Signed Graph Games: Coalitional Games with Friends, Enemies and Allies

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ABSTRACT

The ability to cooperate is one of the key features of many multi-agent systems. In this paper, we extend the well-known model of graph-restricted games due to Myerson to signed graphs, where the link between any two players may be either positive or negative. Hence, in our model, it is possible to explicitly define not only that some players are friends (as in Myerson's model) but also that some other players are enemies. As such our games can express a wider range of situations, e.g., animosities between political parties. We say that a coalition is feasible if every two players are connected by a path of positive edges and no two players are connected by a negative edge. We define the value for signed graph games using the axiomatic approach that closely follows the celebrated characterisation of the Myerson value. Furthermore, we propose an algorithm for computing an arbitrary semivalue, including the one proposed by us. Moreover, we consider signed graph games with a priori defined alliances (unions) between players and propose an algorithm for the extension of the Owen value to this setting.

KEYWORDS

Coalitional Games; Graph Games; Myerson Value; Signed Graphs

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1 INTRODUCTION

The ability to cooperate is one of the key features of many multi-agent systems [5, 32]. The conventional model of cooperative games is defined by the set of players and the characteristic function that assigns to each coalition (subset) of players a numerical value that reflects the performance of this coalition. The explicit assumption here is that all coalitions are feasible, i.e., each player is able and willing to cooperate in any possible group of players. Since this assumption does not hold in various realistic scenarios, a number of cooperative games with restrictions on the set of coalitions have been proposed in the literature [9, 21].

The most popular such model is a graph-restricted game due to Myerson [16]. In this formalism, cooperation between any two agents is possible if and only if there exists a direct or indirect (i.e., through intermediaries) link between them. This rule naturally extends to coalitions—a coalition can cooperate if and only if all the agents involved induce a connected subgraph. While such model of restricted cooperation seems natural in some settings, it is too simple to describe certain others. For instance, it is hard to expect that, if political parties A and B want to cooperate with each other, as well as parties B and C, then it automatically means that A and C also want to do so. In fact, political parties are often so polarized that any attempt to start cooperation would be very badly received by their supporters [11, 12]. This is the case, for example, in some European countries, where various non-liberal election winners find it hard to create a government without having the overall majority, as more moderate political parties refuse to cooperate. A recent example is the Polish senate, where the ruling party PiS won the most seats (48%) but failed to choose the speaker from PiS, as no other party decided to support them [22]. This was consistent with the election-campaign declarations as all the major opposition parties declared that they would not form a coalition with PiS. A similar situation can be observed in other countries, such as Italy, German and Dutch local governments [4, 28].

Compatibility games, the model introduced by See et al. [23], proposes a way to elevate this restrictive assumption of Myerson's model by requiring that only coalitions that induce cliques can collaborate. Intuitively, while Myerson's games focus on modeling positive relations between players, compatibility games focus on negative relation between players. In more detail, in these latter games, the lack of edge represents a strong negative relation between players as it means that these two players cannot work together. This approach, however, has problems on its own, as its not possible to specify whether a relation is neutral or positive.

In this paper, we propose *signed graph games*—a formalism that encompasses graph-restricted games and compatibility games and is more general than both of them. Specifically, in our model, a game is restricted by a signed graph in which every edge is either positive or negative. Positive edges represent the *friendship* or ability of the adjacent players to directly cooperate, just like in graph-restricted games. In turn, negative edges represent the *animosity* or lack of

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such ability, just like the lack of an edge in compatibility games. The lack of an edge in our formalism represents the ability to cooperate but only through (friendly) intermediaries. For this model:

- We extend to our formalism the concept of the Myerson value [16] which is arguably the most well-known solution concepts for graph-restricted games. Our axiomatic characterization is a direct extension of to the celebrated axiomatization of the standard Myerson value;
- Next, we propose an algorithm for computing the Myerson value for signed graphs in time $O(|\mathcal{F}|(|N| + |E_+|))$ where $|\mathcal{F}|$ is the number of feasible coalitions, $|N|$ is the number of players and $|E_+|$—the number of positive edges. Our algorithm is more general as it can compute any semivalue for signed-graph games, where semivalues are a broad family of values which Myerson value belongs to [6];
- Finally, we propose an algorithm for computing the Owen value extended to signed-graph games with an aim to model a priori *alliances* between players. Our algorithm is the first one in the literature for computing the Owen value for any graph-restricted game.

2 PRELIMINARIES

In this section, we introduce basic definitions and notation.

Coalitional games: Let $N = \{1, \dots, n\}$ be a fixed set of players. A *game* is a pair (N, v), where $v : 2^N \to \mathbb{R}$ is the characteristic function which assigns a real number to each subset of players, or *coalition*, S. A solution concept, or *a value*, of the game is a function that for every game assigns a vector of players' payoffs. For a solution concept φ the payoff of player i in game (N, v) is denoted by $\varphi_i(N, v)$. Arguably, the most well-known solution concept for coalitional games is the Shapley value [24]. It is denoted by SV and defined as follows: for every game (N, v) and every player $i \in N$,

$$SV_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \zeta_n(|S|) \left(v(S \cup \{i\}) - v(S)\right),$$

where $\zeta_n(s) = s(n - s - 1)!/n!$. The expression $v(S \cup \{i\}) - v(S)$ is known as the *marginal contribution* of player i to coalition S and denoted $mc_i(S)$ when game is known from the context.

The Shapley value belongs to the class of *semivalues* [6]. Specifically, a solution concept φ is a *semivalue* if it is of the form:

$$\varphi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{\beta(|S|)}{\binom{n-1}{|S|}} \left(v(S \cup \{i\}) - v(S)\right), \quad (1)$$

for some $\beta : \{0, \dots, n - 1\} \to [0, 1]$ such that $\sum_{k=0}^{n-1} \beta(k) = 1$. We will denote a semivalue based on weights β by φ^β and write $\beta^*(k)$ as a shorthand notation for $\beta(k)/\binom{n-1}{k}$. For the Shapley value, we have $\beta(k) = \frac{1}{n}$. For the Banzhaf value—another well-known semivalue—we have $\beta(k) = \binom{n-1}{k}/2^{n-1}$.

Owen value: Owen [18] generalized the Shapley value to games with a priori alliances (or unions). Let such a priori alliances form a coalition structure, $P = \{T_1, \dots, T_k\}$. Each coalition in P should be considered an inseparable group which naturally affects the value of players. In particular, Owen proposed to assess player i by i's marginal contributions not to all coalitions, but to coalitions that consist of some entire coalitions in P and a subset of players from the coalition in P that contains i. Specifically, assume $i \in T_j$ and for $M \subseteq \{1, \dots, k\}$ let us define $T_M = \bigcup_{l \in M} T_l$. Owen proposed the following value of player i:

$$OV_i^P(N, v) = \sum_{M \subseteq \{1,\dots,k\} \setminus \{j\}} \sum_{R \subseteq T_j \setminus \{i\}} \zeta_i^P(T_M \cup R) mc_i(T_M \cup R), \quad (2)$$

where $\zeta_i^P(T_M \cup R) = \zeta_k(|M|)\zeta_{|T_j|}(|R|)$. This solution concept is now known as the *Owen value*.

Note that the Owen value is not a semivalue since not all coalitions of the same size are treated equally.

Graphs: A *graph* (undirected, unweighted) is a pair $G = (N, E)$, where N is the set of nodes, and E is the set of edges, i.e., unordered pairs of nodes. Two nodes are *adjacent* if there is an edge between them. Nodes adjacent to node i are called *neighbors* of i; the set of neighbors of i is denoted by $\mathcal{N}(i)$. For a set of nodes $S \subseteq N$, we define the set of neighbors as follows: $\mathcal{N}(S) = \bigcup_{i \in S} \mathcal{N}(i) \setminus S$.

A graph is a *clique* if every two nodes are adjacent. A graph is an *empty graph* if it has no edges (i.e., $E = \emptyset$). For any subset of edges $M \subseteq E$ we define $M[S] = \{\{i, j\} \in M : i, j \in S\}$. Now, for any subset of nodes $S \subseteq N$ the *subgraph induced by* S is denoted by $G[S]$ and is defined as follows: $G[S] = (S, E[S])$.

A *path* is a sequence of nodes $(i_1, \dots, i_k)$ such that every two consecutive nodes are adjacent. A graph is *connected* if there exists a path between every two nodes in the graph. If graph G is not connected, then its nodes can be uniquely partitioned into maximal connected subsets of nodes, called *(connected) components*; we denote this partition by $K(G)$.

A *signed graph* is a graph (N, E) with a label function $l : E \to \{+, -\}$. All of the above concepts naturally apply to signed graphs by ignoring the labels. For notational convenience, we will denote a signed graph by a triple $G^\pm = (N, E_+, E_-)$, where N is the set of nodes, E_+ is the set of *positive edges* (edges labeled +) and E_- is the set of *negative edges* (edges labeled −) with the assumption that $E_+ \cap E_- = \emptyset$. We also define $\mathcal{N}_+(i) = \{j \in N : \{i, j\} \in E_+\}$, $\mathcal{N}_-(i) = \{j \in N : \{i, j\} \in E_-\}$ and $\mathcal{N}_\pm(i) = \mathcal{N}_+(i) \cup \mathcal{N}_-(i)$. For $S \subseteq N$, we define $G^\pm[S] = (S, E_+[S], E_-[S])$ and $\mathcal{N}_\pm(S) = \bigcup_{i \in S} \mathcal{N}_\pm(i) \setminus S$.

Graph Games: A *graph-restricted game* [16], or shortly a *graph game*, is a tuple (N, v, E) where (N, v) is a coalitional game and $G = (N, E)$ is a *communication graph*. In graph games, a coalition S is feasible if $G[S]$ is a connected graph.

For a game (N, v) and a graph $G = (N, E)$, Myerson [16] defined game v/G as follows:

$$v/G(S) = \sum_{C \in K(G[S])} v(C), \quad (3)$$

and proved that the Shapley value of this game is the only solution concept that satisfies two desirable properties: *Fairness* and *Component Efficiency*. This solution concept is now known as *the Myerson value* and denoted by MV: $MV(N, v, E) = SV(N, v/G)$.

Two Myerson's axioms are defined as follows:

Fairness $\varphi_i(N, v, E) - \varphi_i(N, v, E \setminus \{e\}) = \varphi_j(N, v, E) - \varphi_j(N, v, E \setminus \{e\})$ for every game (N, v, E) and edge $e = \{i, j\} \in E$.

Component Efficiency $\sum_{i \in C} \varphi_i(N, v, E) = v(C)$, for every graph game (N, v, E) and component $C \in K(G)$.

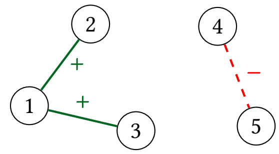

Figure 1: A sample signed graph. Positive edges are green/solid and negative edges are red/dashed.

	graph (N, E_+, E_-) is a clique	graph (N, E_+, E_-) is not a clique
no negative edges $\lvert E_-\rvert = 0$	coalitional games	graph games
negative edges $\lvert E_-\rvert > 0$	compatibility games	**signed graph games**

Table 1: Signed graph games generalize standard coalitional games, graph games and compatibility games.

Fairness states that adding or removing an edge between two players affects values of both of them equally. Component Efficiency states the the sum of values of all players in a connected component is equal to the value of a component.

Other solution concepts have also been defined for graph games by applying their standard definition to game v/G. In particular, the Banzhaf value of game v/G was studied by Owen [19]; it is now called the *restricted Banzhaf value* [29]. On the other hand, the Owen value of game v/G was studied by Vázquez-Brage et al. [30].

Compatibility Games: A *compatibility game* is a tuple (N, v, E), where (N, v) is a coalitional game and (N, E) is a *compatibility graph*. In these games, a coalition S is feasible if $G[S]$ is a clique.

Compatibility games were proposed by See et al. [23], but specifically for weighted voting games. Our definition generalizes this concept to arbitrary coalitional games. See et al. [23] proposed the Shapley-Shubik index and the Banzhaf index for these games. We refer the reader to the original paper for their definition, noting that both values do not satisfy basic properties, such as Fairness.

3 OUR MODEL

Let us now introduce coalitional games restricted by signed graphs:

Definition 3.1. A *signed graph game*, is a quadruple (N, v, E_+, E_-), where (N, v) is a coalitional game and $G^{\pm} = (N, E_+, E_-)$ is a signed graph. In signed graph games, a coalition S is *feasible* if $G^{\pm}[S]$ is connected and does not contain negative edges from E_-. That is, the set of all feasible coalitions, $\mathcal{F}(G^{\pm})$, is defined as:

$$\mathcal{F}(G^{\pm}) = \{S \subseteq N : G^{\pm}[S] \text{ is connected and } E_-[S] = \emptyset\}.$$

We will write simply $\mathcal{F}$ when the graph is known from the context.

In signed graph games, positive edges represent sympathy between players or ability to cooperate directly. In turn, negative edges represent antipathy between players or inability to cooperate. Furthermore, no edge means that the players might cooperate through intermediaries.

Example 3.2. Consider a signed graph game (N, v, E_+, E_-) where the signed graph, $G^{\pm} = (N, E_+, E_-)$, is depicted in Figure 1 and $v(S) = 2|S| - 1$ for every $S \subseteq N$. The set of feasible coalitions is:

$$\mathcal{F}(G^{\pm}) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}.$$

Signed graph games are a combination of graph games by Myerson [16] and compatibility games by See et al. [23]. Let us describe this relation in more detail.

Consider graph game (N, v, E_+) by Myerson [16] with $G^+ = (N, E_+)$. In this game, the set of feasible coalitions consists of connected coalitions:

$$\mathcal{F}_M(G^+) = \{S \subseteq N : G^+[S] \text{ is connected}\}.$$

Now, consider compatibility games. In particular, since in a signed graph game, two players are compatible if they are *not* connected by a negative edge, then let us consider compatibility game $(N, v, \overline{E}_-)$ with $G^- = (N, \overline{E}_-)$, where $\overline{E}_- = \{\{i, j\} \subseteq N : i \neq j, \{i, j\} \notin E_-\}$ is a complement of E_-, i.e., it is the set of pairs of nodes which are not connected with a negative edge. In this game, the set of feasible coalitions consists of coalitions that form a clique in graph G^-, i.e., coalitions without negative edges in graph $G^{\pm}$:

$$\mathcal{F}_c(G^-) = \{S \subseteq N : E_-[S] = \emptyset\}.$$

Now, we observe that the set of feasible coalitions in a signed graph game is the intersection of both sets:

$$\mathcal{F}(G^{\pm}) = \mathcal{F}_M(G^+) \cap \mathcal{F}_c(G^-).$$

In particular, if graph is a clique without negative edges, then $\mathcal{F}_M(G^+) = \mathcal{F}_c(G^-) = 2^N$ and a signed graph game is equivalent to a standard coalitional game (N, v). In turn, if graph does not contain negative edges, but it is not a clique, then $\mathcal{F}_c(G^-) = 2^N$ and a signed graph game is equivalent to graph game (N, v, E_+). Finally, if graph is a clique, but it contains negative edges, then $\mathcal{F}_c(G^-) \subseteq \mathcal{F}_M(G^+)$ and a signed graph game is equivalent to a compatibility game $(N, v, \overline{E}_-)$. Table 1 summarizes these observations.

4 THE MYERSON VALUE FOR SIGNED GRAPHS

The Myerson value is defined as the Shapley value of coalitional game v/G that defines values not only of feasible, but also of infeasible coalitions (Eq. (3)). Following this, one could try to extend the Myerson value to signed graphs by first specifying game $v/G^{\pm}$ for signed graph $G^{\pm}$. However, it is not immediately clear what the value of an infeasible coalition, S, in game $v/G^{\pm}$ should be. We illustrate this in the following example:

Example 4.1. Consider again the signed graph game from Example 3.2. What should be the value of $v/G^{\pm}(S)$ for $S = \{1, 2, 4, 5\}$?

- In the most strict approach, every coalition that contains incompatible players should have value zero. In result, we get that $v/G^{\pm}(S) = 0$.
- In the most liberal approach, every infeasible coalition can partition themselves into smaller parts that can work with each other. In particular, coalition S can partition themselves into three connected parts $\{1, 2\}$, $\{4\}$ and $\{5\}$ and get the value $v/G^{\pm}(S) = 3 + 1 + 1 = 5$. Note, however, that such a partition is not unique and choosing an optimal partition would violate Linearity of the solution. Moreover, according to such a definition, if game is not superadditive, then negative edges

could increase the value of a coalition which is counterintuitive. To see this, assume $v(\{1,2,3\}) < v(\{1,2\}) + v(\{3\})$ (e.g., $v(\{1,2,3\}) = 3$). Now, adding a negative edge between 2 and 3 increases the value of $v/G^{\pm}(\{1,2,3\})$, as coalition $\{1,2,3\}$ partitions itself into two groups: $\{1,2\}$ and $\{3\}$.

Given this, instead of trying to specify game $v/G^{\pm}$, we follow Myerson's approach and focus on axioms. As we will see, the definition of $v/G^{\pm}$ will naturally follow from them.

Consider the following two axioms:

Signed Fairness $\varphi_i(N, v, E_+, E_-) - \varphi_i(N, v, E_+ \setminus \{e\}, E_- \setminus \{e\}) = \varphi_j(N, v, E_+, E_-) - \varphi_j(N, v, E_+ \setminus \{e\}, E_- \setminus \{e\})$ for every signed graph game (N, v, E_+, E_-) and edge $e = \{i, j\} \in E_+ \cup E_-$.

Signed Component Efficiency

$$\sum_{i \in C} \varphi_i(N, v, E_+, E_-) = \begin{cases} v(C) & \text{if } C \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases}$$

for every signed graph game (N, v, E_+, E_-) and component $C \in K(N, E_+, E_-)$.

Both axioms are the direct translations of Myerson's original axioms (see Section 2). Signed Fairness states that adding or removing a positive or a negative edge should have equal impact on the incident nodes. The definition of Signed Component Efficiency focuses on components in $G^{\pm}$, i.e., coalitions such that no player inside is connected to any player outside with a positive or negative edges. We observe that only these coalitions can be considered independent of the rest of the graph. The axiom states that if such a component contains incompatible players, i.e., two players connected by a negative edge, then it is infeasible and it cannot achieve anything; if all players are (pairwise) compatible, then the coalition is feasible and can obtain their value in game v.

In the following theorem, we prove that these axioms uniquely characterize a value. This value, that we call *the Myerson value for signed graphs*, is defined as the Shapley value of game $v/G^{\pm}$, i.e.:

$$MV(N, v, E_+, E_-) = SV(N, v/G^{\pm}), \tag{4}$$

where game $v/G^{\pm}$ restricted by a signed graph $G^{\pm} = (N, E_+, E_-)$ is defined as follows:

$$v/G^{\pm}(S) = \sum_{C \in K(G^{\pm}[S]) \cap \mathcal{F}} v(C). \tag{5}$$

Let us comment on this formulation. Eq. (5) can be interpreted in the following way: coalition S splits into its components $K(G^{\pm}[S])$ and then each component C contributes either its value $v(C)$ or 0 if it contains incompatible players. Hence, groups of players which are not aware of (or not in contact with) an incompatible pair of players are not affected by their conflict.

As an illustration, consider a problem of sitting wedding reception guests at (separate) tables. We consider a table acceptable if it does not contain strongly conflicted guests and every two people know each other directly or through common friends sitting at the same table. Now, consider a hypothetical table with two or more separate groups of friends that do not know each other at all. If there are some conflicted guests in one of these groups, the remaining guests of this group may feel uncomfortable which will minimize their satisfaction. However, this does not affect other groups at the table not aware of this conflict.

The following theorem is our main result in this section:

Theorem 4.2. *The Myerson value for signed graphs (Eq.* (4)*) is the unique value that satisfies Signed Fairness and Signed Component Efficiency.*

Proof. Let us begin by showing that the Myerson value for signed graphs satisfies Signed Fairness and Signed Component Efficiency. We begin with the former axiom. Fix a signed graph game (N, v, E_+, E_-) with $G^{\pm} = (N, E_+, E_-)$ and $e = \{i, j\} \in E_+ \cup E_-$. Consider graph $G^{\pm}_* = (N, E_+ \setminus \{e\}, E_- \setminus \{e\})$. From Eq. (4) and Linearity of the Shapley value (recall that the Shapley value is a semivalue, hence it satisfies Linearity) we get:

$$\begin{aligned} &MV(N, v, E_+, E_-) - MV(N, v, E_+ \setminus \{e\}, E_- \setminus \{e\}) = \\ &\quad SV(N, v/G^{\pm}) - SV(N, v/G^{\pm}_*) = SV(N, v/G^{\pm} - v/G^{\pm}_*). \end{aligned} \tag{6}$$

Consider game $(N, v/G^{\pm} - v/G^{\pm}_*)$. From Eq. (5) we know that for every coalition $S \subseteq N$ values $v/G^{\pm}(S)$ and $v/G^{\pm}_*(S)$ depend solely on the subgraph induced by S; hence, they are equal if $\{i, j\} \nsubseteq S$:

$$v/G^{\pm}(S) = v/G^{\pm}_*(S) \text{ for every } S \subseteq N : \{i, j\} \nsubseteq S.$$

In result, in game $(N, v/G^{\pm} - v/G^{\pm}_*)$ only coalitions containing both i and j have non-zero values. From Symmetry of the Shapley value, we get $SV_i(N, v/G^{\pm} - v/G^{\pm}_*) = SV_j(N, v/G^{\pm} - v/G^{\pm}_*)$ which combined with Eq. (6) implies Signed Fairness.

Let us turn out attention to Signed Component Efficiency. We will prove that the Myerson value for signed graphs is equal to the Myerson value for a (standard) graph game with a modified characteristic function. Specifically, define $\tilde{v} : 2^N \to \mathbb{R}$ as follows:

$$\tilde{v}(S) = \begin{cases} v(S) & \text{if } E_-[S] = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We will prove that $MV(N, v, E_+, E_-) = MV(N, \tilde{v}, E_+ \cup E_-)$.

Note that $\tilde{v}$ is a game obtained from v by assigning value 0 to every coalition that contains players connected by a negative edge. Consider graph $G = (N, E_+ \cup E_-)$ obtained from $G^{\pm}$ by ignoring signs of edges. Now, for an arbitrary coalition $S \subseteq N$ we get that:

$$\tilde{v}/G(S) = \sum_{C \in K(G[S])} \tilde{v}(C) = \sum_{C \in K(G^{\pm}[S]) \cap \mathcal{F}} v(C) = v/G^{\pm}(S).$$

Here, we used Eq. (3), (7) and (5) in this order and the fact that $K(G[S]) = K(G^{\pm}[S])$. Hence, $\tilde{v}/G = v/G^{\pm}$ and $SV(\tilde{v}/G) = SV(v/G^{\pm})$ which implies that $MV(N, v, E_+, E_-) = MV(N, \tilde{v}, E_+ \cup E_-)$.

Now, from the Component Efficiency of the Myerson value we get that for every $C \in K(G^{\pm})$ we have $\sum_{i \in C} MV_i(N, v, E_+, E_-) = \tilde{v}(C)$ which, by the definition, is equal to $v(C)$ if $C \in \mathcal{F}$ and to 0 otherwise. This concludes the proof that the Myerson for signed graph games satisfies both axioms.

In the remainder of the proof, we show uniqueness. We will use the scheme of the proof proposed by Myerson [16] in his seminal work. Specifically, assume φ, φ' both satisfy Signed Fairness and Signed Component Efficiency. We will prove by contradiction that $\varphi(N, v, E_+, E_-) = \varphi'(N, v, E_+, E_-)$ for every game (N, v, E_+, E_-).

Assume otherwise and let (N, E_+, E_-) be a signed graph with the minimal number of edges s.t. $\varphi(N, v, E_+, E_-) \neq \varphi'(N, v, E_+, E_-)$. If $|E_+| + |E_-| = 0$ (i.e., $E_+ = E_- = \emptyset$), then from Signed Component Efficiency for every $i \in N$ we have $\varphi_i(N, v, E_+, E_-) = v(\{i\}) = \varphi'_i(N, v, E_+, E_-)$ which is a contradiction. Assume otherwise that

$|E_+| + |E_-| > 0$ and let $e = \{i, j\} \in E_+ \cup E_-$ be an arbitrary edge. From Signed Fairness we know that for $\varphi^* \in \{\varphi, \varphi'\}$ we have:

$$\varphi_i^*(N, v, E_+, E_-) - \varphi_j^*(N, v, E_+, E_-) = \\ \varphi_i^*(N, v, E_+ \setminus \{e\}, E_- \setminus \{e\}) - \varphi_j^*(N, v, E_+ \setminus \{e\}, E_- \setminus \{e\}). \quad (8)$$

For a player $k \in N$, let us denote by δ_k the difference in payoffs of k according to both values φ and φ' (i.e., $\delta_k = \varphi_k(N, v, E_+, E_-) - \varphi'_k(N, v, E_+, E_-)$). Since the right-hand side of Eq. (8) is equal for both values φ and φ', we get $\delta_i = \delta_j$ for every edge $\{i, j\} \in E_+ \cup E_-$. Moreover, for an arbitrary path $(i, j, \dots, k)$ we get $\delta_i = \delta_j = \dots = \delta_k$. Now, fix an arbitrary $i \in N$ and let $C \in K(G^\pm)$ be a component containing i. Then, we have:

$$|C| \cdot \delta_i = \sum_{j \in C} \delta_j = \sum_{j \in C} \varphi_j(N, v, E_+, E_-) - \sum_{j \in C} \varphi_j(N, v, E_+, E_-) = 0,$$

where the last equality comes from Signed Component Efficiency. We have just showed that $\delta_i = 0$ for every $i \in N$ which yields a contradiction. This concludes the proof of Theorem 4.2. □

Example 4.3. Consider again the signed graph game from Example 3.2. Recall that the feasible coalitions have the following values: $v(\{i\}) = 1$ for $i \in N$, $v(\{1, 2\}) = v(\{1, 3\}) = 3$ and $v(\{1, 2, 3\}) = 5$. Thus, game $v/G^\pm$ is defined as follows:

S	$v/G^\pm$	S	$v/G^\pm$	S	$v/G^\pm$	S	$v/G^\pm$
$\emptyset$	0	$\{2\}$	1	$\{3,4\}$	2	$\{2,3,4\}$	3
$\{1\}$	1	$\{1,2\}$	3	$\{3,5\}$	2	$\{2,3,5\}$	3
$\{3\}$	1	$\{2,3\}$	2	$\{4,5\}$	0	$\{2,4,5\}$	1
$\{4\}$	1	$\{2,4\}$	2	$\{1,3,4\}$	4	$\{1,2,3,4\}$	6
$\{5\}$	1	$\{2,5\}$	2	$\{1,3,5\}$	4	$\{1,2,3,5\}$	6
$\{1,3\}$	3	$\{1,2,3\}$	5	$\{1,4,5\}$	1	$\{1,2,4,5\}$	3
$\{1,4\}$	2	$\{1,2,4\}$	4	$\{3,4,5\}$	1	$\{2,3,4,5\}$	2
$\{1,5\}$	2	$\{1,2,5\}$	4	$\{1,3,4,5\}$	3	$\{1,2,3,4,5\}$	5

As expected, the highest value is obtained by coalitions that contain all three cooperating players $\{1, 2, 3\}$, and—to avoid a conflict—only one of the remaining two players. By looking at the marginal contributions we get that $MV_2 = 3/2$. From Symmetry, we have $MV_3 = 3/2$ and from Signed Component Efficiency: $MV_1 = 2$. Finally, from Symmetry and Signed Component Efficiency we have $MV_4 = MV_5 = 0$.

5 ALGORITHM FOR SEMIVALUES

In this section, we extend the definition of the Myerson value to all semivalues for signed graphs and propose an algorithm to compute every such semivalue in linear time in the size of the (potentially exponential) input. Note that since values of infeasible coalitions will not be taken into account by semivalues, the input for the computational problem is a function v from feasible coalitions into real values: $v : \mathcal{F} \to \mathbb{R}$; hence, the size of the input is $|\mathcal{F}|$.

Recall that the Myerson value for signed graphs was defined as the Shapley value of game $v/G^\pm$. In the same spirit, we define a semivalue for signed graph games as the semivalue of game $v/G^\pm$ that takes into account restrictions of the signed graph:

$$\varphi^\beta(N, v, E_+, E_-) = \varphi^\beta(N, v/G^\pm),$$

We start our analysis with the following lemma that characterizes a marginal contribution of a player in signed graph games.

Lemma 5.1. *For every signed graph game (N, v, E_+, E_-) with graph $G^\pm = (N, E_+, E_-)$, player $i \in N$ and coalition $S \subseteq N \setminus \{i\}$ we have:*

$$v/G^\pm(S \cup \{i\}) - v/G^\pm(S) = \sum_{\substack{C \in K(G^\pm[S \cup \{i\}]) \\ C \in \mathcal{F}, i \in C}} v(C) - \sum_{\substack{C \in K(G^\pm[S]) \\ C \in \mathcal{F}, i \in \mathcal{N}_\pm(C)}} v(C).$$

Proof. Let $mc_i(S) = v/G^\pm(S \cup \{i\}) - v/G^\pm(S)$. From Eq. (5):

$$mc_i(S) = \sum_{C \in K(G^\pm[S \cup \{i\}]) : C \in \mathcal{F}} v(C) - \sum_{C \in K(G^\pm[S]) : C \in \mathcal{F}} v(C).$$

Consider a component $C \in K(G^\pm[S])$. If i is not a neighbor of coalition C, then C is also a component of $K(G^\pm[S \cup \{i\}])$ and $v(C)$ appears in both sums on the right-hand side. Otherwise, C is not a component of $K(G^\pm[S \cup \{i\}])$, but only a part of the component that contains player i. This component can be identified as $C \in K(G^\pm[S \cup \{i\}]) \cap \mathcal{F}$ such that $i \in C$. This concludes the proof. □

We illustrate Lemma 5.1 with the following example.

Example 5.2. Consider the contribution of player 1 to coalition $\{2, 3, 4\}$ in game $v/G^\pm$ from Example 3.2. From the definition of game $v/G^\pm$, we have:

$$mc_1(\{2, 3, 4\}) = v(\{1, 2, 3\}) + v(\{4\}) - v(\{2\}) - v(\{3\}) - v(\{4\}).$$

This simplifies to $v(\{1, 2, 3\}) - v(\{2\}) - v(\{3\})$. Hence, in mc_1: (a) there appears with the positive sign only the value of the component in $G^\pm[\{1, 2, 3, 4\}]$ that player 1 belongs to and (b) there appear with the negative sign all the values of components of $G^\pm[\{2, 3, 4\}]$ adjacent to player 1.

The following theorem characterizes an arbitrary semivalue as a sum over feasible coalitions.

Theorem 5.3. *The semivalue φ^β for signed graph games satisfies the following formula:*

$$\varphi_i^\beta(N, v, E_+, E_-) = \sum_{C \in \mathcal{F}, i \in C} \gamma_i^\beta(C) \cdot v(C) - \sum_{C \in \mathcal{F}, i \in \mathcal{N}_\pm(C)} \gamma_i^\beta(C) \cdot v(C)$$

for every signed graph game (N, v, E_+, E_-) and player $i \in N$, where $\gamma_i^\beta(C) = \sum_{t=0}^{n-|C|-|\mathcal{N}_\pm(C)|} \binom{n-|C|-|\mathcal{N}_\pm(C)|}{t} \beta^*(t + |C \setminus \{i\}|)$.

Proof. From Lemma 5.1 combined with Eq. (1) we get that $\varphi_i^\beta(N, v, E_+, E_-)$ equals:

$$\sum_{\substack{C \in \mathcal{F} \\ i \in C}} \sum_{\substack{S \subseteq N \setminus \{i\} \\ C \in K(G^\pm[S \cup \{i\}])}} \beta^*(|S|) v(C) - \sum_{\substack{C \in \mathcal{F} \\ i \in \mathcal{N}_\pm(C)}} \sum_{\substack{S \subseteq N \setminus \{i\} \\ C \in K(G^\pm[S])}} \beta^*(|S|) v(C).$$

Now, a feasible coalition C that contains player i is a component of graph $G^\pm[S \cup \{i\}]$ if and only if: (1) $C \subseteq S \cup \{i\}$; and (2) $\mathcal{N}_\pm(C) \cap S = \emptyset$. Hence, each such a coalition is of the form $S = (C \setminus \{i\}) \cup T$, where $T \subseteq N \setminus (C \cup \mathcal{N}_\pm(C))$. There are $\binom{n-|C|-|\mathcal{N}_\pm(C)|}{t}$ such coalitions of size $t + |C|$ for every $t \in \{0, \dots, n - |C| - |\mathcal{N}_\pm(C)|\}$. In result, we get that the sum of weights β^* of all such coalitions equals $\gamma_i^\beta(C)$ for function γ_i^β defined in the theorem statement.

The analogous analysis of the second sum on the right-hand side concludes the proof. □

Example 5.4. Consider again the signed graph game from Example 3.2. We will calculate the Myerson value of player 2 again, but this time using Theorem 5.3. Recall, that the Myerson value is, equivalently, the Shapley value of a signed graph game and the Shapley value is a semivalue with weights $\beta^*(k) = \frac{1}{5}/\binom{4}{k}$ for $n = 5$.

There are three feasible coalitions with player 2: $\{2\}$, $\{1,2\}$ and $\{1,2,3\}$. We get that $\gamma_2^\beta(\{2\}) = \binom{3}{0}\frac{1}{5} + \binom{3}{1}\frac{1}{20} + \binom{3}{2}\frac{1}{30} + \binom{3}{3}\frac{1}{20} = 1/2$. Analogously, $\gamma_2^\beta(\{1,2\}) = 1/6$ and $\gamma_2^\beta(\{1,2,3\}) = 1/3$.

Then, there are two feasible coalitions for which player 2 is a neighbor: $\{1\}$ and $\{1,3\}$. For them, we get: $\gamma_2^\beta(\{1\}) = 1/6$ and $\gamma_2^\beta(\{1,3\}) = 1/3$. Eventually, we have:

$$MV_2 = \frac{v(\{2\})}{2} + \frac{v(\{1,2\})}{6} + \frac{v(\{1,2,3\})}{3} - \frac{v(\{1\})}{6} - \frac{v(\{1,3\})}{3} = \frac{3}{2}.$$

From Theorem 5.3 we know that it is enough to traverse all feasible coalitions in order to compute the semivalue of all the players in the game. Every feasible coalition induces a connected subgraph. Hence, to enumerate them all, we will modify the existing algorithm that traverses all induced connected subgraphs used in calculation of the (standard) Myerson value [26]. Note that this algorithm applied directly would yield worse than linear complexity because the number of feasible coalitions (i.e., $|\mathcal{F}|$) can be significantly smaller than the number of induced connected subgraphs.

Algorithm 1: Computing a semivalue for signed graph games

Input: A signed graph $G^\pm = (N, E_+, E_-)$, function $v : \mathcal{F} \to \mathbb{R}$
Output: A semivalue $\varphi_i^\beta(N, v, E_+, E_-)$ for every $i \in N$

1 **begin**
2 **foreach** $i \in N$ **do** $\varphi_i^\beta \leftarrow 0$
3 $color \leftarrow$ array [1,…,n] of {White, Red, RedN} $\cup N$
4 **for** $i \leftarrow 1$ **to** n **do**
5 $color \leftarrow [\underbrace{\text{Red}, \dots, \text{Red}}_{i-1}, i, \underbrace{\text{White}, \dots, \text{White}}_{n-i}]$
6 Rec(*color, i, 1, UpdateSemivalues*)
7 **end**
8 **end**
9 **Function** Rec(*color, i, startIt, updateFun*):
10 **if** $startIt = 1$ **then**
11 **foreach** $j \in \mathcal{N}_-(i) : color[j] \notin N$ **do** $color[j] \leftarrow$ RedN
12 **end**
13 **while** $startIt \leq |\mathcal{N}_+(i)|$ **do**
14 $j \leftarrow \mathcal{N}_+(i).get(startIt)$
15 **if** $color[j] =$ *White* **then**
16 $color[j] \leftarrow i$
17 Rec(*color (array clone), j*, 1, *updateFun*)
18 $color[j] \leftarrow$ RedN
19 **else if** $color[j] =$ *Red* **then** $color[j] \leftarrow$ RedN
20 $startIt \leftarrow startIt + 1$
21 **end**
22 **if** $color[i] \neq i$ **then**
23 $startIt \leftarrow \mathcal{N}_+(color[i]).getIndex(i) + 1$
24 Rec(*color, color[i], startIt, updateFun*)
25 **else** call $updateFun(color)$
26 **Function** UpdateSemivalues(*color*):
27 $C \leftarrow \{j \in N : color[j] \in N\}$
28 $NC \leftarrow \{j \in N : color[j] = \text{RedN}\}$
29 $\gamma_{in}^\beta \leftarrow \sum_{t=0}^{n-|C|-|NC|} \binom{n-|C|-|NC|}{t} \beta^*(t + |C| - 1)$
30 **if** $NC \neq \emptyset$ **then** $\gamma_{out}^\beta \leftarrow \sum_{t=0}^{n-|C|-|NC|} \binom{n-|C|-|NC|}{t} \beta^*(t + |C|)$
31 **foreach** $j \in C$ **do** $\varphi_j^\beta \leftarrow \varphi_j^\beta + \gamma_{in}^\beta \cdot v(C)$
32 **foreach** $j \in NC$ **do** $\varphi_j^\beta \leftarrow \varphi_j^\beta - \gamma_{out}^\beta \cdot v(C)$

Theorem 5.5. *Algorithm 1 computes the semivalue φ^β for signed graph games in time $O(|\mathcal{F}|(|N| + |E_+|))$.*

Proof. The input of Algorithm 1 is the graph G and function v that given a feasible coalition outputs its value. As standard in the literature, we assume that the function v is given by a black box, i.e., an *oracle* [5]. The graph is represented as adjacency lists, $\mathcal{N}_+(i)$ and $\mathcal{N}_-(i)$ for every $i \in N$; these lists provide two functions: $get(it)$ (returns a player at position it) and $getIndex(j)$ (returns a position of player j).

The heart of the algorithm is the recursive function Rec that enumerates all feasible coalitions consistent with the *color* table. The *color* table contains information which players are in the coalition (Gray) and which are not (Red); the remaining players are White. Additionally, among Red players neighbors of Gray players are marked as RedN and for Gray players, instead of a color, the index of a player from which this one was reached (i.e., a "parent") is kept in the *color* table.

To consider all feasible coalitions consistent with the *color* table, Rec focuses on a Gray player i (given by a parameter) and considers i's neighbors. We have two cases:

- If this is the first time Rec is called with i as a parameter, then all players connected with i by a negative edge are marked as forbidden neighbors (lines 10–12)—in this way, we make sure that we list only feasible coalitions. Then all neighbors connected with i by a positive edge are considered one by one (lines 13–21).
- If this is not the first time Rec is called with i as a parameter, then the function goes directly to the first not-yet considered neighbor—its index is given by the parameter $startIt$.

Lines 14–20 consider player j that is a neighbor of i. If j is Gray or RedN, then the algorithm does not do anything. If j is Red, then its color is changed to RedN to mark that it is a neighbor of a Gray player. Now, assume j is White; then the algorithm has to account for all feasible coalitions with and without player j. To enumerate coalitions with j, player j is colored Gray and player i is stored as a parent of player j in the *color* table. Then, function Rec is called with a new *color* table and player j as the starting player. In turn, to enumerate coalition without j, player j is colored RedN to avoid repetitions and the algorithm moves on to the next neighbor of i.

After considering all neighbors, in lines 22-25, function Rec backtracks, i.e., goes back to the parent of player i, i.e., $color[i]$, by calling function Rec with $color[i]$ as the starting player and parameter $startIt$ set in a way that considering neighbors of $color[i]$ will start from a player that appears on the list of neighbors $\mathcal{N}_+$ right after player i (lines 23-24). Note that cloning the table is not necessary here, as *color* table will not longer be used in this Rec call. If $color[i] = i$ (line 25), it means that i was the starting player of the initial call to Rec from line 6. Thus, all neighbors of all Gray players have been considered (and colored RedN), a feasible coalition is found. Hence, function *updateFun* given by a parameter is called to update the values of players. In Algorithm 1, *updateFun* is always equal to UpdateSemivalues, which is based on Theorem 5.3.

Finally, let us discuss the main body of the algorithm (lines 1–8). To enumerate all feasible coalitions, function Rec is called n times: i-th iteration enumerates all feasible coalitions in which player i is the player with the minimal index. This is obtained by coloring all players with smaller indices Red and player i Gray and calling function Rec with player i as the starting player (lines 5–6).

Note that, for every feasible coalition C, the number of steps performed in all the calls of the Rec function with the *color* table in which only players from C are Gray is bounded by $|N|+|E_+|$. Hence, the time complexity of our algorithm is $O(|\mathcal{F}|(|N|+|E_+|))$. □

Example 5.6. Consider Algorithm 1 for the signed graph game from Example 3.2. The consecutive calls of function Rec are:

#	color	i	sIt	#	color	i	sIt
#1	$[1,W,W,W,W]$	1	1	**#8**	$[R,2,W,W,W]$	2	1
. #2	$[1,1,W,W,W]$	2	1	**#9**	$[R,R,3,W,W]$	3	1
.. **#3**	$[1,1,W,W,W]$	1	2	**#10**	$[R,R,R,4,W]$	4	1
... #4	$[1,1,1,W,W]$	3	1	**#11**	$[R,R,R,R,5]$	5	1
.... **#5**	$[1,1,1,W,W]$	1	3				
. #6	$[1,R,1,W,W]$	3	1				
.. **#7**	$[1,R,1,W,W]$	1	3				

Here, we used W to denote White and R to denote Red or RedN in the *color* table. Also, the first column illustrates the call hierarchy. The number in the first column is bold if a given Rec call changes semivalues, i.e., calls the UpdateSemivalues function.

6 ALGORITHM FOR THE OWEN VALUE

Our model allows for the representation of friendship or enmity between players. In this section, we extend our model by considering a priori alliances between players. Specifically, following Owen's model of a priori given coalition structure, we will assume that some groups of players form inseparable groups; these groups form a coalition structure, P that we will take into account when assessing importance of the players in the sign graph game. We will make a natural assumption that all the coalitions from P are feasible, i.e., $T_l \in \mathcal{F}$ for every $T_l \in P$. In words, each coalition induces a connected subgraph with no negative edges.

We define the Owen value for a signed graph game as follows:

$$OV_i^P(N,v,E_+,E_-) = OV_i^P(N, v/G^{\pm}),$$

for $G^{\pm} = (N,E_+,E_-)$. Throughout this section, we will focus on computing the Owen value of a single player $i \in T_j \in P$.

We start by introducing some additional notation. Let us denote by $\mathcal{S}^{(i)}$ the set of coalitions values of which are taken into account by the Owen value:

$$\mathcal{S}^{(i)} = \{T_M \cup R : M \subseteq \{1,\ldots,k\}\setminus\{j\}, R \subseteq T_j\}.$$

Now, consider an intersection of $\mathcal{S}^{(i)}$ and $\mathcal{F}$, denoted by $\mathcal{F}^{(i)}$: $\mathcal{F}^{(i)} = \mathcal{S}^{(i)} \cap \mathcal{F}$. In what follows, we will show that only values of coalitions from $\mathcal{F}^{(i)}$ are taken into account by the Owen value for player i in signed graph games.

For a feasible coalition $C \in \mathcal{F}^{(i)}$, we denote by M^*_C a subset of indexes of coalitions from P that do not overlap with $C \cup \mathcal{N}_{\pm}(C)$,

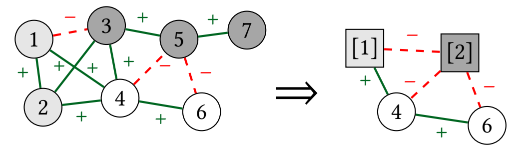

Figure 2: Graph $G^{\pm}$ (on the left) and the corresponding graph $G^{(i)}$ (on the right) for $i = 6$ and partition $P = \{\{1,2\},\{3,5,7\},\{4,6\}\}$ depicted with the colors of nodes: white, light gray and dark gray. Positive edges are green/solid and negative edges are red/dashed.

and by R^*_C the part of T_j which does not belong to $C \cup \mathcal{N}_{\pm}(C)$: $M^*_C = \{l \in \{1,\ldots,k\} : T_l \cap (C \cup \mathcal{N}_{\pm}(C)) = \emptyset\}$ and $R^*_C = T_j \setminus (C \cup \mathcal{N}_{\pm}(C))$.

Example 6.1. Consider a signed graph game (N,v,E_+,E_-) where the signed graph, $G^{\pm} = (N,E_+,E_-)$, is depicted in Figure 2 and let $P = \{\{1,2\},\{3,5,7\},\{4,6\}\}$. Fix $i = 6$. We have $\{5,6\} \notin \mathcal{S}^{(6)}$ and $\{1,2,6\} \in \mathcal{S}^{(6)}$, but $\{1,2,6\} \notin \mathcal{F}^{(6)}$. Set $\mathcal{F}^{(6)}$ consists of 7 coalitions:

$$\mathcal{F}^{(6)} = \{\{4\},\{6\},\{4,6\},\{1,2\},\{1,2,4\},\{1,2,4,6\},\{3,5,7\}\}.$$

Let us focus on coalition $C = \{6\}$. As all coalitions from $\mathcal{F}^{(6)}$, C can be represented as $C = T_M \cup R$ for $M = \emptyset$ and $R = \{6\}$. We have $\mathcal{N}_{\pm}(C) = \{4,5\}$, hence $M^*_C = \{1\}$ (because only coalition $T_1 = \{1,2\}$ from P does not overlap with $C \cup \mathcal{N}_{\pm}(C)$) and $R^*_C = \emptyset$.

In the following theorem, we show that the Owen value of player i can be characterized as a sum over coalitions from $\mathcal{F}^{(i)}$.

Theorem 6.2. *The Owen value for signed graph games satisfies the following formula:*

$$OV_i^P(N,v,E_+,E_-) = \sum_{C\in\mathcal{F}^{(i)}, i\in C} \delta(C)v(C) - \sum_{C\in\mathcal{F}^{(i)}, i\in N(C)} \delta(C)v(C)$$

for every signed graph game (N,v,E_+,E_-), player $i \in N$ and partition P, where $\delta(C) = \sum_{m=0}^{|M^*_C|}\sum_{r=0}^{|R^*_C|}\binom{|M^*_C|}{m}\binom{|R^*_C|}{r}\zeta_k(|M|+m)\zeta_{|T_j|}(|R\setminus\{i\}|+r)$ for every $C = T_M \cup R$.

Proof. Fix $S \in S^{(i)}$ such that $i \notin S$ and consider the marginal contribution of player i to coalition S. From Lemma 5.1 we get:

$$v(S\cup\{i\}) - v(S) = \sum_{\substack{C\in K(G^{\pm}[S\cup\{i\}])\\ C\in\mathcal{F}, i\in C}} v(C) - \sum_{\substack{C\in K(G^{\pm}[S])\\ C\in\mathcal{F}, i\in\mathcal{N}_{\pm}(C)}} v(C).$$

Consider T_l for an arbitrary $l \in \{1,\ldots,k\}\setminus\{j\}$. From the definition of $\mathcal{S}^{(i)}$ we know that either $T_l \subseteq S$ or $T_l \cap S = \emptyset$. If $T_l \subseteq S$, then all players from T_l must belong to one component of $G^{\pm}[S]$ and of $G^{\pm}[S\cup\{i\}]$. This is because we assumed that T_l is feasible which implies that it induces a connected subgraph. Hence, we get that $C \in \mathcal{F}^{(i)}$ for every $C \in K(G^{\pm}[S\cup\{i\}])$ and $C \in K(G^{\pm}[S])$. This fact combined with Eq. (2) implies that $OV_i^P(N,v,E_+,E_-)$ equals:

$$\sum_{\substack{C\in\mathcal{F}^{(i)}\\ i\in C}} \sum_{\substack{S\in\mathcal{S}^{(i)}:i\notin S\\ C\in K(G^{\pm}[S\cup\{i\}])}} \zeta_i^P(S)v(C) - \sum_{\substack{C\in\mathcal{F}^{(i)}\\ i\in\mathcal{N}_{\pm}(C)}} \sum_{\substack{S\in\mathcal{S}^{(i)}:i\notin S\\ C\in K(G^{\pm}[S])}} \zeta_i^P(S)v(C).$$

Now, it remains to prove that the sum of weights $\zeta_i^P(S)$ equals $\delta(C)$ from the thesis of the theorem. To this end, fix $C \in \mathcal{F}^{(i)}$ such that $i \in C$ and assume $C = T_M \cup R \cup \{i\}$. Consider arbitrary

$S = T_{M'} \cup R' \in \mathcal{S}^{(i)}$ such that $i \notin S$. The necessary and sufficient conditions so that $C \in K(G^{\pm}[S \cup \{i\}])$ are:

- $M \subseteq M'$ (otherwise $C \nsubseteq S \cup \{i\}$);
- $R \subseteq R'$ (otherwise $C \nsubseteq S \cup \{i\}$);
- $M' \subseteq M \cup M^*$ (if $T_{M'} \cap \mathcal{N}_{\pm}(C) \neq \emptyset$, then C is not a component);
- $R' \subseteq R \cup R^*$ (if $R' \cap \mathcal{N}_{\pm}(C) \neq \emptyset$, then C is not a component).

We get that $M' = M \cup M''$ for arbitrary subset $M'' \subseteq M^*$ and $R' = R \cup R''$ for arbitrary subset $R'' \subseteq R^*$. In result, we have that $\sum_{S \in \mathcal{S}^{(i)}: i \notin S, C \in K(G^{\pm}[S \cup \{i\}])} \zeta_i^P(S) = \delta(C)$.

An analogous analysis for $C \in \mathcal{F}^{(i)}$ such that $i \in \mathcal{N}_{\pm}(C)$ of the form $C = T_M \cup R$ concludes the proof. □

Example 6.3. Consider again the signed graph game from Example 6.1. We will calculate the Owen value of player 6 using Theorem 6.2. Note that $\zeta_3(0) = \zeta_3(2) = 1/3$, $\zeta_3(1) = 1/6$ and $\zeta_2(0) = \zeta_2(1) = 1/2$.

There are three feasible coalitions from $\mathcal{F}^{(i)}$ with player 6: $\{6\}$, $\{4, 6\}$, $\{1, 2, 4, 6\}$. We get that $\delta(\{6\}) = \binom{1}{0}\binom{0}{0}\frac{1}{3}\frac{1}{2} + \binom{1}{1}\binom{0}{0}\frac{1}{6}\frac{1}{2} = \frac{1}{4}$ (note that for $C = \{6\}$ we have $|M| = 0$, $|R| = 1$, $|M^*_C| = 1$ and $|R^*_C| = 0$, as determined in Example 6.1). Analogously, $\delta(\{4, 6\}) = 1/6$ and $\delta(\{1, 2, 4, 6\}) = 1/12$.

Then, there are three feasible coalitions from $\mathcal{F}^{(i)}$ for which player 6 is a neighbor: $\{4\}$, $\{1, 2, 4\}$ and $\{3, 5, 7\}$. For them, we get: $\delta(\{4\}) = 1/6$, $\delta(\{1, 2, 4\}) = 1/12$ and $\delta(\{3, 5, 7\}) = 1/12$. Eventually, we have that the Owen value of player 6 equals:

$$\frac{v(\{6\})}{4} + \frac{v(\{4{,}6\})}{6} + \frac{v(\{1{,}2{,}4{,}6\})}{12} - \frac{v(\{4\})}{6} - \frac{v(\{1{,}2{,}4\})}{12} - \frac{v(\{3{,}5{,}7\})}{12}.$$

From Theorem 6.2, we know that the Owen value is based only on values of coalitions from $\mathcal{F}^{(i)}$. Hence, traversing all feasible coalitions no longer leads to an algorithm linear in the size of the input. To cope with this problem we will define an auxiliary graph and traverse the subgraphs therein.

Let us define graph $G^{(i)}$ obtained from $G^{\pm}$ by merging for each coalition $T_l \in P \setminus \{T_j\}$ all the players into a single node, denoted by $[l]$. If some player is adjacent to several players from the merged group we leave only one edge: if all edges were positive the edge between the group and this player will be positive; otherwise, this edge will be negative. See Figure 2 for an illustration. Formally:

Definition 6.4. $G^{(i)} = (N^{(i)}, E_+^{(i)}, E_-^{(i)})$ is a signed graph where:

- $N^{(i)} = T_j \cup \{[l] : l \in \{1, \dots, k\} \setminus \{j\}\}$,
- $E_-^{(i)} = E_-[T_j] \cup \{\{[l], [l']\} \subseteq N^{(i)} : l \neq l', \exists_{a \in T_l} \exists_{b \in T_{l'}} \{a, b\} \in E_-\} \cup \{\{[l], m\} \subseteq N^{(i)} : m \in T_j, \exists_{a \in T_l} \{a, m\} \in E_-\}$,
- $E_+^{(i)} = E_+[T_j] \cup \{\{[l], [l']\} \subseteq N^{(i)} : l \neq l', \exists_{a \in T_l} \exists_{b \in T_{l'}} \{a, b\} \in E_+\} \cup \{\{[l], m\} \subseteq N^{(i)} : m \in T_j, \exists_{a \in T_l} \{a, m\} \in E_+\} \setminus E_-^{(i)}$.

We note that there exists a one-to-one mapping between coalitions from $\mathcal{F}^{(i)}$ and $\mathcal{F}(G^{(i)})$, i.e., feasible coalition in $G^{(i)}$. Specifically, arbitrary coalition $C = T_M \cup R \in \mathcal{F}^{(i)}$ maps to coalition $C' = (\{[l] : l \in M\} \cup R)$ and it can be shown that $C' \in \mathcal{F}(G^{(i)})$. Clearly, this mapping is injective, but it is also surjective, since it is reversible. Now, using Definition 6.4 from Theorem 6.2 we have:

Theorem 6.5. *The Owen value of player i for signed graph games can be computed in time $O(|F|(|N| + |E_+|))$ for $F = \{C \in \mathcal{F}^{(i)} : i \in C \vee i \in \mathcal{N}_{\pm}(C)\}$.*

Proof. Based on Theorem 6.2, our goal is enumerate all feasible coalitions in $G^{\pm}$ that contain player i or any neighbor of i. Following the discussion, we will equivalently enumerate such feasible coalitions in $G^{(i)}$. To enumerate all feasible coalition with player i, we call function Rec from Algorithm 1 with the *color* table in which $color[i] = i$, and all other colors are White. After this, we mark player i as RedN. Then, we consider neighbors of i one by one and call function Rec so that it enumerates all feasible coalitions with this neighbor, but without previously considered neighbors. Function Rec is called with a reference to function UpdateOwen which computes the Owen value based on Theorem 6.2. □

7 RELATED WORK

The model of graph restrictions was extended to weighted graphs by Calvo et al. [3]; however, only weights from $[0, 1]$ are allowed. Khmelnitskaya et al. [13] considered games restricted by directed graphs. So far, however, no one considered a graph-restricted game for sighed-graph games.

Computational properties of the Myerson value were studied in a number of papers. Bilbao [2], Elkind [7] and Skibski et al. [26] all proposed different formulas for the Myerson value that traverse only induced connected subgraphs. Polynomial algorithms for the Myerson/Shapley value for weighted voting games restricted by trees [1] and graphs with bounded treewidth [25] were also developed. In a recent paper, Greco et al. [10] considered the Shapley value of games that results from matching problems; these games can also be considered graph-restricted.

A recent application of coalitional games on graphs is the centrality analysis. In particular, Skibski et al. [27] proposed a new centrality measure, called the Attachment centrality, which is the Myerson value of a specific game. Gangal et al. [8] proposed several Shapley-value based centrality measures for signed graphs; however, they do not comply with Myerson's graph-restrictions.

For non-transferable coalitional games, a model in which every player partitions other players into friends, enemies and neutral agents have recently attracted attention in the literature [14, 17, 20]. Unlike our model, the relations therein are not symmetric.

8 CONCLUSIONS

In this paper, we proposed a model of coalitional games restricted by a signed graph. We extended the Myerson value to this setting using the axiomatic approach and proposed an algorithm that works for an arbitrary semivalue. Also, we considered the Owen value and proposed a dedicated algorithm.

Our work can be extended in a number of ways. In particular, directed graphs or hypergraphs can be considered. Also, following the work by Meir et al. [15], it would be interesting to analyze how negative edges affect the stability in graph games. Finally, coalition structure generation problem can be considered in our model [31].

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