We now study the approximation guarantee of PMMS for MMS. Since these two notions coincide when there are only two agents, we assume there are at least three agents. We first provide a tight bound for $n = 3$ and then give an almost tight bound for general n.

Proposition 4.5. *For $n = 3$, a* PMMS *allocation is also* $\frac{4}{3}$-MMS, *and moreover, this bound is tight.*

For general n, we use the connections between PMMS, EFX and MMS to find the approximation guarantee of PMMS for MMS. According to Proposition 4.1, a PMMS allocation is also EFX, and by Proposition 3.5, EFX implies $\frac{2n}{n+1}$-MMS. As a result, we can claim that PMMS also implies $\frac{2n}{n+1}$-MMS. With the following proposition we show that this guarantee is almost tight.

Proposition 4.6. *For $n \geq 4$, a* PMMS *allocation is* $\frac{2n}{n+1}$-MMS *but not necessarily* $(\frac{2n+2}{n+3} - \epsilon)$-MMS *for any $\epsilon > 0$.*

Next, we investigate the approximation guarantee of approximate PMMS for MMS. Let us start with an example of six chores $E = \{e_1, \dots, e_6\}$ and three agents. We focus on agent 1 and the cost function of agent 1 is $c_1(e_j) = 1$ for $j = 1, 2, 3$ and $c_1(e_j) = 0$ for $j = 4, 5, 6$, thus clearly, $\text{MMS}_1(3, E) = 1$. Consider an allocation $\mathbf{A} = (A_1, A_2, A_3)$ with $A_1 = \{e_1, e_2, e_3\}$. It is not hard to verify that allocation $\mathbf{A}$ is a $\frac{3}{2}$-PMMS allocation and also a 3-MMS allocation. Combining the result in Lemma 2.9, we observe that allocation $\mathbf{A}$ only has a trivial guarantee on the notion of MMS. Motivated by this example, we focus on α-PMMS allocations with $\alpha < \frac{3}{2}$.

Proposition 4.7. *For any $n \geq 3$ and $1 < \alpha < \frac{3}{2}$, an α-*PMMS *allocation is* $\frac{n\alpha}{\alpha+(n-1)(1-\frac{\alpha}{2})}$-MMS, *but not necessarily* $(\frac{n\alpha}{\alpha+(n-1)(2-\alpha)} - \epsilon)$-MMS *for any $\epsilon > 0$.*

Before we can prove the above proposition, we need the following two lemmas.

Lemma 4.8. *For any $i \in N$ and bundle $S \subseteq E$, suppose* $\text{MMS}_i(2, S)$ *is defined by a 2-partition $\mathbf{T} = (T_1, T_2)$ with $c_i(T_1) = \text{MMS}_i(2, S)$. If the number of chores in T_1 is at least two, then* $\frac{c_i(S)}{\text{MMS}_i(2,S)} \geq \frac{3}{2}$.

Lemma 4.9. *For any $i \in N$ and bundles $S_1, S_2 \subseteq E$, if* $\text{MMS}_i(2, S_1 \cup S_2) > \text{MMS}_i(2, S_1)$, *then* $\text{MMS}_i(2, S_1 \cup S_2) \leq \frac{1}{2}c_i(S_1) + c_i(S_2)$.

Proof of Proposition 4.7. We first prove the upper bound. Let $\mathbf{A} = (A_1, ..., A_n)$ be an α-PMMS allocation and we focus our analysis on agent i. Let $\alpha^{(i)} = \max_{j \neq i} \frac{c_i(A_i)}{\text{MMS}_i(2, A_i \cup A_j)}$ and $j^{(i)}$ be the index such that $\text{MMS}_i(2, A_i \cup A_{j^{(i)}}) \leq \text{MMS}_i(2, A_i \cup A_j)$ for any $j \in N$ (tie breaks arbitrarily). By these constructions, clearly, $\alpha = \max_{i \in N} \alpha^{(i)}$ and $c_i(A_i) = \alpha^{(i)} \cdot \text{MMS}_i(2, A_i \cup A_{j^{(i)}})$. Then, we split our proof into two different cases.

Case 1: $\exists j \neq i$ such that $\text{MMS}_i(2, A_i \cup A_j) = \text{MMS}_i(2, A_i)$. Then $\alpha^{(i)} = \frac{c_i(A_i)}{\text{MMS}_i(2,A_i)}$ holds. Suppose $\text{MMS}_i(2, A_i)$ is defined by the 2-partition (T_1, T_2) with $c_i(T_1) = \text{MMS}_i(2, A_i)$. If $|T_1| \geq 2$, by Lemma 4.8, we have $\alpha^{(i)} = \frac{c_i(A_i)}{\text{MMS}_i(2,A_i)} \geq \frac{3}{2}$, contradicting to $\alpha^{(i)} \leq \alpha < \frac{3}{2}$. As a result, we can further assume $|T_1| = 1$. By the first point of Lemma 2.8, we have $\text{MMS}_i(n, E) \geq c_i(T_1)$ and accordingly, $\frac{c_i(A_i)}{\text{MMS}_i(n,E)} \leq \frac{c_i(A_i)}{c_i(T_1)} = \alpha^{(i)} \leq \alpha$. For $1 < \alpha < \frac{3}{2}$ and $n \geq 3$, it is not hard to verify that $\alpha \leq \frac{n\alpha}{\alpha+(n-1)(1-\frac{\alpha}{2})}$, completing the proof for this case.

Case 2: $\forall j \neq i$, $\text{MMS}_i(2, A_i \cup A_j) > \text{MMS}_i(2, A_i)$ holds. According to Lemma 4.9, for any $j \neq i$, the following holds

$$\text{MMS}_i(2, A_i \cup A_j) \leq \frac{1}{2}c_i(A_i) + c_i(A_j). \quad (5)$$

Due to the construction of $\alpha^{(i)}$, for any $j \neq i$, we have $c_i(A_i) \leq \alpha^{(i)} \cdot \text{MMS}_i(2, A_i \cup A_j)$. Combining Inequality (5), we have $c_i(A_j) \geq \frac{2-\alpha^{(i)}}{2\alpha^{(i)}} c_i(A_i)$ for any $j \neq i$. Thus, the following holds,

$$\frac{c_i(A_i)}{\text{MMS}_i(n, E)} \leq \frac{nc_i(A_i)}{c_i(E)} \leq \frac{nc_i(A_i)}{c_i(A_i) + (n-1)\frac{2-\alpha^{(i)}}{2\alpha^{(i)}}c_i(A_i)}. \quad (6)$$

The last expression in (6) is monotonically increasing in $\alpha^{(i)}$, and accordingly, we have

$$\frac{c_i(A_i)}{\text{MMS}_i(n, E)} \leq \frac{n\alpha}{\alpha + (n-1)(1 - \frac{\alpha}{2})}.$$

As for the lower bound, consider an instance of n agents with $\frac{n}{2} \in \mathbb{N}^+$ and a set $E = \{e_1, ..., e_{n^2}\}$ of n^2 chores. Agents have identical cost functions. The cost function of agent 1 is as follows: $c_1(e_j) = \alpha$ for $j = 1, ..., n$ and $c_1(e_j) = 2 - \alpha$ for $j = n + 1, ..., n^2$. Now, consider an allocation $\mathbf{B} = (B_1, ..., B_n)$ with $B_i = \{e_{(n-1)i+1}, ..., e_{ni}\}$ for $i = 1, ..., n$. Since $\alpha > 1$, it is easy to see that, except for agent 1, no one else will violate the condition of PMMS, and moreover, the approximation guarantee on MMS is determined by agent 1. For agent 1, since $\frac{n}{2} \in \mathbb{N}^+$, $\text{MMS}_1(2, B_1 \cup B_j) = n$ holds for any $j \geq 2$, and due to $c_1(B_1) = n\alpha$, we can claim that the allocation $\mathbf{B}$ is α-PMMS. It is not hard to verify that $\text{MMS}_1(n, E) = \alpha + (n-1)(2-\alpha)$, yielding the ratio $\frac{n\alpha}{\alpha+(n-1)(2-\alpha)}$, completing the proof. □

The motivating example right before Proposition 4.7, unfortunately, only works for the case of $n = 3$. When n becomes larger, an α-PMMS allocation with $\alpha \geq \frac{3}{2}$ is still possible to provide a non-trivial approximation guarantee on the notion of MMS.

We remain to consider the approximation guarantee of MMS for other fairness criteria. Notice that all of EFX, EF1 and PMMS can have non-trivial guarantee for MMS (i.e., better than n-MMS). However, the converse is not true and even the exact MMS does not provide any substantial guarantee for the other three criteria.

Proposition 4.10. *For any $n \geq 3$, there exists an* MMS *allocation that is only 2-*PMMS.

Proposition 4.11. *An* MMS *allocation is not necessarily β-*EF1 *or β-*EFX *for any $\beta \geq 1$.*

5 PRICE OF FAIRNESS

After having compared the fairness criteria between themselves, in this section we study the efficiency of these fairness criteria in terms of the price of fairness with respect to social optimality of an allocation.

5.1 Two Agents

We start with the case of two players. Our first result concerns EF1.

Proposition 5.1. *The price of* EF1 *is* 5/4 *when there are two agents.*

According to Propositions 3.4 and 3.6, EF1 implies 2-MMS and $\frac{3}{2}$-PMMS. The following two propositions confirm an intuition — if one relaxes the fairness condition, then less efficiency will be sacrificed.

Proposition 5.2. *The price of 2-*MMS *is 1 when there are two agents.*

The above proposition is implied directly by Lemma 2.9.

Proposition 5.3. *The price of* $\frac{3}{2}$-PMMS *is* 7/6 *when there are two agents.*

Proof. We first prove the upper bound. Given an instance I, let $\mathbf{O} = (O_1, O_2)$ be an optimal allocation of I. If the allocation $\mathbf{O}$ is already $\frac{3}{2}$-PMMS, we are done. For the sake of contradiction, we assume that agent 1 violates the condition of $\frac{3}{2}$-PMMS in allocation $\mathbf{O}$, i.e., $c_1(O_1) > \frac{3}{2}\mathrm{MMS}_1(2, E)$. Suppose $O_1 = \{e_1, \ldots, e_h\}$ and the index satisfies the following rule; $\frac{c_1(e_1)}{c_2(e_1)} \geq \frac{c_1(e_2)}{c_2(e_2)} \geq \cdots \geq \frac{c_1(e_h)}{c_2(e_h)}$. In this proof, for simplicity, we write $L(k) := \{e_1, ..., e_k\}$ for any $1 \leq k \leq h$ and $L(0) = \emptyset$. Then, let s be the index such that $c_1(O_1 \setminus L(s)) \leq \frac{3}{2}\mathrm{MMS}_1(2, E)$ and $c_1(O_1 \setminus L(s-1)) > \frac{3}{2}\mathrm{MMS}_1(2, E)$. In the following, we divide our proof into two cases.

Case 1: $c_1(L(s)) \leq \frac{1}{2}c_1(O_1)$. Consider allocation $\mathbf{A} = (A_1, A_2)$ with $A_1 = O_1 \setminus L(s)$ and $A_2 = O_2 \cup L(s)$. We first show allocation $\mathbf{A}$ is $\frac{3}{2}$-PMMS. For agent 1, due to the construction of index s, he does not violate the condition of $\frac{3}{2}$-PMMS. As for agent 2, we claim that $c_2(A_2) = 1 - c_2(O_1 \setminus L(s-1)) + c_2(e_s) < \frac{1}{4} + c_2(e_s)$ because $c_2(O_1 \setminus L(s-1)) \geq c_1(O_1 \setminus L(s-1)) > \frac{3}{2}\mathrm{MMS}_1(2, E) \geq \frac{3}{4}$ where the first inequality transition is due to the fact that O_1 is the bundle assigned to agent 1 in the optimal allocation. If $c_2(e_s) < \frac{1}{2}$, then clearly, $c_2(A_2) < \frac{3}{4} \leq \frac{3}{2}\mathrm{MMS}_2(2, E)$. If $c_2(e_s) \geq \frac{1}{2}$, then $c_2(e_s) = \mathrm{MMS}_1(2, E)$ and accordingly, it is not hard to verify that $c_2(A_2) \leq \frac{3}{2}\mathrm{MMS}_1(2, E)$. Thus, $\mathbf{A}$ is a $\frac{3}{2}$-PMMS allocation.

Next, based on allocation $\mathbf{A}$, we derive an upper bound on the price of $\frac{3}{2}$-PMMS. First, by the order of index, $\frac{c_1(L(s))}{c_2(L(s))} \geq \frac{c_1(O_1)}{c_2(O_1)}$ holds, implying $c_2(L(s)) \leq \frac{c_2(O_1)}{c_1(O_1)}c_1(L(s))$. Since $A_1 = O \setminus L(s)$ and $A_2 = O_2 \cup L(s)$, we have the following:

$$\begin{aligned}\text{Price of } \frac{3}{2}\text{-PMMS} &\leq 1 + \frac{c_2(L(s)) - c_1(L(s))}{c_1(O_1) + c_2(O_2)}\\ &\leq 1 + \frac{c_1(L(s))(\frac{c_2(O_1)}{c_1(O_1)} - 1)}{c_1(O_1) + c_2(O_2)}\\ &= 1 + \frac{\frac{c_1(L(s))}{c_1(O_1)}(1 - c_2(O_2) - c_1(O_1))}{c_1(O_1) + c_2(O_2)}\\ &\leq 1 + \frac{\frac{1}{2} - \frac{1}{2}(c_1(O_1) + c_2(O_2))}{c_1(O_1) + c_2(O_2)}\\ &\leq 1 - \frac{1}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{7}{6},\end{aligned}$$

where the second inequality due to $c_2(L(s)) \leq \frac{c_2(O_1)}{c_1(O_1)}c_1(L(s))$; the third inequality due to the condition of *Case 1*; and the last inequality is because $c_1(O_1) > \frac{3}{2}\mathrm{MMS}_1(2, E) \geq \frac{3}{4}$.

Case 2: $c_1(L(s)) > \frac{1}{2}c_1(O_1)$. We first derive a lower bound on $c_1(e_s)$. Since $c_1(e_s) = c_1(O_1 \setminus L(s-1)) + c_1(L_s) - c_1(O_1)$, combine which with the condition of Case 2 implying $c_1(e_s) > c_1(O_1 \setminus L(s-1)) - \frac{1}{2}c_1(O_1)$, and consequently we have $c_1(e_s) > \frac{3}{2}\mathrm{MMS}_1(2, E) - \frac{1}{2}c_1(O_1) \geq \frac{1}{4}$ where the last transition is due to $\mathrm{MMS}_1(2, E) \geq \frac{1}{2}$ and $c_1(O_1) \leq 1$. Then, we consider two subcases.

If $0 \leq c_2(e_s) - c_1(e_s) \leq \frac{1}{8}$, consider an allocation $\mathbf{A} = (A_1, A_2)$ with $A_1 = O_1 \setminus \{e_s\}$ and $A_2 = O_2 \cup \{e_s\}$. We first show the allocation $\mathbf{A}$ is $\frac{3}{2}$-PMMS. For agent 1, since $c_1(e_s) > \frac{1}{4}$, $c_1(A_1) = c_1(O_1) - c_1(e_s) < \frac{3}{4} \leq \frac{3}{2}\mathrm{MMS}_1(2, E)$. As for agent 2, $c_2(A_2) = c_2(O_2) + c_2(e_s) \leq 1 - c_1(O_1) + c_2(e_s) < \frac{1}{4} + c_2(e_s)$. If $c_2(e_s) < \frac{1}{2}$, then clearly, $c_2(A_2) \leq \frac{3}{4} < \frac{3}{2}\mathrm{MMS}_2(2, E)$ holds. If $c_2(e_s) \geq \frac{1}{2}$, we have $c_2(e_s) = \mathrm{MMS}_2(2, E)$ and accordingly, it is not hard to verify that $c_2(A_2) \leq \frac{3}{2}\mathrm{MMS}_2(2, E)$. Thus, the allocation $\mathbf{A}$ is $\frac{3}{2}$-PMMS. Next, based on the allocation $\mathbf{A}$, we derive an upper bound regarding the price of $\frac{3}{2}$-PMMS,

$$\begin{aligned}\text{Price of } \frac{3}{2}\text{-PMMS} &\leq \frac{c_1(O_1) - c_1(e_s) + c_2(O_2) + c_2(e_s)}{c_1(O_1) + c_2(O_2)}\\ &\leq 1 + \frac{1}{8} \times \frac{4}{3} = \frac{7}{6},\end{aligned}$$

where the second inequality due to $0 \leq c_2(e_s) - c_1(e_s) \leq \frac{1}{8}$ and $c_1(O_1) > \frac{3}{4}$.

If $c_2(e_s) - c_1(e_s) > \frac{1}{8}$, consider an allocation $\mathbf{A}' = (A'_1, A'_2)$ with $A'_1 = \{e_s\}$ and $A'_2 = E \setminus \{e_s\}$. We first show that the allocation $\mathbf{A}'$ is $\frac{3}{2}$-PMMS. For agent 1, due to Lemma 2.8, $c_1(e_s) \leq \mathrm{MMS}_1(2, E)$ holds. As for agent 2, since $c_2(e_s) \geq c_1(e_s) > \frac{1}{4}$, we have $c_2(A'_2) = c_2(E) - c_2(e_s) < \frac{3}{4} \leq \frac{3}{2}\mathrm{MMS}_2(2, E)$. Thus, the allocation $\mathbf{A}'$ is $\frac{3}{2}$-PMMS. In the following, we first derive an upper bound for $c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\})$, then based on the bound, we provide the target upper bound for the price of fairness. Since $c_1(O_1) > \frac{3}{4}$ and $c_2(e_s) - c_1(e_s) > \frac{1}{8}$, we have $c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\}) = c_2(O_1) - c_1(O_1) - (c_2(e_s) - c_1(e_s)) < \frac{1}{8}$, and then, the following holds,

$$\begin{aligned}\text{Price of } \frac{3}{2}\text{-PMMS} &\leq 1 + \frac{c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\})}{c_1(O_1) + c_2(O_2)}\\ &\leq 1 + \frac{1}{8} \times \frac{4}{3} = \frac{7}{6}.\end{aligned}$$

Up to here, we complete the proof of upper bound.

Regarding lower bound, consider an instance I with two agents and a set $E = \{e_1, e_2, e_3, e_4\}$ of four chores. The cost function for agent 1 is: $c_1(e_1) = \frac{3}{8}, c_1(e_2) = \frac{3}{8} + \epsilon, c_1(e_3) = \frac{1}{8} - \epsilon, c_1(e_4) = \frac{1}{8}$ where $\epsilon > 0$ takes arbitrarily small value. For agent 2, here cost function is: $c_2(e_1) = c_2(e_2) = \frac{1}{2}, c_2(e_3) = c_2(e_4) = 0$. It is not hard to verify that $\mathrm{MMS}_i(2, E) = \frac{1}{2}$ for any $i = 1, 2$. In the optimal allocation, the assignment is; e_1, e_2 to agent 1 and e_3, e_4 to agent 2, resulting in $\mathrm{OPT}(I) = \frac{3}{4} + \epsilon$. Observe that to satisfy $\frac{3}{2}$-PMMS, agent 1 cannot receive both chores e_1, e_2, and accordingly, the minimum social cost of a $\frac{3}{2}$-PMMS allocation is $\frac{7}{8}$ by assigning e_1 to agent 1 and the rest chores to agent 2. Based on this instance, when $n = 2$, the price of $\frac{3}{2}$-PMMS is at least $\frac{\frac{7}{8}}{\frac{6}{8}+\epsilon} \to \frac{7}{6}$ as $\epsilon \to 0$. □

We remark that if we have an *oracle* for the maximin share, then our constructive proof of Proposition 5.3 can be transformed into an efficient algorithm for finding a 3/2-PMMS allocation whose cost is at most $\frac{7}{6}$ times the optimal social cost. Moving to other fairness criteria, we have the following uniform bound.

Proposition 5.4. *The price of* PMMS, MMS*, and* EFX *are all 2 when there are two agents.*

5.2 More than Two Agents

Note that the existence of an MMS allocation is not guaranteed in general [7, 31] and the existence of PMMS or EFX allocation is still open when $n \geq 3$. Nonetheless, we are still interested in the prices of fairness in case such a fair allocation does exist. Observe that when the number of chore $m \leq 2$, the price of EF1, EFX, PMMS is trivially 1. If $m = 1$, assigning the unique chore to any agent satisfies all these three fairness criteria, so does the optimal allocation. If $m = 2$, in an optimal allocation, it never happens that both of the two chores are assigned to the same agent. The reason is that if an agent has the smallest cost on one chore, then his cost on another chore is higher than someone else due to the normalized cost function. In the following, we settle down the case of $m \geq 3$.

Proposition 5.5. *For $n \geq 3$ and $m \geq 3$, the price of* EF1, EFX *and* PMMS *are all infinite.*

In the context of goods allocation, Barman et al. [8] present an asymptotically tight price of EF1, $O(\sqrt{n})$. However, as shown by Proposition 5.5, when allocating chores, the price of EF1 is infinite, which shows a sharp contrast between goods and chores allocation.

By using a similar construction to the one in the proof of Proposition 5.5, we can establish the following proposition.

Proposition 5.6. *For $n \geq 3$, the price of $\frac{3}{2}$-*PMMS *is infinite.*

We are now left with MMS fairness. Let us first provide upper and lower bounds on the price of MMS.

Proposition 5.7. *For $n \geq 3$, the price of* MMS *is at most n^2 and at least $\frac{n}{2}$.*

As mentioned earlier, the existence of MMS allocation is not guaranteed. So we also provide an asymptotically tight price of 2-MMS.

Proposition 5.8. *For $n \geq 3$, the price of 2-*MMS *is $\Theta(n)$*

6 CONCLUSIONS

In this paper, we are concerned with fair allocations of indivisible chores among agents under the setting that the cost functions are additive. First we have established pairwise connections between several relaxations of the envy-free fairness in allocating, which look at how an allocation under one fairness criterion provides an approximation guarantee for fairness under another criterion. Some of our results have shown a sharp contrast to what is known in allocating indivisible goods, reflecting the difference between goods and chores allocation. Then we have studied the trade-off between fairness and efficiency, for which we have established the price of fairness for all these fairness notions. We hope our results have provided an almost complete picture for the connections between these chores fairness criteria together with their individual efficiencies relative to social optimum.

ACKNOWLEDGMENTS

We would like to thank Bo Li for helpful discussions. We also thank the anonymous AAMAS reviewers for their thoughtful comments and helpful suggestions.

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