac{k}{k+2}$, the property holds and we are done. Otherwise, by way of contradiction, assume that $\mu_u(C) < \frac{k}{k+2}$.

If $\beta_u \geq \frac{k}{2}$, notice first of all that, since $\mu_u(C) < \frac{k}{k+2}$, other agents (not connected to u) have to belong to coalition $C(u)$, i.e., $|C(u)| > \beta_u + 1$. All nodes in $T = \{u\} \cup B_u$ strictly prefer coalition T to their current one: a contradiction. In fact, $\frac{\delta_T(u)}{|T|} = \frac{\beta_u}{\beta_u+1} \geq \frac{\frac{k}{2}}{\frac{k}{2}+1} = \frac{k}{k+2} > \mu_u(C)$ and, for any $v \in B_u$, $\frac{\delta_T(v)}{|T|} = \frac{1}{\beta_u+1} > \frac{1}{|C(u)|} = \mu_v(C)$.

If $\beta_u < \frac{k}{2}$, all nodes in $T = \{u\} \cup \bar{B}_u$ strictly prefer coalition T to their current one: a contradiction. In fact, $\frac{\delta_T(u)}{|T|} = \frac{k-\beta_u}{k-\beta_u+1} \geq \frac{\frac{k}{2}}{\frac{k}{2}+1} = \frac{k}{k+2} > \mu_u(C)$ and, for any $v \in \bar{B}_u$, $\frac{\delta_T(v)}{|T|} = \frac{1}{k-\beta_u+1} > 0 = \mu_v(C)$. This concludes the proof of Property 5.1.

Let us assume, by way of contradiction, that coalition $\{1, \ldots, x\}$ does not belong to C. For any $u \in [x]$, let $A_u = C(u) \cap [x]$ and $\alpha_u = |A_u|$. Moreover, for any $u \in [x]$, let B_u be the set containing all and only the leaf nodes v connected in G to u and such that $C(v) = C(u)$; moreover, let $\beta_u = |B_u|$. The proof is now divided into three disjoint cases:

- If, for all $u \in [x]$, $\alpha_u = 1$, then, for all $u \in [x]$, $\mu_u(C) = \frac{\beta_u}{\beta_u+1} \leq \frac{k}{k-1}$, because $\beta_u \leq k$. Therefore, all nodes in $T = [x]$ strictly prefer coalition T to their current one: a contradiction. In fact, for any $u \in T$, $\mu_u(C) \leq \frac{k}{k+1} < \frac{k+1}{k+2} = \frac{\delta_T(u)}{|T|}$.
- If there exists $i \in [x]$ such that $\alpha_i \geq 2$ and, for all $u \in C(i) \cap [x]$, $\beta_u \geq 1$, then let $v \in C(i) \cap [x]$ be an agent for which $\beta_v = \min_{u \in C(i)} \beta_u$. Notice that $\alpha_v = \alpha_i \geq 2$ because $C(v) = C(i)$.

  It holds that $\mu_v(C) = \frac{\alpha_v - 1 + \beta_v}{|C(v)|} \leq \frac{\alpha_v - 1 + \beta_v}{\alpha_v + \alpha_v \beta_v} \leq \frac{1}{2}$, where the last inequality holds because $\alpha_v \geq 2$ and $\beta_v \geq 1$: a contradiction to the fact that, by Property 5.1, it holds that $\mu_v(C) \geq \frac{k}{k+2} \geq \frac{2}{3}$ (the last inequality holds because $k \geq 4$).
- Otherwise, i.e., if there exist $i \in [x]$ such that $\alpha_i \geq 2$ and $u \in C(i) \cap [x]$ such that $\beta_u = 0$ (notice that also in this case $\alpha_u = \alpha_i \geq 2$), we have to distinguish among two disjoint subcases:
  - If there exists $i \in [x]$ with $\alpha_i \geq 2$ such that (i) there exists $u \in C(i) \cap [x]$ with $\beta_u = 0$ and (ii) there exists $v \in C(i) \cap [x]$ with $\beta_v \geq 1$, then all nodes in $T = \{u\} \cup \bar{B}_u$ strictly prefer coalition T to their current one: a contradiction. In fact, $\mu_u(C) = \frac{\alpha_u - 1}{|C(i)|} \leq \frac{\alpha_u - 1}{\alpha_u + 1} < \frac{k}{k+1} = \frac{\delta_T(u)}{|T|}$ (the last inequality holds because $\alpha_u \leq k+2 < 2k$) and, for any $j \in \bar{B}_u$, $\mu_j(C) = 0 < \frac{1}{k+1} = \frac{\delta_T(j)}{|T|}$.
  - If, for all $i \in [x]$ with $\alpha_i \geq 2$, all $u \in C(i) \cap [x]$ are such that $\beta_u = 0$, first of all notice that $\alpha_u \leq x - 1 = k + 1$ because we are assuming that coalition $\{1, \ldots, x\}$ does not belong to C. We obtain that all nodes in $T = [x]$ strictly prefer coalition T to their current one: a contradiction. In fact, for all $u \in [x]$ with $\alpha_u \geq 2$, it holds that $\mu_u(C) = \frac{\alpha_u - 1}{\alpha_u} \leq \frac{k}{k+1} < \frac{k+1}{k+2} = \frac{\delta_T(u)}{|T|}$. Finally, for all $j \in [x]$ with $\alpha_j = 1$, it holds that $\mu_j(C) \leq \frac{k}{k+1}$, because agent j can be connected in C only to agents corresponding to the leaf nodes adjacent to her in graph G. Thus, $\mu_j(C) \leq \frac{k}{k+1} < \frac{k+1}{k+2} = \frac{\delta_T(j)}{|T|}$.

Consider now coalition structure $\bar{C}$ in which each agent $u \in [x]$ in the clique K_x is grouped together with the k leaf nodes adjacent to u in graph G, that is $\bar{C} = \{\bar{C}_1, \ldots, \bar{C}_x, \emptyset, \ldots, \emptyset\}$ where, for any $i \in [x]$, $\bar{C}_i = \{i, x+k(i-1)+1, \ldots, x+k(i-1)+k\}$. Hence, the social welfare of an optimal coalition structure is $SW(C^*) \geq SW(\bar{C}) = \frac{2kx}{k+1} = \frac{2k(k+2)}{k+1}$.

Therefore, the ratio between $SW(C^*)$ and $SW(C)$ is at least $\frac{2k(k+2)}{(k+1)^2} \geq 2 - \epsilon$ for a sufficiently large value of k. ◻

We already know from Theorem 4.2 that $\text{LCPoS}(\mathcal{G}(G)) \leq \text{LCPoA}(\mathcal{G}(G)) \leq 4$. We improve this upper bound to $\frac{8}{3}$ in Theorem 5.4. In order to prove it, we first introduce the following definition and Lemma 5.3.

Definition 5.2. Given a graph G, a $\{K_{\leq 3}, P_3\}$-coalition structure is a coalition structure $C = (C_1, \ldots, C_n)$ in which, for $i \in [n]$, every non-empty C_i is such that $G(C_i)$ is isomorphic either to a clique of at most 3 nodes (i.e., to a single node K_1, a matching K_2 or a triangle K_3) or to a path of 3 nodes P_3.

The following lemma shows that it is possible to convert any coalition structure C in a $\{K_{\leq 3}, P_3\}$-coalition structure C' without losing too much with respect to the social welfare.

Lemma 5.3. *Given any graph G and any coalition structure C for game $\mathcal{G}(G)$, there exists a $\{K_{\leq 3}, P_3\}$-coalition structure C' such that $SW(C) \leq \frac{3}{2} SW(C')$.*

Proof (Sketch). Let $C = (C_1, \ldots, C_n)$. For each coalition $C_i \in C$, consider the induced subgraph $G(C_i)$. Let C'_i be a $\{K_{\leq 3}, P_3\}$-coalition structure for game $\mathcal{G}(G(C_i))$ with the highest possible

	A	**B**	**D**	**F**
A	$\frac{a^2}{2}$ for $a \geq 6$ $a(a-1)$ for $a \leq 4$	$\frac{2}{3}ab$ for $a \geq 6$ ab for $a \leq 4$	$\frac{2}{3}ad$	0
B	$\frac{2}{3}ab$ for $a \geq 6$ ab for $a \leq 4$	$b(b-1)$	$\frac{7}{9}bd$	0
D	$\frac{2}{3}ad$	$\frac{7}{9}bd$	$\frac{5}{9}d^2$	$\frac{df}{3}$
F	0	0	$\frac{df}{3}$	0

Table 1: Recap of the maximum number of edges that in $C_i = A_i \cup B_i \cup D_i \cup F_i$ all agents in the row set can have towards all agents belonging to the column sets.

social welfare. Without loss of generality, assume that, among the $\{K_{\leq 3}, P_3\}$-coalition structures with the highest social welfare, C'_i is the one that maximizes the number of coalitions isomorphic to K_2.

Given any coalition $C_i \in C$ and the corresponding coalition structure C'_i, we can partition the agents into four groups A_i, B_i, D_i and F_i, as follows:

- A_i is the set of agents that make part of a coalition isomorphic to a matching K_2 in C'_i,
- B_i is the set of agents that make part of a coalition isomorphic to a triangle K_3 in C'_i,
- D_i is the set of agents that make part of a coalition isomorphic to a path P_3 in C'_i, and
- F_i is the set of agents that are isolated in C'_i, that is they are in a coalition isomorphic to K_1.

In the following we will fix a coalition $C_i \in C$, and we will drop the subscript i in order to improve the readability of the proof; moreover, let a, b, d and f, be equal to $|A|$, $|B|$, $|D|$ and $|F|$, respectively. Thus, there are $\frac{a}{2}$ matching coalitions, $\frac{b}{3}$ triangle coalitions, and $\frac{d}{3}$ path coalitions. Since each matching, triangle, and path coalition contributes 1, 2, and $\frac{4}{3}$ to the social welfare respectively, the social welfare is $SW(C'_i) = \frac{a}{2} + \frac{2b}{3} + \frac{4d}{9}$.

We aim at estimating the maximum possible degree that nodes in A, B, D and F can have towards the various possible coalitions in C'_i. In particular, let δ_u^Y be the number of edges in C_i that can connect node u to all nodes in set Y, and let $\delta_X^Y = \sum_{u \in X} \delta_u^Y$ be the total degree of all nodes in X with respect to edges connecting them to other nodes belonging to set Y. Notice that $\delta_X^Y = \delta_Y^X$ and that $|E(C_i)| = \frac{\sum_{X,Y \in \{A_i,B_i,D_i,F_i\}} \delta_X^Y}{2}$. The upper bounds to δ_X^Y, with $X, Y \in \{A, B, C, F\}$, are summarized in Table 1. Due to space constraints, the proofs of these upper bounds have been omitted.

Since the social welfare of a coalition is equal to the sum of the agents' degrees in that coalition divided by the number of agents, we are able to bound from above $SW(C_i) = \frac{\sum_{X,Y \in \{A_i,B_i,D_i,F_i\}} \delta_X^Y}{|C_i|}$.

Thus, when $a \geq 6$, the ratio $\dfrac{SW(C_i)}{SW(C'_i)}$ between the social welfare of coalition C_i and the social welfare of the coalition structure C'_i is at most:

$$\alpha_1 = \frac{\dfrac{a\left(\frac{a}{2} + \frac{2b}{3} + \frac{2d}{3}\right) + b\left(\frac{2a}{3} + b - 1 + \frac{7d}{9}\right) + d\left(\frac{2a}{3} + \frac{7b}{9} + \frac{5d}{9} + \frac{2f}{3}\right)}{a+b+d+f}}{\frac{a}{2} + \frac{2b}{3} + \frac{4d}{9}}$$

Moreover, when $a \leq 4$, the ratio $\dfrac{SW(C_i)}{SW(C'_i)}$ between the social welfare of coalition C_i and the social welfare of the coalition structure C'_i is at most:

$$\alpha_2 = \frac{\dfrac{a\left(a - 1 + b + \frac{2d}{3}\right) + b\left(a + b - 1 + \frac{7d}{9}\right) + d\left(\frac{2a}{3} + \frac{7b}{9} + \frac{5d}{9} + \frac{2f}{3}\right)}{a+b+d+f}}{\frac{a}{2} + \frac{2b}{3} + \frac{4d}{9}}$$

We first notice that $\alpha_1 \leq \frac{3}{2}$: in fact, by standard calculation, we obtain that it is equivalent to

$$a^2 + \frac{2d^2}{9} + \frac{5ab}{6} + \frac{ad}{6} + \frac{2bd}{9} + 2b + \frac{3af}{2} + 2bf \geq 0.$$

Similarly, it also holds that $\alpha_2 \leq \frac{3}{2}$: in fact, by standard calculation, we obtain that it is equivalent to

$$a\left(2 - \frac{a}{2}\right) + \frac{2d^2}{9} + b\left(2 - \frac{a}{2}\right) + \frac{ad}{6} + \frac{2bd}{9} + \frac{3af}{2} + 2bf \geq 0.$$

Since $\left(2 - \frac{a}{2}\right)$ is greater or equal than 0 for $a \leq 4$, the last inequality always holds.

Hence, it holds that $\dfrac{SW(C_i)}{SW(C'_i)} \leq \frac{3}{2}$.

Let C' be the $\{K_{\leq 3}, P_3\}$-coalition structure obtained by putting together all $\{K_{\leq 3}, P_3\}$-coalition structures C'_i, for all $i \in [n]$.

We have shown that, for every $i \in [n]$, $SW(C_i) \leq \frac{3}{2} SW(C'_i)$. By summing over all the coalitions in C, we get $SW(C) \leq \frac{3}{2} SW(C')$. □

We are now ready to prove the $\frac{8}{3}$ upper bound to the local core price of stability.

Theorem 5.4. *Given any graph G, $LCPoS(\mathcal{G}(G)) \leq \frac{8}{3}$.*

Proof. Let C^* an optimal coalition structure for $\mathcal{G}(G)$. In order to prove the bound, we provide an algorithm that returns a local core stable coalition structure whose social welfare is at least $\frac{3}{8} SW(C^*)$.

First of all, by Lemma 5.3 (applied to coalition structure C^*) we know that there exists a $\{K_{\leq 3}, P_3\}$-coalition structure C' such that $SW(C') \geq \frac{2}{3} SW(C^*)$.

Starting from coalition structure C', let us apply Algorithm 1 described below.

It is worth noticing that lines 2–4 and 10–33 of Algorithm 1 are not needed for the computation of the coalition structure returned, but they will be useful for proving the efficiency of the algorithm in terms of the social welfare. To this respect, in the remaining of this proof we will refer to variables a_i, b_i, d_j, $a_{i,j}$ and $b_{i,j}$ (for any $i \geq 1$ and $j \in [n]$) assuming that their value is the one they have at the end of the algorithm.

By Theorem 3.1, since SS-FHG has the finite improvement path property, it is guaranteed that Algorithm 1 terminates returning a local core stable coalition structure C''.

Algorithm 1 It takes as input a $\{K_{\leq 3}, P_3\}$-coalition structure C' and returns a local core stable coalition structure.

1: $C^0 \leftarrow C'$
2: **for** each $j \in [n]$ **do**
3: $\quad d_j \leftarrow SW(C'_j)$
4: **end for**
5: $i \leftarrow 0$
6: **while** C^i is not local core stable **do**
7: $\quad i \leftarrow i + 1$ ▷ Beginning of step i
8: $\quad$ Let $T_i \subseteq N$ be a set of agents of *maximum size* with a local core improving deviation
9: $\quad C^i$ is obtained from C^{i-1} by a local core improving deviation j_i of agents in T_i, i.e, $C^i_{j_i} = T_i$.
10: $\quad a_i \leftarrow 0$
11: $\quad b_i \leftarrow 0$
12: $\quad$ **for** each $j \in [n]$ **do**
13: $\qquad a_{i,j} \leftarrow 0$
14: $\qquad b_{i,j} \leftarrow 0$
15: $\quad$ **end for**
16: $\quad$ **if** $|T_i| \geq 3$ **then**
17: $\qquad$ **for** each $j \in [n] : C^{i-1}_j \cap T_i \neq \emptyset$ **do**
18: $\qquad\quad$ **if** $j_i = j$ **then**
19: $\qquad\qquad d_j \leftarrow 0$
20: $\qquad\quad$ **else**
21: $\qquad\qquad d_j \leftarrow SW(C^i_j)$
22: $\qquad\quad$ **end if**
23: $\qquad\quad a_{i,j} \leftarrow SW(C^{i-1}_j) - d_j$
24: $\qquad\quad a_i \leftarrow a_i + a_{i,j}$
25: $\qquad\quad b_{i,j} \leftarrow \sum_{u \in T_i \cap C^{i-1}_j} \mu_u(C^i) = |T_i \cap C^{i-1}_j| \frac{|T_i|-1}{|T_i|}$
26: $\qquad\quad b_i \leftarrow b_i + b_{i,j}$
27: $\qquad$ **end for**
28: $\quad$ **else** ▷ In this case $|T_i| = 2$
29: $\qquad$ **for** each $j \in [n] : C^{i-1}_j \cap T_i \neq \emptyset$ **do**
30: $\qquad\quad d_j \leftarrow SW(C^i_j)$
31: $\qquad$ **end for**
32: $\qquad d_{j_i} \leftarrow SW(C^i_{j_i})$
33: $\quad$ **end if**
34: **end while**
35: **return** C^i

It remains to prove that $SW(C'') \geq \frac{9}{16} SW(C')$. In fact, it directly implies the claim, given that $SW(C'') \geq \frac{9}{16} SW(C') \geq \frac{9}{16} \cdot \frac{2}{3} SW(C^*) = \frac{3}{8} SW(C^*)$.

Let us call *crowded deviation* a local core improving deviation involving at least 3 agents.

In the following we prove some properties that will be useful for proving the claim.

Property 5.2. *If an agent u performs a crowded deviation at step i, then she cannot deviate anymore.*

We prove this property by induction on the steps of the algorithm, assuming that in step 0 no agent moves. Namely, we prove that all agents moving at step i by a crowded deviation will never move at any step $i' > i$.

The base of the induction for $i = 0$ is trivially verified.

For the induction step, the induction hypothesis is that for each step $k < i$ it holds that all agents moving at step k through a crowded deviation never move at any step $k' > k$. We have to prove the induction claim for step i.

Assume, by way of contradiction, that u is among the first agents that deviate also at step $i' > i$; it follows that $|T_{i'}| > |T_i|$ because in each deviation (i) u is among the first agents of T_i to move at a step $i' > i$ and (ii) u must join a clique strictly larger than the previous one (otherwise her utility would not increase): hence, $|T_{i'}| \geq 4$. Given that for all agents $v \in [n]$ it holds that (i) $\mu_v(C') \leq \frac{2}{3}$, (ii) agents have utility at most $\frac{1}{2}$ after a non-crowded deviation, and (iii) by the induction hypothesis, no agent involved in a crowded deviation at any step $k < i$ can belong to $T_{i'}$, it follows that the deviation of agents in $|T_{i'}|$ was a local core improving deviation also before step i of algorithm, because in a clique of at least 4 agents each of them has utility at least $\frac{|T_{i'}|-1}{|T_{i'}|} \geq \frac{3}{4}$: a contradiction to the fact that the algorithm considers at each iteration i the set T_i of *maximum size* with a local core improving deviation.

This concludes the proof of Property 5.2.

Property 5.3. $SW(C'') = \sum_{i \geq 1} b_i + \sum_{j \in [n]} d_j$.

Let $A = \bigcup_{i:|T_i| \geq 3} T_i$ be the set of agents performing a crowded deviation at any step i of Algorithm 1. Since (i) b_i is defined in Algorithm 1 as the sum of the utilities of the moving agents and (ii) by Property 5.2 these agents do not move again after step i, we have that $\sum_{u \in A} \mu_u(C'') = \sum_{i \geq 1} b_i$.

Furthermore, for any $j \in [n]$, d_j is initialized to $SW(C'_j)$ and its value, at every step i in which the composition of coalition C^i_j changes and no agent in C^i_j has never performed a crowded deviation at any step $i' \leq i$, is updated to $SW(C^i_j) = \sum_{u \in C^i_j} \mu_u(C^i)$ (at lines 21, 30 and 32 of Algorithm 1). Conversely, for any $j \in [n]$, d_j is set to zero at a step i in which agents T_i, performing a crowded deviation, select strategy j (at line 19 of Algorithm 1). Therefore, it holds that $\sum_{u \notin A} \mu_u(C'') = \sum_{j \in [n]} d_j$.

Hence, $SW(C'') = \sum_{u \in A} \mu_u(C'') + \sum_{u \notin A} \mu_u(C'') = \sum_{i \geq 1} b_i + \sum_{j \in [n]} d_j$ and this concludes the proof of Property 5.3.

Property 5.4. $SW(C') \leq \sum_{j \in [n]} d_j + \sum_{i \geq 1} a_i$.

Notice that for any $j \in [n]$, d_j is initialized to $SW(C'_j)$. Moreover, at each step i in which variables d_j are lowered, of an amount equal to x, from $SW(C^{i-1}_j)$ to either 0 or to $SW(C^i_j)$ (at steps 19 and 21 of Algorithm 1 and with $x = SW(C^{i-1}_j)$ and $x = SW(C^{i-1}_j) - SW(C^i_j)$, respectively), a_i is increased (at lines 23 and 24) exactly of x.

Hence, in order to prove the property it is sufficient to show that lines 30 and 32 never decrease the value of the sum of all involved d_j variables.

To this aim, let u and v be a pair of agents that performs a non-crowded deviation to a matching coalition. In order to perform such a deviation, it must be that the utilities of u and v before the deviation are both strictly less than $\frac{1}{2}$. For this to happen, they must be either leaves of a path coalition, or isolated nodes. Let us analyze the possible deviations at any step i such that $|T_i| = 2$:

- If u and v were both isolated nodes, then the sum of the involved d_j would increase from 0 to 1, because a new matching coalition $\{u, v\}$ is obtained;

- If u was an isolated node and v was a leaf of a path coalition $\{v, x, y\}$ (or vice versa), then the sum of the involved d_j would increase from $\frac{4}{3}$ to 2, because two new matching coalitions $\{u, v\}$ and $\{x, y\}$ are obtained;
- If u and v were both leaves of two path coalitions $\{x, y, u\}$ and $\{v, w, z\}$, then the sum of the involved d_j would increase from $2 \cdot \frac{4}{3} = \frac{8}{3}$ to 3, because three new matching coalitions $\{x, y\}$, $\{u, v\}$ and $\{w, z\}$ are obtained.

This concludes the proof of Property 5.4.

Property 5.5. *For any step $i \geq 1$, it holds that $b_i \geq \frac{9}{16} a_i$.*

First of all, notice that, for any $i \geq 1$, $a_i = \sum_{j \in [n]} a_{i,j}$ and $b_i = \sum_{j \in [n]} b_{i,j}$. In order to prove the property, in the following we show that, for any $i \geq 1$ and any $j \in [n]$, it holds that

$$b_{i,j} \geq \frac{9}{16} a_{i,j}. \tag{2}$$

In fact, fixed any $i \geq 1$, by summing over all j the property directly follows.

Given any $i \geq 1$ and $j \in [n]$, consider $a_{i,j}$ and $b_{i,j}$ that are assigned a value at lines 23 and 25 of Algorithm 1, respectively. Roughly speaking, $a_{i,j}$ is the amount of social welfare that agents in C_j^{i-1}, that have performed no crowded deviation at any step $i' < i$, are losing at step i (notice that among these agents there are also, but not solely, all agents of $|T_i|$), while $b_{i,j}$ is the amount of social welfare that agents in $|T_i|$ will maintain till the end of the algorithm, because, by Property 5.2, any agent $u \in [n]$ can perform at most one crowded deviation. Consider now set $X = C_j^{i-1} \cap T_i$. We distinguish among the following disjoint cases, in which $a_{i,j} > 0$ (notice that if $a_{i,j} = 0$ inequality (2) trivially holds):

- If C_j^{i-1} is a matching coalition (i.e., $SW(C_j^{i-1}) = 1$), each node $u \in X$ is such that $\mu_u(C^i) \geq \frac{2}{3}$, because agent u can improve her utility only joining a coalition of size at least 3: it follows that $b_{i,j} \geq \frac{2}{3}|X|$. Since $a_{i,j} = 1$, we obtain $b_{i,j} \geq \frac{2}{3}|X| \geq \frac{2}{3} \geq \frac{9}{16} = \frac{9}{16} a_{i,j}$.
- If C_j^{i-1} is a triangle coalition (i.e., $SW(C_j^{i-1}) = 2$), each node $u \in X$ is such that $\mu_u(C^i) \geq \frac{3}{4}$, because agent u can improve her utility only joining a coalition of size at least 4: it follows that $b_{i,j} \geq \frac{3}{4}|X|$.
  If $|X| = 1$, since a matching remains in C_j^i, we have $a_{i,j} = 2 - 1 = 1$, and therefore we obtain $b_{i,j} \geq \frac{3}{4} \geq \frac{9}{16} = \frac{9}{16} a_{i,j}$.
  If $|X| = 2$ or $|X| = 3$, we have $a_{i,j} = 2$, and therefore we obtain $b_{i,j} \geq \frac{3}{4}|X| \geq \frac{3}{2} \geq \frac{9}{8} = \frac{9}{16} a_{i,j}$.
- If C_j^{i-1} is a path coalition $\{u, v, w\}$ (i.e., $SW(C_j^{i-1}) = \frac{4}{3}$), if nodes u and/or w are in X, then $\mu_u(C^i) \geq \frac{1}{2}$ and/or $\mu_w(C^i) \geq \frac{1}{2}$, while if node v is is X, then $\mu_v(C^i) \geq \frac{3}{4}$ because agent v can improve her utility only joining a coalition of size at least 4.
  If $X = \{u\}$ or $X = \{w\}$, it holds $b_{i,j} \geq \frac{1}{2}$ and, since a matching remains in C_j^i, we have $a_{i,j} = \frac{4}{3} - 1 = \frac{1}{3}$. Therefore we obtain $b_{i,j} \geq \frac{1}{2} \geq \frac{3}{16} = \frac{9}{16} a_{i,j}$.
  If $X \ni v$, it holds $b_{i,j} \geq \frac{3}{4}$ and we have $a_{i,j} = \frac{4}{3}$. Therefore we obtain $b_{i,j} \geq \frac{3}{4} = \frac{9}{16} a_{i,j}$.
  Notice that u and w cannot belong both to X because C_j^{i-1} is a path coalition and there is no edge between u and w.

This concludes the proof of Property 5.5.

Therefore, it holds that

$$\begin{aligned} SW(C'') &= \sum_{i \geq 1} b_i + \sum_{j \in [n]} d_j && (3) \\ &\geq \frac{9}{16} \sum_{i \geq 1} a_i + \sum_{j \in [n]} d_j && (4) \\ &\geq \frac{9}{16} \left(\sum_{i \geq 1} a_i + \sum_{j \in [n]} d_j \right) \\ &\geq \frac{9}{16} SW(C'), && (5) \end{aligned}$$

where equality (3) holds by Property 5.3, inequality (4) holds by Property 5.5 and inequality (5) holds by Property 5.4. □

6 CONCLUSIONS AND FUTURE WORKS

We addressed the existence of core stable outcomes in SS-FHG. More specifically, we introduced a relaxation of core stability, called local core stability, which takes into account social connection aspects among agents. We showed that a local core stable coalition structure always exists for SS-FHG. Up to our knowledge, before this work, no type of coalition resilient outcomes had been proven to exist in SS-FHG. Moreover, we proved that the local core price of anarchy is at most 4 and that this bound is tight, while the local core price of stability is between 2 and 8/3.

There are some open problems suggested by our work. First, it would be nice to close the gap between the lower and upper bound of the local core price of stability. Next, it could be interesting to study whether any optimal coalition structure is also 2-core stable. In fact, while our lower bound instance in Theorem 5.1 shows that an optimal coalition structure may not be resilient to cliques of at least three nodes, we were not able to prove if this holds also for deviations performed by coalitions which are matchings. It would also be interesting to address the complexity of computing a local core in SS-FHG.

Furthermore, it is worth considering complexity issues that have not been addressed in this work; for instance, even checking whether a coalition structure is in the local core or not is an open problem.

Another interesting research direction could be that of considering the x-local core. More specifically, a coalition structure is in the x-local core if there is no subset of agents which induces a subgraph of G of diameter at most x that can all improve their utility by forming a new coalition together. We notice that in our paper we considered 1-local core. A further research direction could be that of considering x-local core stability to fractional hedonic games in which the input is a weighted undirected graph. In fact, already in a slight modification of SS-FHG admitting edges with weight either 1 or −1, it is not clear whether or not 1-local core outcomes are guaranteed to exist. More generally, it is worth studying the notion of local core stability in general hedonic games.

REFERENCES

[1] Haris Aziz, Felix Brandt, and Paul Harrenstein. 2014. Fractional hedonic games. In *Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems, AAMAS*. 5–12.

[2] Haris Aziz, Serge Gaspers, Joachim Gudmundsson, Julián Mestre, and Hanjo Täubig. 2015. Welfare Maximization in Fractional Hedonic Games. In *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI*. 461–467.

[3] Haris Aziz and Rahul Savani. 2016. Hedonic Games. In *Handbook of Computational Social Choice*. 356–376. https://doi.org/10.1017/CBO9781107446984.016

[4] Davide Bilò, Luciano Gualà, Stefano Leucci, and Guido Proietti. 2016. Locality-Based Network Creation Games. *TOPC* 3, 1 (2016), 6:1–6:26. https://doi.org/10.1145/2938426

[5] Vittorio Bilò, Angelo Fanelli, Michele Flammini, Gianpiero Monaco, and Luca Moscardelli. 2015. On the Price of Stability of Fractional Hedonic Games. In *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems, AAMAS*. 1239–1247.

[6] Vittorio Bilò, Angelo Fanelli, Michele Flammini, Gianpiero Monaco, and Luca Moscardelli. 2018. Nash Stable Outcomes in Fractional Hedonic Games: Existence, Efficiency and Computation. *Journal of Artificial Intelligence* 62 (2018), 315–371.

[7] Florian Brandl, Felix Brandt, and Martin Strobel. 2015. Fractional Hedonic Games: Individual and Group Stability. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems, AAMAS*. 1219–1227.

[8] Raffaello Carosi, Simone Fioravanti, Luciano Gualà, and Gianpiero Monaco. 2019. To appear. Coalition Resilient Outcomes in Max k-Cut Games. In *Proceedings of the 45th International Conference on Current Trends in Theory and Practice of Computer Science, SOFSEM*.

[9] Dinko Dimitrov, Peter Borm, Ruud Hendrickx, and Shao Chin Sung. 2006. Simple Priorities and Core Stability in Hedonic Games. *Social Choice and Welfare* 26, 2 (2006), 421–433. https://doi.org/10.1007/s00355-006-0104-4

[10] J. H. DrÃÍze and J. Greenberg. 1980. Hedonic Coalitions: Optimality and Stability. *Econometrica* 48, 4 (1980), 987–1003. http://www.jstor.org/stable/1912943

[11] Edith Elkind, Angelo Fanelli, and Michele Flammini. 2016. Price of Pareto Optimality in Hedonic Games. In *Proceedings of the 13th Conference on Artificial Intelligence, AAAI*. 475–481.

[12] Michele Flammini, Gianpiero Monaco, Luca Moscardelli, Mordechai Shalom, and Shmuel Zaks. 2018. Online Coalition Structure Generation in Graph Games. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS*. 1353–1361.

[13] Michele Flammini, Gianpiero Monaco, and Qiang Zhang. 2017. Strategyproof Mechanisms for Additively Separable Hedonic Games and Fractional Hedonic Games. In *Proceedings of the 15th International Workshop on Approximation and Online Algorithms, WAOA*. 301–316. https://doi.org/10.1007/978-3-319-89441-6_22

[14] Martin Hoefer, Daniel Vaz, and Lisa Wagner. 2015. Hedonic Coalition Formation in Networks. In *Proceedings of the Twenty-Ninth Conference on Artificial Intelligence, AAAI*. 929–935.

[15] Ayumi Igarashi and Edith Elkind. 2016. Hedonic Games with Graph-restricted Communication. In *Proceedings of the 2016 International Conference on Autonomous Agents & Multiagent Systems, AAMAS*. 242–250.

[16] Christos Kaklamanis, Panagiotis Kanellopoulos, and Konstantinos Papaioannou. 2016. The Price of Stability of Simple Symmetric Fractional Hedonic Games. In *Proceedings of the 9th International Symposium on Algorithmic Game Theory, SAGT*. 220–232. https://doi.org/10.1007/978-3-662-53354-3_18

[17] Stefano Leonardi and Piotr Sankowski. 2007. Network formation games with local coalitions. In *Proceedings of the 26th Annual ACM Symposium on Principles of Distributed Computing, PODC*. 299–305. https://doi.org/10.1145/1281100.1281143

[18] Gianpiero Monaco, Luca Moscardelli, and Yllka Velaj. 2018. Stable Outcomes in Modified Fractional Hedonic Games. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS, 2018*. 937–945.

[19] Martin Olsen. 2012. On Defining and Computing Communities. In *Proceedings of the 18th Conference on Computing: The Australasian Theory Symposium, CATS*. 97–102.

[20] Dominik Peters and Edith Elkind. 2015. Simple Causes of Complexity in Hedonic Games. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence, IJCAI*. 617–623.

[21] Gerhard J. Woeginger. 2013. A hardness result for core stability in additive hedonic games. *Mathematical Social Sciences* 65, 2 (2013), 101–104. https://doi.org/10.1016/j.mathsocsci.2012.10.001