by the fourth inequality. Therefore, the solution of the ILP instance gives a transformation of V, resulting in n votes, which eliminate the candidates according to the order e.

The algorithm works for both constructive and destructive shift bribery. The only difference lies in the elimination orders to be examined by ILP: the constructive case considers only the orders, where only the specific candidate p is eliminated in the last round, whereas the destructive case examines all other orders. Note that the enumeration of elimination orders and the ILP formulation are the same for Hare, Coombs, and Baldwin rules. Only $\text{score}(c, X, h)$ is calculated according to respective definitions of scores. For the Nanson rule, we need to replace the third and forth (in)equalities by the following inequality.

$$\sum_{h' \in H} \sum_{h \in H} (x_{h,h'} \cdot \mathtt{score}(c, \overline{C_i'}, h')) \cdot |\overline{C_i'}|$$
$$< \sum_{c' \in \overline{C_i'}} \sum_{h' \in H} \sum_{h \in H} (x_{h,h'} \cdot \mathtt{score}(c', \overline{C_i'}, h'))$$
$$\forall\ 0 \le i \le r_e - 1, \forall\ c \in C_{i+1}.$$

Since the number of the variables of the ILP instance is bounded by $(m!)^2$, it is solvable in FPT time with m as parameter [20]. The theorem follows then from the number $m! \cdot 2^m$ of elimination orders. □

3.2 The number of swap operations

In the following, we consider the case with the number B of swap operations as parameter. We show $W[1]$-hard results of both constructive and destructive cases for all four iterative voting systems. We consider first Hare and Coombs.

Theorem 3.2. *Hare-CSB, Hare-DSB, Coombs-CSB, and Coombs-DSB are $W[1]$-hard with respect to the parameter B.*

Proof. We prove the theorem for Hare-CSB by giving a reduction from **Independent Set**. The hardness results of other problems can be proven by similar reductions. An independent set I in an undirected graph $\mathcal{G}$ is a set of pairwisely non-adjacent vertices. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and an integer k, **Independent Set** asks for a size-k independent set and is $W[1]$-hard with respect to k [10]. Let $\mathcal{V} = \{v_1, \cdots, v_{n'}\}$ and $\mathcal{E} = \{e_1, \cdots, e_{m'}\}$. Without loss of generality, assume $k \ge 2, n' > 2k+1$ and $m' \ge n'$. We use $\mathsf{deg}(v)$ to denote the degree of v in $\mathcal{G}$. We construct a Hare-CSB instance (C, V, B) from $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), k)$ as follows.

For each vertex $v_i \in \mathcal{V}$, we create a vertex candidate $c_i \in C_1$, and for each edge $e_\ell \in \mathcal{E}$, we create an edge candidate $d_\ell \in C_2$. We also create two dummy candidate sets C_3 and C_4 with $|C_3| = |C_4| = k$. Let $C := C_1 \cup C_2 \cup C_3 \cup C_4 \cup \{p\}$ and $B := k$. The vote set V consists of five subsets $V := V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$, which are constructed as follows. Hereby, if we do not give an explicit order for the candidates in a set, then the candidates in the set can be ordered arbitrarily.

- For each vertex $v_i \in \mathcal{V}$, we create one vote v_i^1 in V_1:
  $v_i^1 : c_i > p > C_3 > C_4 > (C_1 \setminus \{c_i\}) \cup C_2$,
  and $n' - \mathsf{deg}(v_i) - 1$ identical votes v_i^j in V_2:
  $v_i^j : c_i > (C_1 \setminus \{c_i\}) > C_2 > C_3 > C_4 > p$ for $2 \le j \le n' - \mathsf{deg}(v_i)$.
- For each edge $e_\ell = \{v_i, v_j\} \in \mathcal{E}$, we create two votes in V_3:
  $v_\ell^1 : c_i > c_j > d_\ell > (C_1 \setminus \{c_i, c_j\}) > (C_2 \setminus \{d_\ell\}) > C_3 > C_4 > p$,
  $v_\ell^2 : c_j > c_i > d_\ell > (C_1 \setminus \{c_i, c_j\}) > (C_2 \setminus \{d_\ell\}) > C_3 > C_4 > p$;
  and n' identical votes in V_4:
  $v_\ell^r : d_\ell > (C_2 \setminus \{d_\ell\}) > C_3 > p > C_4 > C_1$ for $3 \le r \le n' + 2$.
- In addition, we create $n' + 1 - k$ identical votes in V_5:
  $v^j : p > C_3 > C_4 > C_1 > C_2$ for $1 \le j \le n' + 1 - k$.

The current plurality scores of the candidates are: $\mathtt{score}(p) = n' + 1 - k$, $\mathtt{score}(c) = n'$ for $c \in C_1 \cup C_2$, $\mathtt{score}(c) = 0$ for $c \in C_3 \cup C_4$. Thus, the dummy candidates in $C_3 \cup C_4$ are eliminated in the first round. Note that removing the candidates in $C_3 \cup C_4$ from the votes does not change the plurality scores of the remaining candidates. Thus, the candidate p is eliminated in the second round. The candidate p is not the unique winner. Now, we show the equivalence between the **Independent Set** instance and the instance of Hare-CSB.

"$\Longrightarrow$": Suppose that there exists a size-k independent set $I = \{v_{i_j} : 1 \le j \le k\}$. We swap p with each vertex candidates c_{i_j} in the corresponding vote $v_{i_j}^1$ of V_1. Clearly, we need $k = B$ swap operations, satisfying the budget restriction. Now, we calculate the plurality scores of the candidates: $\mathtt{score}(p) = n' + 1$, $\mathtt{score}(c_i) = n' - 1$ for $c_i \in C_1$ with $i \in \{i_1, \cdots, i_k\}$, $\mathtt{score}(c_i) = n'$ for $c_i \in C_1$ with $i \notin \{i_1, \cdots, i_k\}$, $\mathtt{score}(c) = n'$ for $c \in C_2$; $\mathtt{score}(c) = 0$ for $c \in C_3 \cup C_4$.

Again, the candidates in $C_3 \cup C_4$ are eliminated in the first round. Afterwards, the candidates $c_i \in C_1$ with $i \in \{i_1, \cdots, i_k\}$ are eliminated. Note that, since the corresponding vertices form an independent set, removing these k candidates increases the plurality scores of some candidates in C_1, but has no influence on the scores of the edge candidates in C_2. Thus, the next round eliminates all candidates in C_2 and some candidates in C_1. With the candidates in C_2 being removed, all votes in V_4 rank p on the top. Then, $\mathtt{score}(p) = n' \cdot m' + n' + 1$. Since in total, there are $n' \cdot m' + n'^2 - k + 1 + n'$ many votes, $\text{score}(p) > \frac{1}{2}|V|$. Moreover, since removing candidates cannot decrease the plurality score of any remaining candidate. The scores of other candidates are always less than $\text{score}(p)$, and thus, p is the unique winner.

"$\Longleftarrow$": Suppose that we can make p the unique winner with at most B swap operations. Since $|C_3| = |C_4| = B$, we cannot swap p with the candidates in C_2 in any vote in V. Let C_1' be the set of vertex candidates, which are swapped with p, and let $\mathcal{V}'$ be the corresponding vertex set. Let $\alpha = |C_1'|$ and β be the number of the candidates in $C_3 \cup C_4$, who are swapped with p. Note that given $|C_3| = |C_4| = B$, applying swap operations in the votes in $V_2 \cup V_3 \cup V_4$ does not change the scores of the candidates. However, swapping p with a candidate c in $C_3 \cup C_4$ in a vote in V_5 increases the score of c by at most one and decreases the score of p by at most one. Therefore, since we have $n' > 2k + 1$, the candidates in $C_3 \cup C_4$ have scores less than $\text{score}(p)$ and thus, are first eliminated. To guarantee that p is not eliminated afterwards, it must hold $\alpha \ge B - 1$. In this way, the candidates in C_1' are eliminated with the lowest score $n' - 1$. Then, the score of p remains $n' + 1 - B + \alpha$, and the scores of other candidates are at least n'. Since p is the unique winner, it implies $\alpha = B$. Suppose that there is an edge $e_\ell = \{v, v'\}$

with $v, v' \in \mathcal{V}'$. Then, after the candidates in C_1' are eliminated, we have $\text{score}(d_\ell) = n' + 2$ and $\text{score}(p) = n' + 1$. Then, in the next round where the candidates with a score n' are eliminated, the candidate d_ℓ is not eliminated. As a consequence, the score of p remains $n' + 1$ and thus, p is eliminated before d_ℓ, contradicting that p is the unique winner. Therefore, there is no edge $e_\ell = \{v, v'\}$ with $v, v' \in \mathcal{V}'$, and $\mathcal{V}'$ forms an independent set of size k. □

In the following, we consider the parameterized complexity of shift bribery of Baldwin and Nanson with respect to the parameter B. We first show hardness for the constructive case.

Theorem 3.3. *Baldwin-CSB and Nanson-CSB are $W[1]$-hard with respect to the parameter B.*

Proof. We prove the theorem for Baldwin-CSB by giving a reduction from **Independent Set** on D-regular graphs. The result for Nanson-CSB can be shown by a similar reduction. A D-regular graph is a graph, where all vertices have the same degree D. **Independent Set** remains W[1]-hard with respect to the size k of independent sets on D-regular graphs. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, \cdots, v_{n'}\}$ and $\mathcal{E} = \{e_1, \cdots, e_{m'}\}$. We construct a Baldwin-CSB instance (C, V, B) as follows. Recall that $\overrightarrow{X}$ denotes an arbitrary but fixed ordering of the candidates in a set X and $\overleftarrow{X}$ the reversed ordering of $\overrightarrow{X}$. For a set of candidates C and $c_1, c_2 \in C$, we construct the following *vote pair*:

$$W(c_1, c_2) = (c_1 > c_2 > \overrightarrow{C \setminus \{c_1, c_2\}},\\ \overleftarrow{C \setminus \{c_1, c_2\}} > c_1 > c_2).$$

Similarly, for a set of candidates C and $c_1, c_2, c_3 \in C$, we construct the following vote pair:

$$W(c_1, c_2, c_3) = (c_1 > c_2 > c_3 > \overrightarrow{C \setminus \{c_1, c_2, c_3\}},\\ \overleftarrow{C \setminus \{c_1, c_2, c_3\}} > c_1 > c_3 > c_2).$$

According to the Borda rule, from the two votes in $W(c_1, c_2)$, candidate c_1 gets $|C|$ points, c_2 gets $|C| - 2$ points, and each of the other candidates gets $|C| - 1$ points. For simplicity, we compare the score of each candidate with the average Borda score. Thus, from the two votes in $W(c_1, c_2)$, we say that candidate c_1 gets 1 point, c_2 gets -1 point, and all other candidates get 0 point. Moreover, we say that with respect to $W(c_1, c_2)$, candidate c_2 "gains" one point by eliminating c_1, since c_2 gets the same points as other candidates in $C \setminus \{c_1, c_2\}$ after the elimination of c_1. Similarly, c_1 "loses" one point by eliminating c_2. By a similar analysis, each of c_2 and c_3 "gains" one point from the vote pair in $W(c_1, c_2, c_3)$ by eliminating c_1 and candidate c_1 "loses" one point by eliminating c_2 or c_3.

For each vertex $v_i \in \mathcal{V}$, we construct a vertex candidate $c_i \in C_1$ $(1 \le i \le n')$. For each edge $e_j \in \mathcal{E}$, we construct an edge candidate $d_j \in C_2$ $(1 \le j \le m')$. Moreover, we construct four special candidates $\{c_s, c_h, c_t, c_\ell\}$ and two dummy candidate sets C_3 and C_4 where $|C_3| = |C_4| = B$. Let $C := C_1 \cup C_2 \cup C_3 \cup C_4 \cup \{c_s, c_h, c_t, c_\ell, p\}$ and $B := k$. We construct the set of votes as follows:

- For each vertex v_i with $1 \le i \le n'$, we add two votes to V_1: $c_i > p > \overrightarrow{C_3} > \overrightarrow{C_4} > \overrightarrow{C_1 \setminus \{c_i\}} > \overrightarrow{C_2} > c_s > c_h > c_t > c_\ell$, and $c_\ell > c_t > c_h > c_s > \overleftarrow{C_2} > \overleftarrow{C_1 \setminus \{c_i\}} > c_i > \overleftarrow{C_4} > \overleftarrow{C_3} > p$, and the following votes to V_2: two identical vote pairs $W(c_s, c_i)$, $D+3$ identical vote pairs $W(c_i, c_t)$, and $D+4$ identical vote pairs $W(c_\ell, c_i)$;
- For each edge $e_j = \{v_i, v_{i'}\}$ with $1 \le j \le m'$, we add the following votes to V_2: one vote pair $W(d_j, c_i, c_{i'})$ with c_i and $c_{i'}$ corresponding to v_i and $v_{i'}$ respectively, two identical vote pairs $W(d_j, c_h)$, three identical vote pairs $W(c_t, d_j)$, $D + 2$ identical vote pairs $W(c_\ell, d_j)$;
- Furthermore, we add the following votes to V_2: two identical vote pairs $W(c_h, c_s)$, $n' - D - B - 2$ identical vote pairs $W(p, c_\ell)$, $2n' - 2B + D + 1$ identical vote pairs $W(c_\ell, c_s)$, $(n' - B)(D + 3) - D - 3$ identical vote pairs $W(c_t, c_\ell)$, and $2m' - D - 3$ identical vote pairs $W(c_h, c_\ell)$.
- We create m'^2 groups of votes in V_3, each containing the following four votes:
  - $\overrightarrow{C \setminus \{C_3 \cup C_4 \cup \{p\}\}} > \overrightarrow{C_3} > p > \overrightarrow{C_4}$,
  - $p > \overleftarrow{C_3} > \overleftarrow{C \setminus \{C_3 \cup C_4 \cup \{p\}\}} > \overleftarrow{C_4}$,
  - $\overrightarrow{C \setminus \{C_3 \cup C_4 \cup \{p\}\}} > \overrightarrow{C_4} > p > \overrightarrow{C_3}$,
  - $p > \overleftarrow{C_4} > \overleftarrow{C \setminus \{C_3 \cup C_4 \cup \{p\}\}} > \overleftarrow{C_3}$.

Note that in all votes of V_2, candidate p is always in the middle of C_3 and C_4, that is, $\overrightarrow{C_3} > p > \overrightarrow{C_4}$ or $\overleftarrow{C_4} > p > \overleftarrow{C_3}$. Let $V := V_1 \cup V_2 \cup V_3$.

The current Borda scores of the candidates are as follows. Again, we compute the difference between the score of a candidate and the average Borda score: $\texttt{score}(p) = -D - k - 2, \texttt{score}(c_i) = -D - 2$ for $c_i \in C_1, \texttt{score}(d_j) = -D - 1$ for $d_j \in C_2, \texttt{score}(c_s) = 2k - D - 3, \texttt{score}(c_h) = -D - 1, \texttt{score}(c_t) = 3m' - k(D + 3) - 3 - D, \texttt{score}(c_\ell) = D(m' + k + 4) + 2n' + 2k + 9$. The candidate p is eliminated in the second round after the candidates in $C_3 \cup C_4$ are eliminated in the first round, and p is not the unique winner. Now, we show the equivalence between the **Independent Set** instance and the instance of Baldwin-CSB. Note that, according to the m'^2 groups of votes in V_3, the candidates in $C_3 \cup C_4$ are eliminated before other candidates, no matter where the at most B swap operations apply. Furthermore, the operations that swapping p with the candidates in $C_3 \cup C_4$ have no influence on the election results.

"$\Longrightarrow$": Suppose that there is a size-k independent set I in $\mathcal{G}$. Let C' contain the candidates in C_1 corresponding to the vertices in I. We swap the candidates in C' with p in the votes in V_1, which are created corresponding to the vertices in I. That is, we perform exactly $k = B$ swap operations in exactly k votes, one operation in each vote. After the operations, the scores of the candidates in C' and p are changed: $\texttt{score}(p) = -D-2,\ \texttt{score}(c_i) = -D-3$ for $c_i \in C'$. The candidates in $C_3 \cup C_4$ are still eliminated first and the candidates in C' are eliminated in the second round. Since the corresponding vertices of the candidates in C' are independent, the score of each edge candidate d_j is $-D - 1$ or $-D - 2$. There are still $n' - k$ candidates of C_1 remaining. The candidate c_s is eliminated in the third round with a score of $-D - 3$. The following elimination sequence is $c_h, C_2, c_t, C', c_\ell, p$ and, the candidate p is the unique winner. The scores of the candidates in each round are shown in Table 2.

"$\Longleftarrow$": Suppose that there is no size-k independent set in $\mathcal{G}$. According to the construction of votes, no matter how p is swapped with other candidates, the candidates in $C_3 \cup C_4$ are eliminated before other candidates. The operations swapping p with the candidates in $C_3 \cup C_4$ do not change the final winner. Therefore, it is

Table 2: The scores of the candidates in each round in the proof of Theorem 3.3. The set C' contains the candidates, who are swapped with p. With d_j^1 we denote the edge candidates, whose corresponding edges are incident to some vertices in I, and d_j^2 denotes the other edge candidates. The candidates in $C_3 \cup C_4$ are eliminated before C' and their scores are omitted.

	p	c_i		d_j		c_s	c_h	c_t	c_ℓ
Initial	-D-k-2	-D-2		-D-1		2k-D-3	-D-1	3m'-(k+1)(D+3)	D(m'+k'+4)+2k'+2n'+9
		$c_i \notin C'$	$c_i \in C'$						
After swapping	-D-2	-D-2	-D-3	-D-1		2k-D-3	-D-1	3m'-(k+1)(D+3)	D(m'+k'+4)+2k'+2n'+9
				d_j^1	d_j^2				
Eliminating C'	-D-2	-D-2	—	-D-1	-D-2	-D-3	-D-1	3m'-3-D	D(m'+4)-2k'+2n'+9
Eliminating c_s	-D-2	-D	—	-D-1	-D-2	—	-D-3	3m'-3-D	Dm'+3D+8
Eliminating c_h	-D-2	-D	—	-D-3	-D-4	—	—	3m'-3-D	Dm'+2D+2m'+5
Eliminating d_j	-D-2	0	—	—	—	—	—	-3-D	2D+5
Eliminating c_t	-D-2	-D-3	—	—	—	—	—	—	(n'-k')(D+3)+D+2
Eliminating c_i	n'-k-D-2	—	—	—	—	—	—	—	-n'+k'+D+2
Eliminating c_ℓ and p win	$\lvert V \rvert$								

only meaningful to swap p with the candidates of C_1 in the votes of V_1. Furthermore, if the number of swap operations is less than k, p will be eliminated after $C_3 \cup C_4$. In order to make p the unique winner, we have to swap p with exactly $k = B$ vertex candidates in V_1. Clearly, these operations happen in exactly B votes in V_1, one in each vote. Let C' denote the set of vertex candidates, who are swapped with p. Since there is no size-k independent set in $\mathcal{G}$, the elimination of the candidates in C' results in an edge candidate d_j of a score $-D-3$. Then, d_j and c_s are eliminated together in the third round, which makes c_h gain at least one point and leads to the elimination of p in the fourth round. Then, p is not the unique winner. In summary, if there is no size-k independent set in $\mathcal{G}$, then candidate p cannot be an unique winner by at most k swap operations. ◻

Next, we consider the destructive cases of Baldwin and Nanson.

Theorem 3.4. *Baldwin-DSB and Nanson-DSB are $W[1]$-hard with respect to the parameter B.*

Proof. We prove the theorem for Nanson-DSB by giving a reduction from **Clique** on D-regular graphs. The result for the Baldwin-DSB can be shown in a similar way. Given a D-regular graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and an integer k, asking for a size-k clique is $W[1]$-hard with respect to k [10]. Let $\mathcal{V} = \{v_1, \cdots, v_{n'}\}$ and $\mathcal{E} = \{e_1, \cdots, e_{m'}\}$. We construct a Nanson-DSB instance (C, V, B) as follows. Again, we compare the score of each candidate with the average Borda score. Thus, from the two votes in $W(c_1, c_2)$ in the proof of Theorem 3.3, candidate c_1 gets 1 point, c_2 gets -1 point, and all other candidates get 0 point. Moreover, we say that with respect to $W(c_1, c_2)$, candidate c_2 gains one point by eliminating c_1, and c_1 loses one point by eliminating c_2. Similarly, each of c_2 and c_3 gains one point from $W(c_1, c_2, c_3)$ by eliminating c_1 and c_1 loses one point by eliminating c_2 or c_3.

For each vertex $v_i \in \mathcal{V}$, we create a vertex candidate $c_i \in C_1$ ($1 \leq i \leq n'$). For each edge $e_j \in \mathcal{E}$, we create an edge candidate $d_j \in C_2$ ($1 \leq j \leq m'$). Moreover, we create six special candidates $C_5 = \{c_{t_1}, c_{t_2}, c_{q_1}, c_{q_2}, c_{q_3}, c_h\}$ and two dummy candidate sets C_3 and C_4 with $|C_3| = |C_4| = B$. Let $C := C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup \{p\}$ and $B := k$. We construct the set of votes as follows:

- For each vertex v_i with $1 \leq i \leq n'$, add two votes $c_i > p > \overrightarrow{C_3} > \overrightarrow{C_4} > \overrightarrow{C_1 \setminus \{c_i\}} > \overrightarrow{C_2} > \overrightarrow{C_5}$ and $\overleftarrow{C_5} > \overleftarrow{C_2} > \overleftarrow{C_1 \setminus \{c_1\}} > c_i > \overleftarrow{C_4} > \overleftarrow{C_3} > p$ to the set V_1. Add D identical vote pairs $W(c_i, c_h)$ to V_2;
- For each edge $e_j = \{v_i, v_{i'}\}$ with $1 \leq j \leq m'$: Add one vote pair $W(d_j, c_i, c_{i'})$ and two identical vote pairs $W(c_{t_1}, d_j)$ to V_2;
- Add the following votes to V_2: $m' + k(k-1)$ identical vote pairs $W(c_{t_2}, c_{t_1})$, m' identical vote pairs $W(c_{t_1}, p)$, and one vote pair $W(p, c_{t_2})$;
- Add the following votes to V_3: m' identical vote pairs $W(p, c_{q_1})$, m' identical vote pairs $W(c_{q_1}, c_{q_2})$, and m' identical vote pairs $W(c_{q_2}, c_{q_3})$.

Note that in all votes in $V \setminus V_1$, candidate p is always in the middle of C_3 and C_4, that is, $\overrightarrow{C_3} > p > \overrightarrow{C_4}$ or $\overleftarrow{C_4} > p > \overleftarrow{C_3}$. Let $V := V_1 \cup V_2 \cup V_3$. The candidates $c_{q_1}, c_{q_2}, c_{q_3}$ and the votes in V_3 make sure that the candidate p has a score greater than the average Borda score in the first four rounds. The equivalence between the two instances can be proved in a similar but more tricky way as in the proof in Theorem 3.3. ◻

3.3 The number of votes

In the following, we show FPT results for Hare-CSB and Hare-DSB, and $W[1]$-hard results for Baldwin-CSB, Nanson-CSB, and Nanson-DSB with the number of votes n as parameter.

Theorem 3.5. *Hare-CSB and Hare-DSB are FPT with respect to the number of votes n.*

Proof. Let $V = \{v_1, \cdots, v_n\}$ and $C = \{c_1, \cdots, c_m\}$ denote the vote and candidate sets, respectively. We prove only the constructive case. The destructive case can be proved in a similar way. The case of $m \leq n$ follows directly from Theorem 3.1. For the case $m > n$, observe that there are at most n candidates, who can get at least one point. Other candidates are eliminated with 0 point in the first round. Therefore, we enumerate all 2^n subsets of votes, which represent the votes in a possible solution, that rank p at the top. For each subset $\{i_1, \cdots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n$, we calculate the number of swap operations needed to shift p in v_{i_j} to the first position for each $1 \leq j \leq k$. If the total number of swap operations exceeds B, we exclude this subset from further consideration; otherwise, we calculate the plurality scores of the candidates with the votes $v'_1, \cdots, v'_n$, where $v'_i = v_i$ for $i \notin \{i_1, \cdots, i_k\}$ and for $i \in \{i_1, \cdots, i_k\}$, v'_i ranks the candidates in the same orders as v_i with the possible exception that p is in the first position. Then, after eliminating the candidates with the least plurality score in the first round, there remain at most n candidates. The optimal shift strategy in the votes v_i with $i \notin \{i_1, \cdots, i_k\}$ can be then computed with the ILP approach given in the proof of Theorem 3.1. In summary, we solve at most 2^n ILP's, each solvable in FPT time. This completes the proof. □

In the following, we show the hardness results for Baldwin-CSB and Nanson-CSB.

Theorem 3.6. *Baldwin-CSB and Nanson-CSB are W[1]-hard with respect to the parameter n.*

Proof. We prove the theorem for Nanson-CSB by giving a reduction from the **Multi-colored Independent Set** problem. The result for Baldwin-CSB follows from a similar but more tricky reduction. Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with each vertex being colored by one of k colors, **Multi-colored Independent Set** asks for a colorful independent set of size k. A colorful set contains no two vertices with the same color. A simple reduction from **Independent Set** shows that **Multi-colored Independent Set** is W[1]-hard with respect to k. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be our input instance. Without loss of generality, we assume that the number of vertices of each color is the same, the degree of each vertex is D, and there is no edge between vertices of the same color. Further, let $\mathcal{V}^i = \{v^i_1, \ldots v^i_q\}$ denote the set of vertices of color i and $\mathcal{E}^i$ be the set of edges incident to the vertices of color i. It is clear that each edge is in two $\mathcal{E}^i$'s. For each vertex v, let $\mathcal{E}(v)$ denote the set of edges that are incident to v.

We construct an instance of Nanson-CSB as follows. Let $B := k(q+(q-1)D)$. The candidate set is $C = \mathcal{V}(\mathcal{G}) \cup \mathcal{E}(\mathcal{G}) \cup \{c_t, p\} \cup D \cup D' \cup F \cup F'$, where D, D', F, and F' are sets of dummy candidates and $|D| = |D'| = |F| = |F'| = B$. For each vertex v, we define an ordering $\overrightarrow{S(v)}$ as $v > \overrightarrow{\mathcal{E}(v)}$. For each color i, we define $\overrightarrow{R(i)}$ as $\overrightarrow{D'} > \overrightarrow{\mathcal{V} \setminus \mathcal{V}^i} > \overrightarrow{\mathcal{E} \setminus \mathcal{E}^i} > t > \overrightarrow{D}$. Then $\overleftarrow{\mathcal{E}(v)}$ and $\overleftarrow{R(i)}$ denote the reversed orderings of $\overrightarrow{\mathcal{E}(v)}$ and $\overrightarrow{R(i)}$, respectively. We construct the set of votes as follows.

For each color $1 \leq i \leq k$, we create four votes in V^i:

$$x_i : \overrightarrow{S(v^i_1)} > \cdots > \overrightarrow{S(v^i_q)} > p > \overrightarrow{R(i)} > \overrightarrow{F} > \overrightarrow{F'},$$
$$x'_i : \overleftarrow{S(v^i_q)} > \cdots > \overleftarrow{S(v^i_1)} > p > \overrightarrow{R(i)} > \overrightarrow{F} > \overrightarrow{F'},$$
$$y_i : \overleftarrow{R(i)} > p > \overleftarrow{S(v^i_q)} > \cdots > \overleftarrow{S(v^i_1)} > \overleftarrow{F'} > \overleftarrow{F},$$
$$y'_i : \overleftarrow{R(i)} > p > \overleftarrow{S(v^i_1)} > \cdots > \overleftarrow{S(v^i_q)} > \overleftarrow{F'} > \overleftarrow{F}.$$

Further, we create the following six votes in V':

$$z_1 : \overrightarrow{\mathcal{E}(\mathcal{G})} > \overrightarrow{\mathcal{V}(\mathcal{G})} > \overrightarrow{F} > p > \overrightarrow{F'} > c_t > \overrightarrow{D} > \overrightarrow{D'},$$
$$z'_1 : c_t > \overleftarrow{\mathcal{V}(\mathcal{G})} > \overleftarrow{\mathcal{E}(\mathcal{G})} > \overleftarrow{F'} > p > \overleftarrow{F} > \overleftarrow{D'} > \overleftarrow{D},$$
$$z_2 : c_t > \overrightarrow{\mathcal{E}(\mathcal{G})} > \overrightarrow{\mathcal{V}(\mathcal{G})} > \overrightarrow{F} > p > \overrightarrow{F'} > \overrightarrow{D} > \overrightarrow{D'},$$
$$z'_2 : \overleftarrow{F'} > p > \overleftarrow{F} > c_t > \overleftarrow{\mathcal{V}(\mathcal{G})} > \overleftarrow{\mathcal{E}(\mathcal{G})} > \overleftarrow{D'} > \overleftarrow{D},$$
$$z_3 : p > c_t > \overrightarrow{F} > \overrightarrow{F'} > \overrightarrow{\mathcal{E}(\mathcal{G})} > \overrightarrow{\mathcal{V}(\mathcal{G})} > \overrightarrow{D} > \overrightarrow{D'},$$
$$z'_3 : \overleftarrow{\mathcal{V}(\mathcal{G})} > \overleftarrow{\mathcal{E}(\mathcal{G})} > \overleftarrow{F'} > \overleftarrow{F} > p > c_t > \overleftarrow{D'} > \overleftarrow{D}.$$

The vote set V is set equal to $(\bigcup_{i=1}^{k} V^i) \cup V'$. By the construction of the votes, we can observe that each candidate in $C \setminus (F \cup F')$ receives $|C| + 2B$ points from the votes x_i and y_i for each $1 \leq i \leq k$, while each candidate in $F \cup F'$ receives $2B - 1$ points from these two votes. In total, each candidate in $C \setminus (F \cup F')$ receives $2k(|C| + 2B)$ points from the votes in $\bigcup_{i=1}^{k} V^i$, while each candidate in $F \cup F'$ receives $2k(2B - 1)$ points. Thus, concerning the votes in $\bigcup_{i=1}^{k} V^i$, each candidate in $F \cup F'$ receives a Borda score less than the average Borda score. From the remaining six votes, we can conclude that the candidates in $D \cup D'$ have scores less than the average Borda score. It is obvious that the scores of candidates in $D \cup D' \cup F \cup F'$ are less than the average score, and thus, the first round eliminates these candidates. Afterwards, all candidates receive the same points from $\bigcup_{i=1}^{k} V^i$. However, p receives the least point from the remaining six votes. Thus, p is not the unique winner and is eliminated in the second round.

"$\Longrightarrow$": Suppose there is a colorful independent set I for $\mathcal{G}$ and for each color $1 \leq i \leq k$, let $v^i_{s_i}$ be the vertex of color i in I. For each pair of x_i and x'_i, we shift p in x_i by swap operations to the position directly in front of the candidate $v^i_{s_i+1}$ and in x'_i directly in front of the candidate $v^i_{s_i}$. For every pair of votes x_i and x'_i, $q + (q-1) \cdot D$ swaps are needed and in total B swaps are needed. Thus, in this way, all edge and vertex candidates have been swapped with p, and the score of each edge and vertex candidate is decreased by at least one, resulting in that instead of p, the vertex and edge candidates are eliminated in the second round. Then, the candidate c_t is eliminated next and p is the unique winner.

"$\Longleftarrow$": Suppose that p is the unique winner after swapping p with other candidates. Since $|D| = |D'| = |F| = |F'| = B$, p can be swapped with the dummy candidates, or the vertex and edge candidates in x_i or x'_i. No matter which candidates p is swapped with, the dummy candidates have always scores less than the average Borda score and are eliminated in the first round. Suppose that there exist vertex candidates or edge candidates, which are not swapped with p. Let C' denote the set of these candidates. Then, the candidates in $(\mathcal{V}(\mathcal{G}) \cup \mathcal{E}(\mathcal{G})) \setminus C'$ are eliminated next. If $|C'| = 1$, then all remaining candidates have the same Borda score and are eliminated together. Then p is not the unique winner. If $|C'| > 1$, the score of p is less than the average Borda score and p is eliminated after eliminating dummy candidates. Then, p is not the unique winner. To guarantee that p is the unique winner, it must satisfy $|C'| = 0$. It also means that all vertex and edge candidates have been swapped with p. For each color, p has to be swapped with at least $q+(q-1) \cdot D$ candidates and in total, exact B swaps for all colors. After these swaps, p lies directly in front of $v^i_{s_i+1}$ in x_i and directly in front of $v^i_{s_i}$ in x'_i. On the other hand, for each color i, there is a set of edge candidates $\mathcal{E}(v^i_{s_i})$, that are not swapped with p. Thus, to guarantee that the score of each edge candidate is decreased by one, there cannot be any edge between the vertices of $v^i_{s_i}$ with $1 \leq i \leq k$. It also means that the corresponding k vertices form an independent set. Therefore, Nanson-CSB is $W[1]$-hard with respect to the parameter n. □

Finally, we prove W[1]-hardness of the destructive case of Nanson.

Theorem 3.7. *Nanson-DSB is $W[1]$-hard with respect to the parameter n.*

Proof. We prove the theorem by a similar but more tricky reduction from the **Multi-colored Clique** problem. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be our input instance, i.e., an undirected graph with each vertex being colored with one of k colors. Without loss of generality, we assume that the number of vertices of each color is the same, the degree of each vertex is D, and there is no edge between vertices of the same color. Further, let $\mathcal{V}^i = \{v^i_1, \dots v^i_q\}$ denote the set of vertices of color i and $\mathcal{E}^i$ be the set of edges incident to vertices of color i. For each vertex v, let $\mathcal{E}(v)$ denote the set of edges that are incident to v.

We construct an instance of Nanson-DSB as follows. Let $B := k(q + (q + 1)D)$. The candidate set is $C := \mathcal{V}(\mathcal{G}) \cup \mathcal{E}(\mathcal{G}) \cup C_1 \cup C_2 \cup C_3 \cup \{p, c_s, c_d\} \cup H$, where H is a set of dummy candidates with $|H| = 4B$, $C_1 = \{c_{t_1}, c_{t_2}, c_{t_3}, c_{t_4}\}$, $C_2 = \{c_{r_1}, c_{r_2}, c_{r_3}, c_{r_4}\}$, and $C_3 = \{c_{u_1}, c_{u_2}, c_{u_3}, c_{u_4}, c_{u_5}\}$. Again, for each vertex v, we define $\overrightarrow{S(v)}$ as $v > \overrightarrow{\mathcal{E}(v)}$. For each color i, we define $\overrightarrow{R(i)}$ as $\overrightarrow{\mathcal{V} \setminus \mathcal{V}^i} > \overrightarrow{\mathcal{E} \setminus \mathcal{E}^i} > \overrightarrow{C_1} > \overrightarrow{C_2} > \overrightarrow{C_3} > c_s > c_d$. We construct the set of votes as follows. The set H plays the same role as the dummy candidates in the proof of Theorem 3.6. To simplify the presentation, we omit these candidates in the votes.

For each color $1 \leq i \leq k$, we create four votes in V^i:

$$\overrightarrow{x_i} : \overrightarrow{R(i)} > p > \overrightarrow{S(v^i_1)} > \cdots > \overrightarrow{S(v^i_q)},$$
$$\overrightarrow{x'_i} : \overrightarrow{R(i)} > p > \overleftarrow{S(v^i_q)} > \cdots > \overleftarrow{S(v^i_1)},$$

and $\overrightarrow{y_i} := \overleftarrow{x_i}, \overrightarrow{y'_i} := \overleftarrow{x'_i}$ ($\overleftarrow{x_i}$ and $\overleftarrow{x'_i}$ denote the reversed orderings of $\overrightarrow{x_i}$ and $\overrightarrow{x'_i}$, respectively).

We create the following seven vote pairs in V_2:

$$z_1 : \overrightarrow{C_1} > \overrightarrow{\mathcal{E}(\mathcal{G})} > \overrightarrow{C \setminus (\mathcal{E}(\mathcal{G}) \cup C_1)},$$
$$\overleftarrow{C \setminus (\mathcal{E}(\mathcal{G}) \cup C_1)} > \overleftarrow{C_1} > \overleftarrow{\mathcal{E}(\mathcal{G})};$$
$$z_2 : \overrightarrow{\mathcal{V}(\mathcal{G})} > \overrightarrow{C_1} > \overrightarrow{C \setminus (\mathcal{V}(\mathcal{G}) \cup C_1)},$$
$$\overleftarrow{C \setminus (\mathcal{V}(\mathcal{G}) \cup C_1)} > \overleftarrow{\mathcal{V}(\mathcal{G})} > \overleftarrow{C_1};$$
$$z_3 : \overrightarrow{C_2} > \overrightarrow{\mathcal{V}(\mathcal{G})} > \overrightarrow{C \setminus (\mathcal{V}(\mathcal{G}) \cup C_2)},$$
$$\overleftarrow{C \setminus (\mathcal{V}(\mathcal{G}) \cup C_2)} > \overleftarrow{C_2} > \overleftarrow{\mathcal{V}(\mathcal{G})};$$
$$z_4 : p > \overrightarrow{C_2} > \overrightarrow{C \setminus (\{p\} \cup C_2)},\ \overleftarrow{C \setminus (\{p\} \cup C_2)} > p > \overleftarrow{C_2};$$
$$z_5 : \overrightarrow{C_1} > p > \overrightarrow{C \setminus (\{p\} \cup C_1)},\ \overleftarrow{C \setminus (\{p\} \cup C_1)} > \overleftarrow{C_1} > p;$$
$$z_6 : \overrightarrow{\mathcal{E}(\mathcal{G})} > c_s > \overrightarrow{C \setminus (\mathcal{E}(\mathcal{G}) \cup \{c_s\})},$$
$$\overleftarrow{C \setminus (\mathcal{E}(\mathcal{G}) \cup \{c_s\})} > \overleftarrow{\mathcal{E}(\mathcal{G})} > c_s;$$
$$z_7 : p > c_s > \overrightarrow{C \setminus \{p, c_s\}},\ \overleftarrow{C \setminus \{p, c_s\}} > p > c_s.$$

Further, V_3 contains the following seven vote pairs:

$$z_8 : p > c_d > \overrightarrow{\mathcal{E}(\mathcal{G})} > \overrightarrow{C \setminus \mathcal{E}(\mathcal{G})},\ \overleftarrow{C \setminus \mathcal{E}(\mathcal{G})} > p > \overleftarrow{\mathcal{E}(\mathcal{G})} > c_d;$$
$$z_9 : c_d > c_{u_1} > \overrightarrow{C \setminus \{c_d, c_{u_1}\}},\ \overleftarrow{C \setminus \{c_d, c_{u_1}\}} > c_d > c_{u_1};$$
$$z_{10} : c_{u_4} > c_{u_2} > \overrightarrow{C \setminus \{c_{u_2}, c_{u_4}\}},\ \overleftarrow{C \setminus \{c_{u_2}, c_{u_4}\}} > c_{u_4} > c_{u_2};$$
$$z_{11} : c_{u_2} > p > \overrightarrow{C \setminus \{c_{u_2}, p\}},\ \overleftarrow{C \setminus \{c_{u_2}, p\}} > c_{u_2} > p;$$
$$z_{12} : c_{u_2} > c_{u_3} > \overrightarrow{C \setminus \{c_{u_2}, c_{u_3}\}},\ \overleftarrow{C \setminus \{c_{u_2}, c_{u_3}\}} > c_{u_2} > c_{u_3};$$
$$z_{13} : p > c_{u_4} > \overrightarrow{C \setminus \{p, c_{u_4}\}},\ \overleftarrow{C \setminus \{p, c_{u_4}\}} > p > c_{u_4};$$
$$z_{14} : c_{u_3} > c_{u_5} > \overrightarrow{C \setminus \{c_{u_3}, c_{u_5}\}},\ \overleftarrow{C \setminus \{c_{u_3}, c_{u_5}\}} > c_{u_3} > c_{u_5}.$$

Note that in V_3, there are four identical copies of z_8 and z_9, three identical copies of z_{10}, two identical copies of z_{11}; and one copy for each of other votes. Let $V_1 = \bigcup V^i$ and $V := V_1 \cup V_2 \cup V_3$ and $|V| = 4k + 46$. It is easy to verify that p is the unique winner, as $\mathcal{E}(\mathcal{G}) \cup \{c_s, c_{u_1}, c_{u_5}\}$ are eliminated in the first round, $C_1 \cup \{c_d, c_{u_3}\}$ in the second round, and $\mathcal{V}(\mathcal{G}) \cup C_2 \cup \{c_{u_2}\}$ in the third round. The role of $C_1 \cup C_2 \cup \{c_s\}$ is to control in which round $\mathcal{E}(\mathcal{G}) \cup \mathcal{V}(\mathcal{G})$ are eliminated, and $C_3 \cup \{c_d\}$ is to control in which round p is eliminated. The proof of the equivalence between the two instances is deferred to the long version. □

4 CONCLUSION

We achieved FPT and $W[1]$-hard results for both constructive and destructive shift bribery problems on the iterative voting systems of Hare, Coombs, Baldwin, and Nanson. There remain some open problems. For instance, the parameterized complexity of Baldwin-DSB, Coombs-CSB, and Coombs-DSB are open with respect to the number of votes. Moreover, we only considered the shift bribery with the unit price function. It would be interesting to study other price functions such as the all-or-nothing price function. Some of our results hold for other price functions (all FPT results), but some do not. One might think that the shift bribery problems with all-or-nothing price function could be easier to solve than the ones with unit price function, because with the all-or-nothing function, it seems reasonable to shift p to the first position in the constructive case and to the last position in the destructive case. However, there exist concrete examples, where the optimal shift strategy is to leave p in the middle of some votes. Another direction for future work can be the approximability of shift bribery problems for these systems. Furthermore, the shift bribery behavior of other iterative voting systems such as Plurality with Runoff could be an interesting research topic. Finally, we are not aware of any computational complexity result for controlling iterative voting systems.

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