

On Stable Matchings with Pairwise Preferences and Matroid Constraints

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ABSTRACT

In this paper, we consider the following generalization of the stable matching problem. We are given a set of doctors and a set of hospitals. In the classical model, each doctor has a strict total order over the hospitals. On the other hand, in our model, each doctor has a pairwise preference over the hospitals, which was introduced by Farzadi, Georgiou, and Könemann. Roughly speaking, in a pairwise preference, transitivity does not necessarily hold, and a comparison between some hospitals is not relevant to stability. Furthermore, we generalize capacity constraints on the hospitals to matroid constraints. Especially, we focus on the situation in which we are given a master list over the doctors, and the preference list of each hospital over the doctors is derived from this master list. For this problem, we give several hardness results and polynomial-time solvable cases.

KEYWORDS

stable matching, pairwise preference, matroid

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1 INTRODUCTION

The stable matching problem, which was introduced by Gale and Shapley [10], is one of the most important mathematical models for real-world assignment problems. The National Resident Matching Program in the U.S. is a famous real-world matching scheme, which assigns graduating medical students to hospitals (see, e.g., [33]). In this problem, there exist two groups of agents, and each agent has a preference list over the agents on the other side. Then the goal is to find a stable matching between these groups for the preference lists. Many generalizations of the stable matching problem have been studied (see, e.g., [27]). In this paper, we consider the following two directions of generalization of the stable matching problem.

The first direction is generalization of preference lists. In this direction, Irving [11] started the study of ties in preference lists. It is known that if there exist ties in preference lists, then the situation dramatically changes (see, e.g., [15] and [27, Chapter 3] for a survey of stable matchings with ties). For the stable matching problem with ties, Irving [11] proposed three stability concepts,

which are called weak stability, strong stability, and super-stability. Irving [11] proved that there always exists a weakly stable matching, and a weakly stable matching can be found in polynomial time by slightly modifying the algorithm of Gale and Shapley [10]. However, a strongly stable matching and a super-stable matching do not necessarily exist. Farzadi, Georgiou, and Könemann [6] introduced a more general preference, which is called a pairwise preference (see also [4] for its motivation). Roughly speaking, in a pairwise preference, transitivity does not necessarily hold, and a comparison between some partners is not relevant to stability.

The second direction is generalization of capacity constraints to matroid constraints. Matroids can represent not only capacity constraints but also more complex constraints including hierarchical capacity constraints (see, e.g., [2]). Thus, matroid constraints are important from not only the theoretical viewpoint but also the practical viewpoint. Matroid generalizations of the stable matching problem [7–9, 16, 19, 20, 23, 28, 35] and the popular matching problem [17, 18] have been extensively studied.

In this paper, we consider the stable matching problem with pairwise preferences and matroid constraints. Especially, we focus on the situation in which we are given a master list, and the preference list of each agent on one side is derived from this master list. There exist two reasons to consider this situation. First, a master list is practically useful (see [14]). Second, to the best of our knowledge, the computational complexity of the general version of the stable matching problem with ties and matroid constraints is open. However, the stable matching problem with ties, master lists, and matroid constraints can be solved in polynomial time [17, 20].

Our contributions. Our contributions are summarized as follows. In this paper, we assume that we are given a set of doctors, a set of hospitals, and a master list over the doctors (i.e., each doctor has a pairwise preference over the hospitals). In Section 3, we prove that the decision versions of the weakly stable matching problem, the super-stable matching problem, and the strongly stable matching problem are **NP**-complete. Especially, we prove that the decision versions of the weakly stable matching problem and the strongly stable matching problem are **NP**-complete even if every hospital is indifferent between every pair of doctors. In Section 4, we first prove that if every hospital is indifferent between every pair of doctors, then the super-stable matching problem can be solved in polynomial time. Furthermore, we prove that if every preference is asymmetric (see Section 4 for its definition), then the super-stable matching problem can be solved in polynomial time. In Section 5, we prove that if every preference is asymmetric, then the strongly stable matching problem can be solved in polynomial time.

Technical highlights. In the proofs of the **NP**-completeness of the problems of checking the existence of a weakly stable matching,

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a super-stable matching, and a strongly stable matching with pairwise preferences in [6, Theorem 1] and [4, Theorem 4], preference lists are not derived from a master list. Thus, our technical highlight in the proofs of hardness results is how to reduce an NP-complete problem to an instance with a master list.

Our technical highlight in polynomial-time algorithms can be described as follows. Our algorithms are based on the results in [17]. In the algorithm proposed in [17] for the strongly stable matching problem, when some doctor applies, this doctor simply chooses the hospitals that are not dominated by any other hospital. However, in our setting, this is not sufficient. We have to carefully select a subset of the hospitals that are not dominated (see Algorithm 4).

Related work. In the one-to-one setting, Irving [11] proposed polynomial-time algorithms for finding a super-stable matching and a strongly stable matching (see also [26]). In the many-to-one setting, Irving, Manlove, and Scott [12] proposed a polynomial-time algorithm for finding a super-stable matching, and Irving, Manlove, and Scott [13] and Kavitha, Mehlhorn, Michail, and Paluch [22] proposed polynomial-time algorithms for finding a strongly stable matching. In the many-to-many setting, Scott [34] considered super-stability, and the papers [3, 24, 25] considered strong stability. Olaosebikanand and Manlove [29, 30] considered super-stability and strong stability in the student-project allocation problem.

For the situation in which a master list is given, Irving, Manlove, and Scott [14] gave simple polynomial-time algorithms for finding a super-stable matching and a strongly stable matching. O'Malley [31] gave polynomial-time algorithms for finding a super-stable matching and a strongly stable matching in the many-to-one setting. For matroid constraints, Kamiyama [17, 20] gave polynomial-time algorithms for finding a super-stable matching and a strongly stable matching in the many-to-one and many-to-many settings. It should be noted that Kamiyama [17, 20] considered usual preferences (i.e., complete and transitive preference lists).

Farzadi, Georgiou, and Könemann [6] introduced the stable matching problem with pairwise preferences, and they gave hardness results and a polynomial-time solvable case. Furthermore, in [1], the authors proved that if both sides can have cycles in the preference lists, then the problem of determining the existence of a weakly stable matching is NP-complete. Furthermore, Cseh and Juhos [4] gave a complete picture of the complexity of the stable matching problem with pairwise preferences.

2 PRELIMINARIES

For each set X and each element x , we define $X + x := X \cup \{x\}$ and $X - x := X \setminus \{x\}$. For each positive integer z , we define $[z] := \{1, 2, \dots, z\}$. Define $[0] := \emptyset$. A pair $\mathbf{M} = (U, \mathcal{I})$ of a finite set U and a non-empty family \mathcal{I} of subsets of U is called a *matroid* if for every pair of subsets I, J of U , the following conditions are satisfied.

- (I1) If $I \subseteq J$ and $J \in \mathcal{I}$, then $I \in \mathcal{I}$.
- (I2) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists an element u in $J \setminus I$ such that $I + u \in \mathcal{I}$.

A member of \mathcal{I} is called an *independent set* of \mathbf{M} .

2.1 Problem formulation

In this paper, we are given a finite simple (not necessarily complete) bipartite graph $G = (V, E)$ such that V is partitioned into subsets

D and H , and each edge in E connects a vertex in D and a vertex in H . We call a vertex in D (resp., H) a *doctor* (resp., *hospital*). If there exists an edge in E that connects a doctor d in D and a hospital h in H , then we denote by (d, h) this edge. In this paper, we assume that $|D| \leq |E|$ and $|H| \leq |E|$. For each subset F of E and each doctor d in D (resp., hospital h in H), we denote by $F(d)$ (resp., $F(h)$) the set of edges (d', h') in F such that $d' = d$ (resp., $h' = h$). We are given a complete (i.e., for every pair of doctors d, d' , at least one of $d \succeq_H d'$ and $d' \succeq_H d$ holds) and transitive binary relation \succeq_H on D . For each pair of doctors d, d' in D , we write $d >_H d'$ (resp., $d \sim_H d'$) if $d \succeq_H d'$ and $d' \not\succeq_H d$ (resp., $d \succeq_H d'$ and $d' \succeq_H d$). For each doctor d in D , we are given a subset R_d of $E(d) \times E(d)$ that does not contain (e, e) for any edge e in $E(d)$. Assume that we are given a doctor d in D and a pair of edges e, f in $E(d)$.

- If $(e, f) \in R_d$ and $(f, e) \notin R_d$, then we write $e >_d f$.
- If $(e, f) \notin R_d$ and $(f, e) \notin R_d$, then we write $e \sim_d f$.
- If $(e, f) \in R_d$ and $(f, e) \in R_d$, then we write $e \parallel_d f$.

Notice that if $e >_d f$, then $f \not>_d e$. Furthermore, if $e \sim_d f$ (resp., $e \parallel_d f$), then $f \sim_d e$ (resp., $f \parallel_d e$). For each doctor d in D and each pair of edges e, f in $E(d)$, we write $e \succeq_d f$ if at least one of $e >_d f$ and $e \sim_d f$ holds. Intuitively speaking, $e >_d f$ means that d strictly prefers e to f (i.e., there exists a strong incentive to move from f to e), $e \sim_d f$ means that d does not know the relation between e and f yet (i.e., there exists a weak incentive to move between e and f), and $e \parallel_d f$ means that e and f are equally good for d (i.e., there does not exist an incentive to move between e and f). See also [4]. Lastly, we are given a matroid $\mathbf{N} = (E, \mathcal{J})$ such that for every edge e in E , $\{e\} \in \mathcal{N}$. For example, \mathbf{N} can represent capacity constraints on H (see Section 3). We assume that for every subset I of E , we can determine whether $I \in \mathcal{J}$ in time bounded by a polynomial in the size of G . That is, in this paper, we consider the oracle model. We denote by EO the time required to determine whether $I \in \mathcal{J}$ for each subset I of E . In this paper, we assume that $\text{EO} \geq |E|$.

A subset M of E is called a *matching* in G if

- (M1) $|M(d)| \leq 1$ for every doctor d in D , and
- (M2) $M \in \mathcal{J}$.

For each matching M in G and each doctor d in D such that $M(d) \neq \emptyset$, we denote by $\mu_M(d)$ the unique edge in $M(d)$.

Assume that we are given a matching M in G and an edge $e = (d, h)$ in $E \setminus M$. We say that d *weakly* (resp., *strongly*) *prefers* e on M if one of the following conditions is satisfied.

- (D1) $M(d) = \emptyset$.
- (D2) $M(d) \neq \emptyset$ and $e \succeq_d \mu_M(d)$ (resp., $e >_d \mu_M(d)$).

Furthermore, we say that H *weakly* (resp., *strongly*) *prefers* e on M if one of the following conditions is satisfied.

- (H1) $M + e \in \mathcal{J}$.
- (H2) $M + e \notin \mathcal{J}$, and there exists an edge $f = (d', h')$ in M such that $d \succeq_H d'$ (resp., $d >_H d'$) and $M + e - f \in \mathcal{J}$.

A matching M in G is said to be *weakly stable* if there does not exist an edge $e = (d, h)$ in $E \setminus M$ such that d and H strongly prefer e on M . A matching M in G is said to be *super-stable* if there does not exist an edge $e = (d, h)$ in $E \setminus M$ such that d and H weakly prefer e on M . A matching M in G is said to be *strongly stable* if there does not exist an edge $e = (d, h)$ in $E \setminus M$ such that d and H weakly prefer e on M , and at least one of d and H strongly prefers e on M . In the

problem WEAKLY STABLE MATCHING, we first solve the decision version of WEAKLY STABLE MATCHING, which is defined below, and if a weakly stable matching in G exists, then we find a weakly stable matching in G . The *decision version* of WEAKLY STABLE MATCHING is the problem of determining whether there exists a weakly stable matching in G . Similarly, the problems SUPER-STABLE MATCHING and STRONGLY STABLE MATCHING are defined.

Since \succeq_H is transitive, it is not difficult to see that there exists a unique positive integer k such that we can partition D into the non-empty subsets D_1, D_2, \dots, D_k satisfying the following conditions.

- For every integer i in $[k]$ and every pair of doctors d, d' in D_i , we have $d \sim_H d'$.
- For every pair of integers i, j in $[k]$ such that $i < j$ and every pair of doctors d in D_i and d' in D_j , we have $d \succ_H d'$.

We can find such a partition in $O(|E|^2)$ time. For each integer i in $[k]$, we define $D[i] := \bigcup_{j=1}^i D_j$. Define $D[0] := \emptyset$.

Assume that we are given a subset F of E . Define $\mathcal{U}(F)$ as the family of subsets F' of F such that $|F'(d)| \leq 1$ for every doctor d in D . Define the matroid $\mathbf{U}(F)$ by $\mathbf{U}(F) := (F, \mathcal{U}(F))$.

2.2 Basics of matroids

Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. Then a subset C of U is called a *circuit* of \mathbf{M} if C is not an independent set of \mathbf{M} , but every proper subset of C is an independent set of \mathbf{M} .

LEMMA 2.1 (SEE, E.G., [32, LEMMA 1.1.3]). *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. Then for every pair of distinct circuits C_1, C_2 of \mathbf{M} such that $C_1 \cap C_2 \neq \emptyset$ and every element u in $C_1 \cap C_2$, there exists a circuit C of \mathbf{M} such that $C \subseteq (C_1 \cup C_2) \setminus \{u\}$.*

Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$ and an independent set I of \mathbf{M} . It is not difficult to see that for every element u in $U \setminus I$ such that $I + u \notin \mathcal{I}$, $I + u$ contains a circuit of \mathbf{M} as a subset, and (I1) implies that u belongs to this circuit. Furthermore, Lemma 2.1 implies that such a circuit is uniquely determined. We call this circuit the *fundamental circuit* of u for I and \mathbf{M} , and we denote by $C_{\mathbf{M}}(u, I)$ this circuit. It is well known (see, e.g., [32, p.20, Exercise 5]) that for every element u in $U \setminus I$ such that $I + u \notin \mathcal{I}$, $C_{\mathbf{M}}(u, I)$ is the set of elements u' in $I + u$ such that $I + u - u' \in \mathcal{I}$. For each element u in $U \setminus I$ such that $I + u \notin \mathcal{I}$, we define $D_{\mathbf{M}}(u, I) := C_{\mathbf{M}}(u, I) - u$. Then for each matching M in G and each edge $e = (d, h)$ in $E \setminus M$ such that $M + e \notin \mathcal{J}$, (H2) can be restated as follows. There exists an edge (d', h') in $D_{\mathbf{N}}(e, M)$ such that $d \succeq_H d'$ (resp., $d \succ_H d'$).

We can easily prove the following lemma by using Lemma 2.1.

LEMMA 2.2. *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, independent sets I, J of \mathbf{M} such that $I \subseteq J$, and an element u in $U \setminus J$ such that $I + u \notin \mathcal{I}$. Then $J + u \notin \mathcal{I}$ and $C_{\mathbf{M}}(u, I) = C_{\mathbf{M}}(u, J)$.*

LEMMA 2.3 (SEE, E.G., [17, LEMMA 2]). *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, circuits C, C_1, C_2, \dots, C_x of \mathbf{M} , and distinct elements u_1, u_2, \dots, u_x in U . Furthermore, we assume that the following conditions are satisfied.*

- $u_i \in C \cap C_i$ holds for every integer i in $[x]$.
- $u_{i_1} \notin C_{i_2}$ holds for every pair of distinct integers i_1, i_2 in $[x]$.
- $C \setminus (\bigcup_{i=1}^x C_i) \neq \emptyset$.

Then there exists a circuit C' of \mathbf{M} such that $C' \subseteq (C \cup (\bigcup_{i=1}^x C_i)) \setminus \{u_1, u_2, \dots, u_x\}$.

Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. Then a maximal independent set of \mathbf{M} is called a *base* of \mathbf{M} . Notice that the condition (I2) implies that all the bases of \mathbf{M} have the same size. For each subset X of U , we define $\mathcal{I}|X$ as the family of subsets I of X such that $I \in \mathcal{I}$, and we define $\mathbf{M}|X := (X, \mathcal{I}|X)$. It is known [32, p.20] that for every subset X of U , $\mathbf{M}|X$ is a matroid. For each subset X of U , we define $r_{\mathbf{M}}(X)$ as the size of a base of $\mathbf{M}|X$. Furthermore, for each pair of disjoint subsets X, I of U , we define $p(I; X)$ as $r_{\mathbf{M}}(I \cup X) - r_{\mathbf{M}}(X)$. For each subset X of U , we define \mathcal{I}/X as the family of subsets I of $U \setminus X$ such that $p(I; X) = |I|$, and we define $\mathbf{M}/X := (U \setminus X, \mathcal{I}/X)$. It is known [32, Proposition 3.1.6] that for every subset X of U , \mathbf{M}/X is a matroid.

LEMMA 2.4 (SEE, E.G., [32, PROPOSITION 3.1.25]). *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. Then for every pair of disjoint subsets X, Y of U , $(\mathbf{M}/X)|Y = (\mathbf{M}|(X \cup Y))/X$.*

LEMMA 2.5 (SEE, E.G., [32, PROPOSITION 3.1.7]). *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, a subset X of U , and a base B of $\mathbf{M}|X$. For every subset I of $U \setminus X$, I is an independent set (resp., a base) of \mathbf{M}/X if and only if $I \cup B$ is an independent set (resp., a base) of \mathbf{M} .*

Assume that we are given two matroids $\mathbf{M}_1 = (U, \mathcal{I}_1)$ and $\mathbf{M}_2 = (U, \mathcal{I}_2)$. A subset I of U is called a *common independent set* of \mathbf{M}_1 and \mathbf{M}_2 if $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. If we can determine whether $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ in time bounded by a polynomial in $|U|$ for every subset I of U , then we can find a maximum-size common independent set of \mathbf{M}_1 and \mathbf{M}_2 in time bounded by a polynomial in $|U|$. For example, if we use the algorithm proposed in [5], we can find a maximum-size common independent set of \mathbf{M}_1 and \mathbf{M}_2 in $O(|U|^{2.5} \text{EO}')$ time, where EO' represents the time required to determine whether $I \in \mathcal{I}_i$ for each subset I of U and each integer i in $\{1, 2\}$.

3 HARDNESS RESULTS

The goal of this section is to prove that the decision versions of our problems are **NP**-complete by reduction from VERTEX COVER. In VERTEX COVER, we are given a finite simple undirected graph $Q = (N, L)$ such that $N = \{v_1, v_2, \dots, v_n\}$ and $L = \{e_1, e_2, \dots, e_m\}$, where we define $n := |N|$ and $m := |L|$. We are given a positive integer w such that $w \leq n$. If there exists an edge in L connecting vertices v_i and v_j in N , then we denote by $v_i v_j$ this edge. For each subset C of N , we call C a *vertex cover* in Q if $|\{u, v\} \cap C| \geq 1$ for every edge uv in L . Then the goal of VERTEX COVER is to determine whether there exists a vertex cover C in Q such that $|C| \leq w$. It is known [21] that VERTEX COVER is **NP**-complete. We prove that the decision versions of WEAKLY STABLE MATCHING WITH CAPACITIES, SUPER-STABLE MATCHING WITH CAPACITIES, and STRONGLY STABLE MATCHING WITH CAPACITIES, which are defined below, are **NP**-complete. In these problems, we are given a positive integer $c(h)$ for each hospital h in H , and \mathcal{J} is defined as the family of subsets I of E such that $|I(h)| \leq c(h)$ for every hospital h in H .

THEOREM 3.1. *The decision version of STRONGLY STABLE MATCHING WITH CAPACITIES is **NP**-complete even if $k = 1$.*

PROOF. Clearly, the decision version of STRONGLY STABLE MATCHING WITH CAPACITIES is in **NP**. Assume that we are given an instance (i.e., Q and w) of VERTEX COVER. We construct an instance of STRONGLY STABLE MATCHING WITH CAPACITIES as follows (see

Figure 1). For each vertex v_i in N , D contains a vertex a_i . For each edge e_i in L , D contains a vertex z_i . For each integer i in $[n - w]$, D contains a vertex b_i . For each vertex v_i in N , H contains a vertex p_i . For each edge $e_i = v_{j_1}v_{j_2}$ in L , H contains vertices q_{i,j_1} and q_{i,j_2} . Furthermore, H contains vertices $s, r_1, r_2, \dots, r_{n-w}$. For each vertex v_i in N , E contains edges (a_i, p_i) and (a_i, s) . For each edge $e_i = v_{j_1}v_{j_2}$ in L and each integer j in $\{j_1, j_2\}$, E contains edges (z_i, p_j) and $(z_i, q_{i,j})$. For each integer i in $[n - w]$, E contains edges (b_i, r_i) and (b_i, s) . The capacity of each hospital is defined as follows. For each vertex v_i in N , we define $c(p_i) := 1$. For each edge $e_i = v_{j_1}v_{j_2}$ in L and each integer j in $\{j_1, j_2\}$, we define $c(q_{i,j}) := 1$. Define $c(s) := n - w$. Furthermore, for each integer i in $[n - w]$, we define $c(r_i) := 1$. The preferences are defined as follows.

- (A1) For each vertex v_i in N , $(a_i, p_i) \parallel_{a_i} (a_i, s)$.
- (A2) For each edge $e_i = v_{j_1}v_{j_2}$ in L ,
 - $(z_i, q_{i,j_1}) \sim_{z_i} (z_i, p_{j_1}), (z_i, q_{i,j_2}) \sim_{z_i} (z_i, p_{j_2})$,
 - $(z_i, q_{i,j_1}) \parallel_{z_i} (z_i, p_{j_2}), (z_i, q_{i,j_2}) \parallel_{z_i} (z_i, p_{j_1})$, and
 - $(z_i, q_{i,j_1}) \parallel_{z_i} (z_i, q_{i,j_2}), (z_i, p_{j_1}) \parallel_{z_i} (z_i, p_{j_2})$.
- (A3) For each integer i in $[n - w]$, $(b_i, s) \sim_{b_i} (b_i, r_i)$.
- (A4) For each pair of doctors d, d' in D , $d \sim_H d'$.

Notice that we can construct this instance in polynomial time.

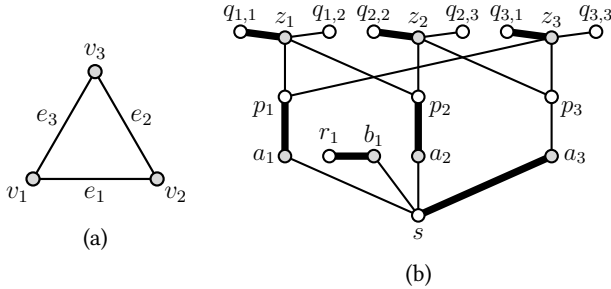


Figure 1: (a) An instance of VERTEX COVER such that $w = 2$. (b) The graph obtained from the instance in (a). The bold edges represent the matching obtained from a solution $\{v_1, v_2\}$ of VERTEX COVER.

First, we assume that there exists a vertex cover C in Q such that $|C| \leq w$. Without loss of generality, we can assume that $|C| = w$ by adding vertices. We construct a matching M in G as follows. For each vertex v_i in C , M contains the edge (a_i, p_i) . For each vertex v_i in $N \setminus C$, M contains the edge (a_i, s) . Furthermore, for each integer i in $[n - w]$, M contains the edge (b_i, r_i) . For each edge $e_i = v_{j_1}v_{j_2}$ in L , M contains the edge $(z_i, q_{i,j})$ for exactly one vertex v_j in $\{v_{j_1}, v_{j_2}\} \cap C$. (Recall that $|\{v_{j_1}, v_{j_2}\} \cap C| \geq 1$.)

What remains is to prove that M is a strongly stable matching in G . Since $|M(s)| = n - w$, M is a matching in G . Next, we prove that M is strongly stable. For every edge (a_i, p_i) in $E \setminus M$, since $(a_i, s) \in M$, (A1) implies that a_i does not weakly prefer (a_i, p_i) on M . For every edge (a_i, s) in $E \setminus M$, since $(a_i, p_i) \in M$, (A1) implies that a_i does not weakly prefer (a_i, s) on M . Let $e_i = v_{j_1}v_{j_2}$ be an edge in L . Assume that $(z_i, q_{i,j}) \in M$ for a vertex v_j in $\{v_{j_1}, v_{j_2}\} \cap C$. Then (A2) implies that it is sufficient to consider the edge (z_i, p_j) in $E \setminus M$. First, (A2) implies that z_i does not strongly prefer (z_i, p_j)

on M . Furthermore, since $(a_j, p_j) \in M$ and $a_j \sim_H z_i$, H does not strongly prefer (z_i, p_j) on M . Lastly, for every integer i in $[n - w]$, since $|M(s)| = n - w$, H does not strongly prefer (b_i, s) on M . For every integer i in $[n - w]$, since $(b_i, r_i) \in M$, (A3) implies that b_i does not strongly prefer (b_i, s) on M . These observations imply that M is strongly stable.

Next, we assume that we are given a strongly stable matching M in G . Define C as the set of vertices v_i in N such that $(a_i, p_i) \in M$.

CLAIM 1. $|M(s)| = c(s)$. Furthermore, M does not contain (b_i, s) for any integer i in $[n - w]$.

PROOF. Assume that $|M(s)| < c(s)$. Then there exists an integer i in $[n - w]$ such that $(b_i, s) \notin M(s)$. Thus, H strongly prefers (b_i, s) on M . Furthermore, b_i weakly prefers (b_i, s) on M . This contradicts the fact that M is strongly stable.

Assume that there exists an integer i in $[n - w]$ such that $(b_i, s) \in M$. In this case, H strongly prefers (b_i, r_i) on M . Furthermore, b_i weakly prefers (b_i, r_i) on M . This contradicts the fact that M is strongly stable. \square

CLAIM 2. For every edge $e_i = v_{j_1}v_{j_2}$ in L and every integer j in $\{j_1, j_2\}$, we have $(z_i, p_j) \notin M$.

PROOF. Assume that there exist an edge $e_i = v_{j_1}v_{j_2}$ in L and an integer j in $\{j_1, j_2\}$ such that $(z_i, p_j) \in M$. Then (A2) implies that z_i weakly prefers $(z_i, q_{i,j})$ on M . Furthermore, since $M(q_{i,j}) = \emptyset$, H strongly prefers $(z_i, q_{i,j})$ on M . This contradicts the fact that M is strongly stable. \square

We are now ready to prove that C is a vertex cover in Q such that $|C| \leq w$. First, we prove that $|C| \leq w$. Define w' as the number of integers i in $[n]$ such that $(a_i, s) \notin M$. Then Claim 1 implies that $|C| \leq w' \leq w$. This completes the proof. Next, we prove that C is a vertex cover in Q . Assume that there exists an edge $e_i = v_{j_1}v_{j_2}$ in L such that $|\{v_{j_1}, v_{j_2}\} \cap C| = 0$. Then $(a_{j_1}, p_{j_1}), (a_{j_2}, p_{j_2}) \notin M$. Thus, Claim 2 implies that $M(p_{j_1}) = M(p_{j_2}) = \emptyset$. First, we assume that $M(z_i) = \emptyset$. In this case, z_i strongly prefers every edge in $E(z_i)$ on M , and H strongly prefers (z_i, p_{j_1}) on M . This contradicts the fact that M is strongly stable. Second, we assume that there exists an integer j in $\{j_1, j_2\}$ such that M contains the edge $(z_i, q_{i,j})$. In this case, (A2) implies that z_i weakly prefers (z_i, p_j) on M . Furthermore, since $M(p_j) = \emptyset$, H strongly prefers (z_i, p_j) on M . This contradicts the fact that M is strongly stable. This completes the proof. \square

THEOREM 3.2. The decision version of SUPER-STABLE MATCHING WITH CAPACITIES is NP-complete even if $k = 2$.

PROOF. By defining \succsim_H so that $D_1 = \{a_1, a_2, \dots, a_n\}$ and

$$D_2 = \{z_1, z_2, \dots, z_m\} \cup \{b_1, b_2, \dots, b_{n-w}\},$$

we can prove this theorem in the same way as Theorem 3.1. \square

THEOREM 3.3. The decision version of WEAKLY STABLE MATCHING WITH CAPACITIES is NP-complete even if $k = 1$ and there do not exist a doctor d in D and a pair of edges e, f in $E(d)$ such that $e \parallel_d f$.

PROOF. We can prove this theorem in a similar way as Theorem 3.1. For each integer i in $[n - w]$, D contains vertices $r_{i,1}$ and $r_{i,2}$ instead of r_i , and E contains edges $(b_i, r_{i,1})$ and $(b_i, r_{i,2})$ instead of (b_i, r_i) . The preferences are defined as follows.

- (B1) For each vertex v_i in N , we have $(a_i, p_i) \sim_{a_i} (a_i, s)$.
 (B2) For each edge $e_i = v_{j_1} v_{j_2}$ in L ,
 - $(z_i, p_{j_1}) \succ_{z_i} (z_i, q_{i, j_1}), (z_i, q_{i, j_2}) \succ_{z_i} (z_i, p_{j_1})$,
 - $(z_i, p_{j_2}) \succ_{z_i} (z_i, q_{i, j_2}), (z_i, q_{i, j_1}) \succ_{z_i} (z_i, p_{j_2})$, and
 - $(z_i, q_{i, j_1}) \sim_{z_i} (z_i, q_{i, j_2}), (z_i, p_{j_1}) \sim_{z_i} (z_i, p_{j_2})$.
 (B3) For each integer i in $[n-w]$, $(b_i, s) \succ_{b_i} (b_i, r_{i,1}), (b_i, r_{i,2}) \succ_{b_i} (b_i, s)$, and $(b_i, r_{i,1}) \succ_{b_i} (b_i, r_{i,2})$.
 (B4) For each pair of doctors d, d' in D , $d \sim_H d'$.

The rest of the proof is almost the same as Theorem 3.1 \square

4 SUPER-STABLE MATCHINGS

4.1 Case of $k = 1$

In this subsection, we assume that $k = 1$ (i.e., $d \sim_H d'$ for every pair of doctors d, d' in D). Then the goal is to prove that in this case, SUPER-STABLE MATCHING can be solved in polynomial time. For each doctor d in D , we define S_d as the set of edges e in $E(d)$ such that $f \succ_d e$ does not hold for any edge f in $E(d) - e$. It is not difficult to see that for every doctor d in D , we can find S_d in $O(|E(d)|^2)$ time. Define $S := \bigcup_{d \in D} S_d$. Our algorithm is described in Algorithm 1 (see Section 2 for the definition of the matroid $\mathbf{U}(S)$). Roughly speaking, we prove that this case can be reduced to the problem of finding a maximum-size common independent set of $\mathbf{U}(S)$ and $\mathbf{N}|S$. We can find a maximum-size common independent set of $\mathbf{U}(S)$ and $\mathbf{N}|S$ in $O(|E|^{2.5} \text{EO})$ time by using the algorithm in [5]. Thus, the time complexity of Algorithm 1 is $O(|E|^{2.5} \text{EO})$.

Algorithm 1: Case of $k = 1$

- 1 Find a maximum-size common independent set K of $\mathbf{U}(S)$ and $\mathbf{N}|S$.
 - 2 **if** $|K| < |\{d \in D \mid E(d) \neq \emptyset\}|$ **then**
 - 3 | Halt. (There does not exist a super-stable matching in G .)
 - 4 **end**
 - 5 Output K , and halt. (K is a super-stable matching in G .)
-

LEMMA 4.1. *Assume that there exists a super-stable matching M in G . Then M is a common independent set of $\mathbf{U}(S)$ and $\mathbf{N}|S$.*

PROOF. Since M is a matching in G , $|M(d)| \leq 1$ for every doctor d in D and $M \in \mathcal{J}$. Thus, it is sufficient to prove that M is a subset of S . Assume that there exists an edge $e = (d, h)$ in $M \setminus S$. Then there exists an edge f in $E(d) - e$ such that $f \succ_d e$. This implies that d weakly prefers f on M . The assumption of this subsection implies that H weakly prefers f on M . This contradicts the fact that M is super-stable. \square

We can easily prove the following theorem by Lemma 4.1.

THEOREM 4.2. *If $k = 1$, then Algorithm 1 can solve SUPER-STABLE MATCHING.*

4.2 Asymmetric preference case

In this subsection, we assume that there do not exist a doctor d in D and a pair of edges e, f in $E(d)$ such that $e \parallel_d f$. We call this case the *asymmetric preference case*. The goal of this subsection is to prove Theorem 4.7. In the rest of this paper, we define $\mathbf{C}(\cdot, \cdot) := \mathbf{C}_{\mathbf{N}}(\cdot, \cdot)$ and $\mathbf{D}(\cdot, \cdot) := \mathbf{D}_{\mathbf{N}}(\cdot, \cdot)$.

For each doctor d in D and each subset F of E , we define $S_d(F)$ as the set of edges e in $F(d)$ such that $e \succ_d f$ for every edge f in $F(d) - e$. Clearly, for every doctor d in D and every subset F of $E(d)$, we can find $S_d(F)$ in $O(|F|^2)$ time. Furthermore, it is not difficult to see that for every doctor d in D and every subset F of E , $|S_d(F)| \leq 1$. Our algorithm is described in Algorithm 2. Roughly speaking, each doctor applies to the hospital that dominates any other hospital. The proposed algorithm is based on the algorithm proposed in [17] for the super-stable matching problem with ties, master lists, and matroid constraints. For each integer t in $[k]$, the time complexity of Steps 4 to 14 is $O(|E| \text{EO} + \sum_{d \in D_t} |E(d)|^2)$. Thus, the time complexity of Algorithm 2 is $O(|E|^2 \text{EO})$ time.

Algorithm 2: Asymmetric preference case

- 1 Define $K_0 := \emptyset, F_0 := E, A_0 := \emptyset$, and $L_0 := \emptyset$.
 - 2 Set $t := 1$.
 - 3 **while** $t \leq k$ **do**
 - 4 | Define $F_t := F_{t-1} \setminus L_{t-1}$ and $A_t := A_{t-1} \cup L_{t-1}$.
 - 5 | **if** there exists a doctor d in D_t such that $F_t(d) \neq \emptyset$ and $S_d(F_t) = \emptyset$ **then**
 - 6 | | Output **null**, and halt.
 - 7 | **end**
 - 8 | Define $e(d)$ as the unique edge in $S_d(F_t)$ for each doctor d in D_t such that $F_t(d) \neq \emptyset$.
 - 9 | Define $K_t := K_{t-1} \cup \{e(d) \mid d \in D_t \text{ such that } F_t(d) \neq \emptyset\}$.
 - 10 | **if** $K_t \notin \mathcal{J}$ **then**
 - 11 | | Output **null**, and halt.
 - 12 | **end**
 - 13 | Define $L_t := \{(d, h) \in F_t \mid d \notin D[t], K_t + (d, h) \notin \mathcal{J}\}$.
 - 14 | Set $t := t + 1$.
 - 15 **end**
 - 16 Output K_k , and halt.
-

In the rest of this subsection, we assume that Algorithm 2 halts when $t = \ell$. Notice that if $\ell = k + 1$ (resp., $\ell \leq k$), then Algorithm 2 outputs K_k (resp., **null**).

LEMMA 4.3. *If Algorithm 2 outputs K_k , then K_k is a super-stable matching in G .*

PROOF. Since K_k is clearly a matching in G , it is sufficient to prove that K_k is super-stable. Assume that we are given a doctor d in D and an edge e in $E(d) \setminus K_k$, and $d \in D_i$ for some integer i in $[k]$. If $e \in A_i$, then there exists an integer j in $[i - 1]$ such that $e \in L_j$. This implies that $K_j + e \notin \mathcal{J}$ and $a \in D[j]$ for every edge (a, p) in $\mathbf{D}(e, K_j)$. Furthermore, since $K_j \subseteq K_k$, Lemma 2.2 implies that $K_k + e \notin \mathcal{J}$ and $\mathbf{D}(e, K_j) = \mathbf{D}(e, K_k)$. Thus, since $a \in D[j]$ for every edge (a, p) in $\mathbf{D}(e, K_j)$, $d \in D_i$, and $j < i$, we have $a \succ_H d$ for every edge (a, p) in $\mathbf{D}(e, K_k)$. This implies that H does not weakly prefer e on K_k . If $e \in F_i$, then $F_i(d) \neq \emptyset$. Thus, we have $S_d(F_i) \neq \emptyset$ and $\mu_{K_k}(d) = e(d)$. This implies that $\mu_{K_k}(d) \succ_d e$. Thus, d does not weakly prefer e on K_k . This completes the proof. \square

LEMMA 4.4. *If $\ell \leq k$, then for every super-stable matching M in G , every integer i in $[\ell]$, and every doctor d in D_i , $M(d) \subseteq S_d(F_i)$.*

PROOF. We call an edge $e = (d, h)$ in E a *bad edge* if (i) $d \in D_i$ for some integer i in $[\ell]$, (ii) there exists a super-stable matching M in G such that $e \in M$, and (iii) $e \notin S_d(F_i)$. Then proving this lemma is equivalent to proving that there does not exist a bad edge in E . Thus, we assume that there exists a bad edge in E . Define Δ as the set of integers i in $[\ell]$ such that there exists a bad edge $e = (d, h)$ in E such that $d \in D_i$. Let j be the minimum integer in Δ . Furthermore, let us fix a bad edge $e = (d, h)$ in E such that $d \in D_j$. Since e is a bad edge, there exists a super-stable matching M in G such that $e \in M$.

First, we assume that $e \in A_j$. In this case, there exists an integer x in $[j - 1]$ such that $e \in L_x$. This implies that $K_x + e \notin \mathcal{J}$. Define $C := C(e, K_x)$. Then since $M \in \mathcal{J}$, $C \setminus M \neq \emptyset$. For every edge (a, p) in $C \setminus M$, since $d \in D_j$, $a \in D[x]$, and $x < j$, we have $a \succ_H d$.

Assume that there exists an edge $f = (a, p)$ in $C \setminus M$ such that $M(a) \neq \emptyset$ and $\mu_M(a) \succ_a f$. Furthermore, we assume that $a \in D_z$ for some integer z in $[x]$. Since $e \in M$, $f \neq e$ holds. This implies that $f \in K_x$. Thus, $f \in S_a(F_z)$. This implies that there does not exist an edge g in $F_z(a) - f$ such that $g \succ_a f$. Thus, $\mu_M(a) \notin F_z(a)$. This implies that $\mu_M(a) \notin S_a(F_z)$. Thus, $\mu_M(a)$ is a bad edge in E . Since $a \in D_z$ and $z \leq x < j$. This contradicts the minimality of j .

Assume that one of $M(a) = \emptyset$ and $\mu_M(a) \not\succeq_a f$ holds for every edge $f = (a, p)$ in $C \setminus M$. Then the assumption in this subsection implies that $f \succeq_a \mu_M(a)$ for every edge $f = (a, p)$ in $C \setminus M$ such that $M(a) \neq \emptyset$. Thus, if there exists an edge $f = (a, p)$ in $C \setminus M$ such that $M + f \in \mathcal{J}$, then a weakly prefers f on M and H strongly prefers f on M . However, this contradicts the fact that M is super-stable. Thus, $M + f \notin \mathcal{J}$ holds for every edge $f = (a, p)$ in $C \setminus M$. For each edge f in $C \setminus M$, we define $C_f := C(f, M)$. Since M is super-stable, $b \succ_H a$ for every edge $f = (a, p)$ in $C \setminus M$ and every edge (b, q) in $C_f - f$. For every edge $f = (a, p)$ in $C \setminus M$, since $a \succ_H d$, we have $e \notin C_f$. Thus, since $f \in C \cap C_f$ for every edge f in $C \setminus M$, Lemma 2.3 implies that there exists a circuit C' of \mathbf{N} such that $C' \subseteq (C \cup C^*) \setminus (C \setminus M)$, where we define $C^* := \bigcup_{f \in C \setminus M} C_f$. Since $C_f - f \subseteq M$ for every edge f in $C \setminus M$, $C' \subseteq M$. This contradicts the fact that $M \in \mathcal{J}$. This completes the proof.

Next, we assume that $e \in F_j$. Since $e \notin S_d(F_j)$, there exists an edge f in $F_j(d) - e$ such that $e \not\succeq_d f$. Then the assumption in this subsection implies that $f \succeq_d e$. Thus, if $M + f \in \mathcal{J}$, then d weakly prefers f on M and H strongly prefers f on M . This contradicts the fact that M is super-stable. Thus, $M + f \notin \mathcal{J}$. Define M_{j-1} as the set of edges (a, p) in M such that $a \in D[j - 1]$. Then the minimality of j implies that $M_{j-1} \subseteq K_{j-1}$. Since $f \in F_j$, $K_{j-1} + f \in \mathcal{J}$. (Notice that since $\{f\} \in \mathcal{J}$, this holds even if $j = 1$.) Thus, (I1) implies that $M_{j-1} + f \in \mathcal{J}$. This implies that there exists an edge (a, p) in $D(f, M)$ such that $a \in D \setminus D[j - 1]$. Thus, H weakly prefers f on M . This contradicts the fact that M is super-stable. \square

LEMMA 4.5. *If $\ell \leq k$, then for every super-stable matching M in G , every integer i in $[\ell]$, and every doctor d in D_i , $M(d) = S_d(F_i)$.*

PROOF. Assume that there exists a super-stable matching M in G , an integer i in $[\ell]$, and a doctor d in D_i such that $M(d) \neq S_d(F_i)$. Then Lemma 4.4 implies that $M(d) \subsetneq S_d(F_i)$. Thus, since $|S_d(F_i)| \leq 1$, we have $S_d(F_i) \neq \emptyset$ and $M(d) = \emptyset$. Let e be the edge in $S_d(F_i)$. Then d strongly prefers e on M . If $M + e \in \mathcal{J}$, then H strongly prefers e on M . This contradicts the fact that M is super-stable.

Thus, $M + e \notin \mathcal{J}$. Define M_{i-1} as the set of edges (a, p) in M such that $a \in D[i - 1]$. Lemma 4.4 implies that $M_{i-1} \subseteq K_{i-1}$. Furthermore, since $e \in F_i$, $K_{i-1} + e \in \mathcal{J}$. Thus, (I1) implies that $M_{i-1} + e \in \mathcal{J}$. This implies that there exists an edge (a, p) in $D(e, M)$ such that $a \in D \setminus D[i - 1]$. This implies that H weakly prefers e on M . This contradicts the fact that M is super-stable. \square

LEMMA 4.6. *If Algorithm 2 outputs **null**, then there does not exist a super-stable matching in G .*

PROOF. Notice that in this case, $\ell \leq k$. Assume that there exists a super-stable matching M in G .

First, we assume that Algorithm 2 outputs **null** in Step 6. Then there exists a doctor d in D_ℓ such that $F_\ell(d) \neq \emptyset$ and $S_d(F_\ell) = \emptyset$. Let e be an edge in $F_\ell(d)$. Since Lemma 4.5 implies that $M(d) = \emptyset$, d strongly prefers e on M . Thus, since M is super-stable, $M + e \notin \mathcal{J}$. Define $M_{\ell-1}$ as the set of edges (a, p) in M such that $a \in D[\ell - 1]$. Lemma 4.5 implies that $M_{\ell-1} = K_{\ell-1}$. Since $e \in F_\ell$, $K_{\ell-1} + e \in \mathcal{J}$. Thus, $M_{\ell-1} + e \in \mathcal{J}$. This implies that there exists an edge (a, p) in $D(e, M)$ such that $a \in D \setminus D[\ell - 1]$. Thus, H weakly prefers e on M . This contradicts the fact that M is super-stable.

Next, we assume that Algorithm 2 outputs **null** in Step 11. Define M_ℓ as the set of edges (d, h) in M such that $d \in D[\ell]$. Lemma 4.5 implies that $M_\ell = K_\ell$. Since $K_\ell \notin \mathcal{J}$, $M_\ell \notin \mathcal{J}$. This contradicts the fact that $M \in \mathcal{J}$. \square

The following theorem follows from Lemmas 4.3 and 4.6.

THEOREM 4.7. *Assume that there do not exist a doctor d in D and a pair of edges e, f in $E(d)$ such that $e \parallel_d f$. Then Algorithm 2 can solve SUPER-STABLE MATCHING.*

5 STRONGLY STABLE MATCHINGS

In this section, we assume that there do not exist a doctor d in D and a pair of edges e, f in $E(d)$ such that $e \parallel_d f$. The goal of this section is to prove Theorem 5.6.

The proposed algorithm is described in Algorithm 3. Our algorithm is based on the algorithm proposed in [17] for the strongly stable matching problem with ties, master lists, and matroid constraints. Roughly speaking, each doctor applies to the set of hospitals that are not dominated by any other hospital. However, we have to carefully select a subset of the hospitals that are not dominated (see Algorithm 4). In Algorithm 3, Lemmas 2.4 and 2.5 imply that by finding a base of $\mathbf{N}|P_{t-1}$, we can determine whether I is an independent set of \mathbf{N}_t for every subset I of T_t . Notice that since $Y_{t, \alpha+1} \subseteq Y_{t, \alpha}$ holds during the course of Algorithm 4, the number of iterations of Algorithm 4 is $O(\sum_{d \in D_t} |E(d)|)$. If we use the algorithm in [5], then since for each integer t in $[k]$, the time complexity of Steps 4 to 19 is $O(|\bigcup_{d \in D_t} E(d)|^{2.5} \text{EO})$, the time complexity of Algorithm 3 is $O(|E|^{2.5} \text{EO})$.

In the rest of this section, we assume that Algorithm 3 halts when $t = \ell$. Notice that if $\ell = k + 1$ (resp., $\ell \leq k$), then Algorithm 3 outputs K_k (resp., **null**).

LEMMA 5.1. *For every integer i in $[\ell - 1]$, K_i is a base of $\mathbf{N}|P_i$.*

PROOF. First, it should be noted that K_0 is a base of $\mathbf{N}|P_0$. Assume that we are given an integer j in $[\ell - 1]$, and this lemma holds when $i = j - 1$ (i.e., K_{j-1} is a base of $\mathbf{N}|P_{j-1} = (\mathbf{N}|P_j)|P_{j-1}$). Lemma 2.4

Algorithm 3: Asymmetric preference case

```

1 Define  $K_0 := \emptyset$ ,  $F_0 := E$ ,  $A_0 := \emptyset$ ,  $L_0 := \emptyset$ , and  $P_0 := \emptyset$ .
2 Set  $t := 1$ .
3 while  $t \leq k$  do
4   Define  $F_t := F_{t-1} \setminus L_{t-1}$  and  $A_t := A_{t-1} \cup L_{t-1}$ .
5   Find the subset  $T_t$  of  $F_t$  by using Algorithm 4.
6   Define  $P_t := P_{t-1} \cup T_t$  and  $N_t := (N/P_{t-1})|T_t$ .
7   if there exists a doctor  $d$  in  $D_t$  such that  $F_t(d) \neq \emptyset$  and
       $T_t(d) = \emptyset$  then
8     Output null, and halt.
9   end
10  if  $|\{d \in D_t \mid F_t(d) \neq \emptyset\}| < r_{N_t}(T_t)$  then
11    Output null, and halt.
12  end
13  Find a maximum-size common independent set  $I_t$  of
       $U(T_t)$  and  $N_t$ .
14  if  $|I_t| < |\{d \in D_t \mid F_t(d) \neq \emptyset\}|$  then
15    Output null, and halt.
16  end
17  Define  $K_t := K_{t-1} \cup I_t$ .
18  Define  $L_t := \{(d, h) \in F_t \mid d \notin D[t], K_t + (d, h) \notin \mathcal{J}\}$ .
19  Set  $t := t + 1$ .
20 end
21 Output  $K_k$ , and halt.

```

Algorithm 4: Subroutine of Algorithm 3

```

1 Define  $Y_{t,1}$  as the set of edges  $e = (d, h)$  in  $F_t$  such that
    $d \in D_t$  and  $f \not>_d e$  holds for every edge  $f$  in  $F_t(d) - e$ .
2 Set  $\alpha := 1$ .
3 do
4   Define  $Z_{t,\alpha}$  as the set of edges  $e = (d, h)$  in  $F_t \setminus Y_{t,\alpha}$ 
      such that  $d \in D_t$  and  $\{e\} \in \mathcal{J}/(P_{t-1} \cup Y_{t,\alpha})$ .
5   Define  $Y_{t,\alpha+1}$  as the set of edges  $e = (d, h)$  in  $Y_{t,\alpha}$  such
      that  $e >_d f$  for every edge  $f$  in  $Z_{t,\alpha}(d)$ .
6   Set  $\alpha := \alpha + 1$ .
7 while  $Y_{t,\alpha-1} \setminus Y_{t,\alpha} \neq \emptyset$ ;
8 Output  $Y_{t,\alpha}$  as  $T_t$ , and halt.

```

implies that $N_j = (N/P_j)/P_{j-1}$. Thus, Lemma 2.5 implies that it is sufficient to prove that I_j is a base of N_j . Since I_j is an independent set of N_j , $|I_j| \leq r_{N_j}(T_j)$. Furthermore, the definitions of Steps 10 and 14 imply that $|I_j| \geq r_{N_j}(T_j)$. Thus, $|I_j| = r_{N_j}(T_j)$. This implies that I_j is a base of N_j . This completes the proof. \square

LEMMA 5.2. *If Algorithm 3 outputs K_k , then K_k is a strongly stable matching in G .*

PROOF. Since Lemma 5.1 implies that K_k is an independent set of N , K_k is a matching in G . Thus, what remains is to prove that K_k is strongly stable. Assume that we are given a doctor d in D and an edge e in $E(d) \setminus K_k$, and $d \in D_i$ for some integer i in $[k]$.

First, we assume that $e \in A_i$. Then there exists an integer j in $[i-1]$ such that $e \in L_j$. This implies that $K_j + e \notin \mathcal{J}$ and $a \in D[j]$

for every edge (a, p) in $D(e, K_j)$. Since $K_j \subseteq K_k$, Lemma 2.2 implies that $K_k + e \notin \mathcal{J}$ and $a >_H d$ for every edge (a, p) in $D(e, K_k)$. Thus, H does not weakly prefer e on K_k . This completes the proof.

Next, we assume that $e \in F_i$. Then $F_i(d) \neq \emptyset$. Since Algorithm 3 does not output **null** in Step 15, $|I_i| \geq |\{d' \in D_i \mid F_i(d') \neq \emptyset\}|$. Since $I_i \subseteq \bigcup_{d' \in D_i} F_i(d')$, $I_i(d') = \emptyset$ for every doctor d' in D_i such that $F_i(d') = \emptyset$. Thus, since $|I_i(d')| \leq 1$ for every doctor d' in D_i , $I_i(d') \neq \emptyset$ holds for every doctor d' in D_i such that $F_i(d') \neq \emptyset$. In addition, $\mu_{K_k}(d) \in T_i$. Thus, $e \not>_d \mu_{K_k}(d)$, i.e., d does not strongly prefer e on K_k . If d does not weakly prefer e on K_k (i.e., $\mu_{K_k}(d) >_d e$), then the proof is done. Assume that $e \sim_d \mu_{K_k}(d)$. First, we consider the case in which $e \in T_i$. In this case, since the proof of Lemma 5.1 implies that I_i is a base of N_i , $I_i + e$ is not an independent set of N_i . Thus, Lemmas 2.5 and 5.1 imply that $K_i + e \notin \mathcal{J}$. Since $K_i \subseteq K_k$, Lemma 2.2 implies that $K_k + e \notin \mathcal{J}$ and $a \succeq_H d$ for every edge (a, p) in $D(e, K_k)$. This implies that H does not strongly prefer e on K_k . Next, we consider the case in which $e \notin T_i$. Since $e \sim_d \mu_{K_k}(d)$ and $\mu_{K_k}(d) \in T_i$, the definition of Algorithm 4 implies that $\{e\}$ is not an independent set of N/P_i . Thus, Lemma 5.1 implies that $K_i + e \notin \mathcal{J}$. Since $K_i \subseteq K_k$, Lemma 2.2 implies that $K_k + e \notin \mathcal{J}$ and $a \succeq_H d$ for every edge (a, p) in $D(e, K_k)$. This implies that H does not strongly prefer e on K_k . This completes the proof. \square

LEMMA 5.3. *If $\ell \leq k$, then for every strongly stable matching M in G , every integer i in $[\ell]$, and every doctor d in D_i , $M(d) \subseteq T_i(d)$.*

PROOF. We call an edge $e = (d, h)$ in E a *bad edge* if (i) $d \in D_i$ for some integer i in $[\ell]$, (ii) there exists a strongly stable matching M in G such that $e \in M$, and (iii) $e \notin T_i$. Then proving this lemma is equivalent to proving that there does not exist a bad edge in E . Thus, we assume that there exists a bad edge in E . Define Δ as the set of integers i in $[\ell]$ such that there exists a bad edge $e = (d, h)$ in E such that $d \in D_i$. Let j be the minimum integer in Δ .

First, we consider the case in which there exists a bad edge $e = (d, h)$ in E such that $d \in D_j$ and $e \in A_j$. Then there exists an integer x in $[j-1]$ such that $e \in L_x$. This implies that $K_x + e \notin \mathcal{J}$. Define $C := C(e, K_x)$. Since e is a bad edge, there exists a strongly stable matching M in G such that $e \in M$. Since $M \in \mathcal{J}$, $C \setminus M \neq \emptyset$. For every edge (a, p) in $C \setminus M$, since $d \in D_j$, $a \in D[x]$, and $x < j$, we have $a >_H d$.

Assume that there exists an edge $f = (a, p)$ in $C \setminus M$ such that $M(a) \neq \emptyset$ and $\mu_M(a) >_a f$. Furthermore, we assume that $a \in D_z$ for some integer z in $[x]$. Since $e \in M$, $f \neq e$ holds. Thus, $f \in K_x$. This implies that $f \in T_z$. Thus, there does not exist an edge g in $F_z(a) - f$ such that $g >_a f$. This implies that $\mu_M(a) \notin F_z(a)$. Thus, $\mu_M(a) \notin T_z$. This implies that $\mu_M(a)$ is a bad edge in E . Since $a \in D_z$ and $z \leq x < j$, this contradicts the minimality of j .

Assume that one of $M(a) = \emptyset$ and $\mu_M(a) \not>_a f$ holds for every edge $f = (a, p)$ in $C \setminus M$. The assumption in this section implies that $f \succeq_a \mu_M(a)$ for every edge $f = (a, p)$ in $C \setminus M$ such that $M(a) \neq \emptyset$. If there exists an edge $f = (a, p)$ in $C \setminus M$ such that $M + f \in \mathcal{J}$, then a weakly prefers f on M and H strongly prefers f on M . This contradicts the fact that M is strongly stable. Thus, $M + f \notin \mathcal{J}$ holds for every edge f in $C \setminus M$. For each edge f in $C \setminus M$, we define $C_f := C(f, M)$. Since M is strongly stable, $b \succeq_H a$ for every edge $f = (a, p)$ in $C \setminus M$ and every edge (b, q) in $C_f - f$. For every edge $f = (a, p)$ in $C \setminus M$, since $a >_H d$, $e \notin C_f$ holds. Thus, since

$f \in C \cap C_f$ for every edge f in $C \setminus M$, Lemma 2.3 implies that there exists a circuit C' of \mathbf{N} such that $C' \subseteq (C \cup C^*) \setminus (C \setminus M)$, where we define $C^* := \bigcup_{f \in C \setminus M} C_f$. Since $C_f - f \subseteq M$ for every edge f in $C \setminus M$, $C' \subseteq M$. However, this contradicts the fact that $M \in \mathcal{J}$.

In the rest of this proof, we consider the case in which $e \in F_j$ for every bad edge $e = (d, h)$ in E such that $d \in D_j$.

First, we assume that there exists a bad edge $e = (d, h)$ in E such that $d \in D_j$ and $e \in F_j \setminus Y_{j,1}$. Let M be a strongly stable matching in G such that $e \in M$. The definition of $Y_{j,1}$ implies that there exists an edge f in $F_j(d) - e$ such that $f \succ_d e$. Notice that since $f \neq e$, $f \notin M$. Define M_{j-1} as the set of edges (a, p) in M such that $a \in D[j-1]$. Since $f \in F_j$, $K_{j-1} + f \in \mathcal{J}$. Lemma 5.1 implies that K_{j-1} is a base of $\mathbf{N}|P_{j-1}$. Furthermore, the minimality of j implies that $M_{j-1} \subseteq P_{j-1}$. Since $M_{j-1} \in \mathcal{J}$, (I2) implies that there exists a base B of $\mathbf{N}|P_{j-1}$ such that $M_{j-1} \subseteq B$. Then Lemma 2.5 implies that $B + f \in \mathcal{J}$. Thus, (I1) implies that $M_{j-1} + f \in \mathcal{J}$. This implies that if $M + f \notin \mathcal{J}$, then there exists an edge (a, p) in $D(f, M)$ such that $d \succ_H a$, i.e., H weakly prefers f on M . This contradicts the fact that M is strongly stable.

Next, we assume that $e \in Y_{j,1}$ holds for every bad edge $e = (d, h)$ in E such that $d \in D_j$. Assume that $T_j = Y_{j,\beta}$ for some positive integer β . The definition of Algorithm 4 implies that $\beta \geq 2$. Let γ be the minimum integer in $[\beta-1]$ such that there exists a bad edge $e = (d, h)$ in $Y_{j,\gamma} \setminus Y_{j,\gamma+1}$. Let M be a strongly stable matching in G such that $e \in M$. Define M_j as the set of edges (a, p) in M such that $a \in D[j]$. Then the minimality of γ and the assumption of this paragraph imply that $M_j \subseteq P_{j-1} \cup Y_{j,\gamma}$. Since e is in $Y_{j,\gamma} \setminus Y_{j,\gamma+1}$, there exists an edge f in $F_j(d) \setminus Y_{j,\gamma}$ such that $f \sim_d e$ and $\{f\}$ is an independent set of $\mathbf{N}/(P_{j-1} \cup Y_{j,\gamma})$. Notice that since $e \in Y_{j,\gamma}$, $f \neq e$. Since $M_j \in \mathcal{J}$, there exists a base B of $\mathbf{N}/(P_{j-1} \cup Y_{j,\gamma})$ such that $M_j \subseteq B$. Then Lemma 2.5 implies that $B + f \in \mathcal{J}$. Thus, (I1) implies that $M_j + f \in \mathcal{J}$. This implies that if $M + f \notin \mathcal{J}$, then there exists an edge (a, p) in $D(f, M)$ such that $d \succ_H a$. This implies that H strongly prefers f on M . Since $\mu_M(d) = e$ and $f \sim_d e$, this contradicts the fact that M is strongly stable. \square

LEMMA 5.4. *If $\ell \leq k$, then for every strongly stable matching M in G , every integer i in $[\ell]$, $M \cap T_i$ is a base of \mathbf{N}_i .*

PROOF. Let M be a strongly stable matching in G . We prove this lemma by induction on i . The assumption in this section, for every integer t in $[\ell]$, every doctor d in D_t , and every pair of edges e, f in $T_t(d)$, we have $e \sim_d f$.

First, we consider the case of $i = 1$. Assume that $M \cap T_1$ is not a base of $\mathbf{N}_1 = \mathbf{N}|T_1$. Since $M \cap T_1 \in \mathcal{J}$, there exists an edge $e = (d, h)$ in $T_1 \setminus M$ such that $(M \cap T_1) + e \in \mathcal{J}$. Since $e \in T_1$, $d \in D_1$. If $M(d) \neq \emptyset$, then since Lemma 5.3 implies that $\mu_M(d) \in T_1$, we have $e \sim_d \mu_M(d)$. Assume that $M + e \notin \mathcal{J}$. Then $C(e, M) \not\subseteq T_1$. This implies that there exists an edge $f = (a, p)$ in $D(e, M) \setminus T_1$. If $a \in D_1$, then Lemma 5.3 implies that $f \in T_1$. This contradicts the fact that $f \notin T_1$. Thus, $a \in D \setminus D[1]$. This implies that H strongly prefers e on M . This contradicts the fact that M is strongly stable.

Next, we assume that we are given an integer j such that $2 \leq j \leq \ell$, and this lemma holds when $i = x$ for every integer x in $[j-1]$. Then we prove that this lemma holds when $i = j$. In this case, Lemma 2.5 implies that $M \cap P_{j-1}$ is a base of $\mathbf{N}|P_{j-1}$. Furthermore, since $M \in \mathcal{J}$ holds, (I1) implies that $M \cap P_j$ is an independent set

of $\mathbf{N}|P_j$. Assume that $M \cap T_j$ is not a base of $\mathbf{N}_j = (\mathbf{N}|P_j)/P_{j-1}$. Then Lemma 2.5 implies that $M \cap P_j$ is not a base of $\mathbf{N}|P_j$. Thus, there exists an edge $e = (d, h)$ in $P_j \setminus M$ such that $(M \cap P_j) + e \in \mathcal{J}$. Since $M \cap P_{j-1}$ is a base of $\mathbf{N}|P_{j-1}$, $e \in T_j$. Notice that since $e \in T_j$, $d \in D_j$. If $M(d) \neq \emptyset$, then since Lemma 5.3 implies that $\mu_M(d) \in T_j$, we have $e \sim_d \mu_M(d)$. Assume that $M + e \notin \mathcal{J}$. Then $C(e, M) \not\subseteq P_j$. This implies that there exists an edge $f = (a, p)$ in $D(e, M) \setminus P_j$. If $a \in D[j]$, then Lemma 5.3 implies that $f \in P_j$. This contradicts the fact that $f \notin P_j$. Thus, $a \in D \setminus D[j]$, and H strongly prefers e on M . This contradicts the fact that M is strongly stable. \square

LEMMA 5.5. *If Algorithm 3 outputs **null**, then there does not exist a strongly stable matching in G .*

PROOF. Notice that in this case, $\ell \leq k$. Assume that there exists a strongly stable matching M in G .

First, we assume that Algorithm 3 outputs **null** in Step 8. There exists a doctor d in D_ℓ such that $F_\ell(d) \neq \emptyset$ and $T_\ell(d) = \emptyset$. Let e be an edge in $F_\ell(d)$. Since Lemma 5.3 implies that $M(d) = \emptyset$, d strongly prefers e on M . Since M is strongly stable, $M + e \notin \mathcal{J}$. Define $M_{\ell-1}$ as the set of edges (a, p) in M such that $a \in D[\ell-1]$. Lemmas 2.5, 5.3, and 5.4 imply that $M_{\ell-1}$ (i.e., $M \cap P_{\ell-1}$) is a base of $\mathbf{N}|P_{\ell-1}$. In addition, since $e \in F_\ell$, $K_{\ell-1} + e \in \mathcal{J}$. Thus, Lemmas 2.5 and 5.1 imply that $\{e\}$ is an independent set of $\mathbf{N}/P_{\ell-1}$. Thus, Lemma 2.5 implies that $M_{\ell-1} + e \in \mathcal{J}$. This implies that there exists an edge (a, p) in $D(e, M)$ such that $a \in D \setminus D[\ell-1]$. Thus, H weakly prefers e on M . This contradicts the fact that M is strongly stable.

Next, we assume that Algorithm 3 outputs **null** in Step 11. Then since Lemma 5.4 implies that $M \cap T_\ell$ is a base of \mathbf{N}_ℓ , $|M \cap T_\ell| = \mathbf{r}_{\mathbf{N}_\ell}(T_\ell)$. Furthermore, since $T_\ell \subseteq F_\ell$, $|M \cap T_\ell| \leq |\{d \in D_\ell | F_\ell(d) \neq \emptyset\}|$. Thus, $|\{d \in D_\ell | F_\ell(d) \neq \emptyset\}| \geq \mathbf{r}_{\mathbf{N}_\ell}(T_\ell)$. However, this contradicts the definition of Step 10.

Lastly, we assume that Algorithm 3 outputs **null** in Step 15. Then since Lemma 5.4 implies that $M \cap T_\ell$ is a common independent set of $\mathbf{U}(T_\ell)$ and \mathbf{N}_ℓ , $|M \cap T_\ell| \leq |I_\ell|$. Notice that Lemma 5.3 implies that $|M \cap T_\ell| = |\{d \in D_\ell | M(d) \neq \emptyset\}|$. Thus, the definition of Step 14 implies that there exists a doctor d in D_ℓ such that $F_\ell(d) \neq \emptyset$ and $M(d) = \emptyset$. The rest of the proof of this case is the same as the proof of the first case. This completes the proof. \square

The following theorem follows from Lemmas 5.2 and 5.5.

THEOREM 5.6. *Assume that there do not exist a doctor d in D and a pair of edges e, f in $E(d)$ such that $e \parallel_d f$. Then Algorithm 3 can solve STRONGLY STABLE MATCHING.*

6 CONCLUSION

It would be interesting to consider whether the results in this paper can be extended to the many-to-many case [20]. Furthermore, it would be interesting to consider strategic issues in our problems.

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