$\mathbf{A}'$, each bundle A'_j is contained in the union of A_j and a neighbor bundle of A_j, which indicates that $v_i(A'_j) \leq v_i(A_j) + v_i(A_{i^*})$. Then

$$\begin{aligned} v_i(A'_i) \geq \frac{1}{2} v_i(A_i) &= \frac{1}{4}(v_i(A_i) + v_i(A_i)) \\ &\geq \frac{1}{4} \cdot \frac{1}{3+o(1)} \cdot (v_i(A_j) + v_i(A_{i^*})) \\ &\geq \frac{1}{12+o(1)} v_i(A'_j), \end{aligned}$$

which implies that allocation $\mathbf{A}'$ is $\frac{1}{12+o(1)}$-envy-free. □

4 EFFICIENCY

In this section, we study the problem of finding valid allocations from the perspective of efficiency, that is, maximizing the utilitarian or egalitarian social welfare. We first present NP-hardness results for both types of welfare, and then show that there is a $4+o(1)$ approximation for maximizing the utilitarian welfare. Here, the facilities' valuation functions are not necessarily have to be normalized over the interval $[0, 1]$.

Theorem 4.1. *The problem of finding a valid allocation maximizing the utilitarian social welfare is NP-hard, even if the valuation functions are piecewise-uniform.*

Proof. We reduce from EXACT-3-COVER (X3C), which is an NP-complete problem [21]. An instance of X3C is given by $I = (X, \mathcal{T})$, where $X = \{x_1, \ldots, x_{3s}\}$ is a set of elements, and $\mathcal{T} = \{T_1, \ldots, T_r\}$ is a family of 3-element subset of X. The answer is "yes" if and only if X can be exactly covered by s sets from $\mathcal{T}$, i.e., each element in X is covered by exactly one of the s sets. For a set $T \in \mathcal{T}$, order the three elements of T in some canonical way (e.g., alphabetically) and write T^1, T^2, T^3 for the elements in that order.

Consider an instance $I = (X, \mathcal{T})$ of X3C, where the elements of T are denoted by x_T^1, x_T^2, x_T^3 for each $T \in \mathcal{T}$. We construct an instance of our problem as follows. There are three subintervals y_T^1, y_T^2, y_T^3 for each set $T \in \mathcal{T}$, and $r-1$ dummy subintervals $D = \{d_1, d_2, \ldots, d_{r-1}\}$. All of these $m = 4r-1$ subintervals have the same length of $\delta < \frac{1}{m^3}$, and are pairwise disjoint. The order of these subintervals in the line segment $[0, 1]$ is

$$y_{T_1}^1 < y_{T_1}^2 < y_{T_1}^3 < d_1 < y_{T_2}^1 < y_{T_2}^2 < y_{T_2}^3 < d_2 < y_{T_3}^1 < \cdots < y_{T_r}^3.$$

Let the left endpoint of interval $y_{T_1}^1$ be 0, and the right endpoint of interval $y_{T_r}^3$ be 1. Define a T-set to be $V_T = \{y_T^1, y_T^2, y_T^3\}$ for each $T \in \mathcal{T}$. We arrange the m subintervals in $[0, 1]$ uniformly such that the distance between every two adjacent subintervals in a T-set is ϵ (for example, the distance between the right endpoint of $y_{T_1}^2$ and the left endpoint of $y_{T_1}^3$ is ϵ), and the distance between every dummy interval and its adjacent interval is 3ϵ. Denote by Y the union of these m subintervals.

There are a total of $n = 2s + 2r - 1$ facilities: $r - s$ identical T-type facilities, one facility F_x for each $x \in X$, and one facility F_d for each dummy subinterval $d \in D$. For each facility $i \in N$, we have $v_i(Y) = 1$, and each facility has uniform valuation over each subinterval. For each subinterval y, the valuations are defined as:

$$v_T(y) = \begin{cases} 1 & \text{if } y \notin D \\ 0 & \text{otherwise} \end{cases}$$

$$v_x(y) = \begin{cases} 3 & \text{if } y = y_T^k \text{ and } x = x_T^k \text{ for some } k, T \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_d(y) = \begin{cases} L & \text{if } y = d \\ 0 & \text{otherwise} \end{cases}$$

where $L > 10mn$ is a large number. That is, every T-type facility values 2 for each non-dummy subinterval, every x-type facility values 3 for each subinterval corresponding element x, and every d-type facility F_{d_i} has a large value L for the corresponding dummy subinterval $d_i \in D$. Clearly, this instance is constructed in polynomial time.

It is easy to see that, in any optimal allocation, every d-type facility F_{d_i} must receive the dummy subinterval $d_i \in D$, and the bundle of every x-type or T-type facility cannot contain any piece in a dummy subinterval, indicating that $v_x(\mathbf{A}) \leq 3$ and $v_T(\mathbf{A}) \leq 3$ by the contiguous constraint. So the optimal social welfare is at most $L(r-1) + 3 \cdot 3s + 3(r-s)$.

We claim that the X3C instance $I = (X, \mathcal{T})$ has a solution iff the optimal social welfare of a valid allocation is $L(r-1)+3 \cdot 3s+3(r-s)$. This gives a reduction.

Suppose there is an exact cover $\mathcal{T}'$. We can construct a valid allocation: every x-type facility receives a corresponding subinterval in the cover $\mathcal{T}'$, every T-type facility receives a T-set not in the cover $\mathcal{T}'$, and every d-type facility F_{d_i} receives the corresponding dummy subinterval $d_i \in D$. This allocation is valid because we can locate each facility on the midpoint of its bundle, satisfying the proximity rule. The social welfare of this allocation is $L(r-1)+3 \cdot 3s+3 \cdot (r-s)$, and thus it is optimal.

Conversely, suppose the optimal social welfare is $L(r-1) + 9s + 3(r-s)$. Consider an optimal valid allocation $\mathbf{A}$. Let $\mathcal{T}' \subseteq \mathcal{T}$ be the family of sets in which at least one element corresponds to a subinterval assigned to an x-type facility, that is,

$$\mathcal{T}' = \{T \in \mathcal{T} | \text{ there is a } y_T^k \text{ assigned to some } F_x \text{ with } x = x_T^k\}.$$

Clearly, the total utility of all x-type facilities is at most $3 \cdot 3s = 9s$, and that of all T-type facilities is at most $3(r-s)$. Then, every x-type facility must receive a corresponding subinterval (as otherwise the maximum social welfare is less than $L(r-1) + 9s + 3(r-s)$), and thus it must be $|\mathcal{T}'| \geq s$. If $|\mathcal{T}'| > s$, then at least one T-type facility cannot receive a full T-set, and the total utility of all T-type facilities is less than $3(r-s)$. It indicates that the maximum social welfare is less than $L(r-1) + 9s + 3(r-s)$, a contradiction. Therefore, it must be $|\mathcal{T}'| = s$, which is an exact cover. □

Using a similar analysis, we can prove that maximizing the egalitarian social welfare is also NP-hard.

Theorem 4.2. *The problem of finding a valid allocation maximizing the egalitarian social welfare is NP-hard, even if the valuation functions are piecewise-uniform.*

Proof sketch. Given an arbitrary instance of X3C, construct an instance of the FLCD as in the proof of Theorem 4.1. We can claim that, the X3C instance has a solution if and only if the optimal egalitarian social welfare of a valid allocation is 3. If there is an exact cover, then every x-type facility and T-type facility is able to receive a utility exactly 3 in an optimal allocation. If there is no exact cover, then in an optimal solution there is a T-type facility has a utility less than 3. This establishes the proof. □

Next, we evaluate the performance of an algorithm on the system efficiency in the standard worst-case approximation framework. Formally, given an instance I, let $opt(I)$ be an optimal valid allocation maximizing the utilitarian welfare, and $\mathcal{A}(I)$ be the allocation output by an algorithm $\mathcal{A}$. Say $\mathcal{A}$ is α-approximate for the objective of maximizing the utilitarian welfare if for every instance I,

$$u(opt(I)) \leq \alpha \cdot u(\mathcal{A}(I)).$$

The approximation ratio for the egalitarian welfare is defined analogously.

We note that, for the contiguous cake cutting, Arunachaleswaran *et al.* [3] provide an algorithm with approximation ratio $2 + o(1)$ for maximing the utilitarian welfare. For our problem, using the algorithm in [3] to obtain a preliminary contiguous allocation, we can construct a valid allocation by locating the facilities in a way as in Proposition 2.6.

Theorem 4.3. *For any n-facility instance, there exists a $(4+o(1))$-approximate algorithm that returns a valid allocation in polynomial time under the objective of maximizing the utilitarian welfare.*

Proof. For any instance $I = (N, \{v_i\}_{i\in N})$, let $\mathbf{A} = (A_1, \ldots, A_n)$ be the contiguous allocation output by the algorithm in the proof of Theorem 6 in [3], which guarantees that $(2+o(1))u(\mathbf{A}) \geq u(opt(I))$. Let $\mathbf{A}' = (A'_1, \ldots, A'_n)$ be the valid allocation obtained by applying Proposition 2.6, based on $\mathbf{A}$. Since $v_i(A'_i) \geq \frac{v_i(A_i)}{2}$ for any $i \in N$, we have

$$u(\mathbf{A}') = \sum_{i\in N} v_i(A'_i) \geq \frac{u(\mathbf{A})}{2} \geq \frac{1}{4+o(1)} u(opt(I)),$$

which completes the proof. □

To end this section, we show that our Algorithm 1 has a bad performance guarantee on both types of social welfare. By Theorem 3.3, Algorithm 1 achieves an egalitarian welfare at least $\frac{1}{2n}$, and thus a utilitarian welfare at least $\frac{1}{2}$. Note that for any instance, the egalitarian and utilitarian welfare are at most 1 and n. So the approximation ratio of Algorithm 1 is $2n$ for both types of social welfare. In the following we give an example to show the bad performance on egalitarian welfare.

Example 4.4. Consider an instance with n facilities. The valuation function of facility 1 satisfies $v_1(0, 0.1) = \frac{1}{n}$, $v_1(0.1, 0.9) = 0$ and $v_1(0.9, 0.9 + \frac{1}{10n}) = \frac{n-1}{n}$. For $i = 2, \ldots, n$, the valuation function of facility i satisfies $v_i(0.9 + \frac{i-1}{10n}, 0.9 + \frac{i}{10n}) = 1$. It is not hard to see that, the allocation induced by location profile $\mathbf{x}^* = (x^*_1, \ldots, x^*_n)$ with $x^*_i = 0.9 + \frac{2i-1}{20n}$ for $i \in N$ guarantees that every facility has a utility 1, and thus the optimal egalitarian welfare is 1. However, Algorithm 1 locates facility 1 at point 0.1, and the utility of facility 1 and the egalitarian welfare is merely $\frac{1}{n}$.

5 PRICE OF FAIRNESS

In this section, we measure the efficiency loss under a fair allocation by the price of fairness. We study two kinds of concepts on price of fairness: best price and worst price, which compare the social welfare of an optimal solution to that of the best/worst fair solution.

Given an instance $I = (N, \{v_i\}_{i\in N})$, its *worst utilitarian price of fairness* w.r.t. criterion F (proportionality or envy-freeness) is defined as the ratio of the utilitarian social welfare of the optimal valid allocation over that of the worst valid allocation satisfying criterion F. Formally,

Definition 5.1. *Let I be an instance, X be the set of all valid allocations, and $X_F \subseteq X$ be the set of valid allocations satisfying criterion F. If $X_F \neq \emptyset$, the* worst utilitarian price of fairness *for instance I w.r.t. criterion F is*

$$\bar{P}^u_F(I) = \frac{\sup_{\mathbf{A}\in X} u(\mathbf{A})}{\inf_{\mathbf{A_F}\in X_F} u(\mathbf{A_F})}.$$

The (overall) worst utilitarian price of fairness w.r.t. criterion F is the supremum over all instances. That is,

$$\bar{P}^u_F = \sup_{I\in\mathcal{I}_n} \bar{P}^u_F(I).$$

As is commonly done, the price of fairness is not defined when there is no valid allocation satisfying criterion F for instance I. The *worst egalitarian price of fairness* $\bar{P}^e_F$ is defined analogously.

We are the first to study the worst price of fairness in FLCD, and derive lower and upper bounds on the worst price of fairness w.r.t. proportionality and envy-freeness, respectively.

Theorem 5.2. *For the FLCD, the worst utilitarian price of proportionality $\bar{P}^u_{pr}$ is in the interval $[n - \frac{1}{n}, n]$.*

Proof. *Upper bound.* Consider an arbitrary instance I. For any proportional valid allocation $\mathbf{A}$, the utility of each facility is at least $\frac{1}{n}$, and thus the utilitarian welfare is $u(\mathbf{A}) \geq 1$. As the optimal utilitarian welfare is at most n, it follows that $\bar{P}^u_{pr} \leq n$.

Lower bound. Consider an n-facility instance, where the valuation density function f_i of each facility $i \in N$ is defined as follows. For facility $i = 1, \ldots, n-1$,

$$f_i(x) = \begin{cases} 2n, & \text{if } x \in [\frac{2i-1}{2n}, \frac{2i}{2n}] \\ 0, & \text{otherwise} \end{cases}$$

For facility n,

$$f_n(x) = \begin{cases} 2(n-1), & \text{if } x \in [0, \frac{1}{2n}] \\ 2, & \text{if } x \in [\frac{2n-1}{2n}, 1] \\ 0, & \text{otherwise} \end{cases}$$

First, consider a valid allocation $\mathbf{A}$ induced by location profile $\mathbf{x} = (x_1, \ldots, x_n)$, where each facility $i \in N \setminus \{n\}$ is located at the point $x_i = \frac{i}{n}$, and facility n is located at the point $x_n = 0$. That is, $A_i = [\frac{2i-1}{2n}, \frac{2i+1}{2n}]$, for any $i \in N \setminus \{n-1, n\}$, $A_{n-1} = [\frac{2n-3}{2n}, 1]$, and $A_n = [0, \frac{1}{2n}]$. It indicates that each facility $i \in N \setminus \{n\}$ will obtain a utility 1, and facility n will obtain a utility $1 - \frac{1}{n}$ under the allocation $\mathbf{A}$. Then the utilitarian welfare of $\mathbf{A}$ is $u(\mathbf{A}) = n - \frac{1}{n}$, which means that the optimal utilitarian welfare is at least $n - \frac{1}{n}$.

Next construct a proportional valid allocation $\mathbf{A}' = (A'_1, \ldots, A'_n)$ induced by location profile of facilities $\mathbf{x}'$, where each facility $i \in N$ is located at the point $x'_i = \frac{i-1}{n} + \frac{1}{2n^2}$. That is, $A'_1 = [0, \frac{1}{2n} + \frac{1}{2n^2}]$, $A'_n = [\frac{2n-3}{2n} + \frac{1}{2n^2}, 1]$, and $A'_i = [\frac{2i-3}{2n} + \frac{1}{2n^2}, \frac{2i-1}{2n} + \frac{1}{2n^2}]$ for $i = 2, \ldots, n-1$. It follows that every facility has utility $\frac{1}{n}$ under the allocation $\mathbf{A}'$. Thus $\mathbf{A}'$ is proportional, and the utilitarian social welfare is $u(\mathbf{A}') = 1$. Therefore, $\bar{P}^u_{pr} \geq \frac{u(\mathbf{A})}{u(\mathbf{A}')} = n - \frac{1}{n}$. □

Theorem 5.3. *For the FLCD, the worst egalitarian price of proportionality $\bar{P}^e_{pr} = n$.*

Proof. *Upper bound.* For any instance I and any proportional valid allocation $\mathbf{A}$, the egalitarian welfare of $\mathbf{A}$ is at least $\frac{1}{n}$, while the optimal egalitarian welfare is at most 1. This implies that $\bar{P}^e_{pr} \leq n$.

Lower bound. Consider an n-facility instance, where the valuation density function f_i of each facility $i \in N$ is defined as

$$f_i(x) = \begin{cases} 2n, & \text{if } x \in [\frac{2i-1}{2n}, \frac{2i}{2n}] \\ 0, & \text{otherwise} \end{cases}$$

First, consider a valid allocation $\mathbf{A} = (A_1, \dots, A_n)$ induced by location profile $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i = \frac{2i-1}{2n}$ for each $i \in N$. That is, $A_i = [\frac{i-1}{n}, \frac{i}{n}]$, and each facility i obtains a utility 1 under $\mathbf{A}$. Then the egalitarian social welfare is $u(\mathbf{A}) = 1$, which is also optimal.

Then construct an proportional valid allocation $\mathbf{A}' = (A'_1, \dots, A'_n)$, induced by location profile $\mathbf{x}'$, where each facility $i \in N$ is located at $x_i = \frac{i-1}{n} + \frac{1}{2n^2}$. Under $\mathbf{A}'$, each facility $i \in N \setminus \{n\}$ obtains a utility $\frac{1}{n}$ and facility n obtains a utility 1. Then $\mathbf{A}'$ is proportional, and the egalitarian social welfare of $\mathbf{A}'$ is $eg(\mathbf{A}') = \frac{1}{n}$. Therefore, $\bar{P}^e_{pr} \geq \frac{eg(\mathbf{A})}{eg(\mathbf{A}')} = n$. □

Theorem 5.4. *For the FLCD, the worst utilitarian price of envy-freeness $\bar{P}^u_{ef}$ is in the interval $[\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2} + 1 - o(1)]$, and the worst egalitarian price of envy-freeness $\bar{P}^e_{ef}$ is in the interval $[\frac{n}{2}, n]$.*

Proof. The bounds for $\bar{P}^u_{ef}$ can be obtained by the proof of Theorem 2.1 in [4]. Next we only show the lower bound of $\bar{P}^e_{ef}$. Consider the instance constructed in the proof of Theorem 2.4 [4], where each facility $i \in N \setminus \{n\}$ has a valuation density function f_i:

$$f_i(x) = \begin{cases} \frac{1/2+\epsilon}{2\epsilon}, & \text{if } x \in [\frac{i}{n} - \epsilon, \frac{i}{n} + \epsilon] \\ \frac{1/2-\epsilon}{2\epsilon}, & \text{if } x \in [1 - \frac{2i+1}{2n} - \epsilon, 1 - \frac{2i+1}{2n} + \epsilon] \\ 0, & \text{otherwise} \end{cases}$$

and facility n has $f_n(x) = 1$ for any $i \in [0, 1]$. Assume that n is odd, then consider a valid allocation induced by location profile $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i = \frac{i}{n}$ for $i = 1, \cdots, \frac{n-1}{2}$, $x_i = 1 - \frac{2i+1}{2n}$ for $i = \frac{n-1}{2} + 1, \cdots, n-1$, and $x_n = \frac{n-1}{2n} + 2\epsilon$, in which, each facility obtains a utility at least $1/2 - \epsilon$. However, in any envy-free valid allocation, facility n has a utility less than $1/n + 2\epsilon$, which implies that $\bar{P}^e_{ef} \geq \frac{1/2-\epsilon}{1/n+2\epsilon} \to \frac{n}{2}$ as ϵ approaches zero. □

We also consider the well-studied measurement: *best price of fairness*, which is defined as the ratio of the social welfare of an optimal valid allocation over that of the best valid allocation satisfying some fairness criterion. Formally, for an instance I, if it admits a valid allocation satisfying criterion F, then its *best utilitarian price of fairness* w.r.t. criterion F is

$$P^u_F(I) = \frac{\sup_{\mathbf{A} \in X} u(\mathbf{A})}{\sup_{\mathbf{A_F} \in X_F} u(\mathbf{A_F})},$$

and the (overall) best utilitarian price of fairness w.r.t. criterion F is

$$P^u_F = \sup_{I \in \mathcal{I}_n} P^u_F(I).$$

The *best egalitarian price of fairness* P^e_F is defined analogously.

Aumann and Dombb [4] derive the upper bounds and lower bounds on the best price of proportionality and envy-freeness for contiguous cake cutting. In their problem, every instance admits proportional and envy-free contiguous allocations, and thus the price of proportionality is no more than the price of envy-freeness. However, this is not true in our problem, because the price of fairness may be not defined for some instances. We also note that partial results in [4] are applicable to our problem, since we can define feasible location profiles of facilities in the instances constructed in their proof to obtain valid allocations. Motivated by these, we present the following results.

Theorem 5.5. *For the FLCD, the best utilitarian price of proportionality P^u_{pr} is in the interval $[\frac{\sqrt{n}}{2}, n-1+\frac{1}{n}]$, and the best egalitarian price of proportionality P^e_{pr} is* 1.

Theorem 5.6. *For the FLCD, the best utilitarian price of envy-freeness P^u_{ef} is in the interval $[\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2} + 1 - o(1)]$, and the best egalitarian price of envy-freeness P^e_{pr} is $\frac{n}{2}$.*

6 CONCLUSION

This paper is devoted to the problem of fairly locating facilities and assigning customers, who are continuously distributed on a line, to the facilities. We consider two fairness criteria of proportionality and envy-freeness, and provide upper and lower bounds on their multiplicative approximation guarantees. Compared with the contiguous cake cutting, the existence of an approximately fair valid allocation that admits a feasible location profile of facilities is much harder to be guaranteed.

Our work opens up a number of new directions for future research. The first one is to narrow the gaps between upper and lower bounds on the approximation guarantees for proportionality and envy-freenss. Second, while the existence of a proportional/envy-free valid allocation is not guaranteed, the hardness of the problem of determining the existence of a proportional/envy-free valid allocation is still unknown. In addition, there are some other fairness criteria (e.g., equitability[8]) and other objectives (e.g., Nash social welfare [3]) deserving to be studied. One can also generalize this model to a graph [8, 26] or two-dimensional space [17], rather than a line segment.

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