Influence Maximization (IM). We are interested in the Group Shapley value for functions that describe information propagation in SNs. Two of the most popular models for describing such information propagation are the *Independent Cascade* (IC) and *Linear Threshold* (LT) models [7]. In both models, we are given a directed graph $G = (V, E)$ where V is a set of n nodes, values $\{p_{uv} \in [0,1] : (u,v) \in E\}$ and an initial node set $A \subseteq V$ called *seed set.* A spread of influence from A is defined as a randomly generated sequence of node sets $(A_t)_{t\in\mathbb{N}}$, where $A_0 = A$ and $A_{t-1} \subseteq A_t$. These sets represent active users, i.e., a node v is *active* at time step t if $v \in A_t$. We say that the sequence converges as soon as $A_{t^*} = A_{t^*+1}$, for some time step $t^* \geq 0$ called the time of quiescence. For a set A, we use $\sigma(A) = \mathbb{E}[|A_{t^*}|]$ to denote the expected number of nodes activated at the time of quiescence when running the process with seed set A. In IM, the standard objective is to find a set A maximizing $\sigma(A)$ under a cardinality constraint. Both the IC and LT model are special cases of the more general *Triggering Model*, see [7, Proofs of Theorem 4.5 and 4.6] that we consider in this work.

Influence-based Group Shapley Centrality. Chen and Teng [4] consider the Shapley value of nodes w.r.t. the influence spread function σ in a SN modeled by the Triggering Model. They use the resulting *Shapley centrality* $\phi^{\text{Sh}}_{\sigma}(i)$ as a measure of centrality of node i. In this work, we consider $\phi^{\text{Sh}}_{\sigma}$, i.e., the Group Shapley value w.r.t. σ. We call this value the *Influence-based Group Shapley* (IGS) centrality of S. We refer to it as $\phi^{\text{Sh}}(S)$ omitting σ as an index.

The central problem of this paper consists in finding sets S of size at most k that have large IGS centrality among all such sets. We call this the Max-Shapley-Group problem. The most important tool for studying this problem are so-called RR sets [2, 12]. In fact, there exists a concise formulation of ϕ^{Sh} in terms of RR sets:

Lemma 2.1 (IGS centrality via RR sets). *For any $S \subseteq V$,*

$$\phi^{\text{Sh}}(S) = n \cdot \mathbb{E}_R\left[\frac{\mathbb{1}_{R\cap S\neq\emptyset}}{|R \setminus S| + 1}\right].$$

3 HARDNESS OF APPROXIMATION

Unfortunately, Max-Shapley-Group in the IC model is, up to a constant factor, as hard to approximate as Densest-k-Subgraph.

Theorem 3.1. *Let $\alpha \in (0,1]$. If there is an α-approximation algorithm for Max-Shapley-Group, then there is an $\alpha/8$-approximation algorithm for Densest-k-Subgraph.*

A number of strong hardness of approximation results are known for Densest-k-Subgraph: (1) Densest-k-Subgraph cannot be approximated within $1/n^{o(1)}$ if the Gap Exponential Time Hypothesis holds [8]. (2) Densest-k-Subgraph cannot be approximated within any constant if the Unique Games with Small Set Expansion conjecture holds [11]. (3) Densest-k-Subgraph cannot be approximated within $n^{-(\log\log n)^{-c}}$ for some constant c if the Exponential Time Hypothesis holds [8]. We provide a reduction which yields the same hardness results for Max-Shapley-Group. In particular, according to (1) and our reduction, it is unlikely to find anything better than an (n^{-c})-approximation for Max-Shapley-Group, where c is a constant. Furthermore, for all settings where $k = O(n^c)$, such an algorithm is implied by our result in Section 4.

4 A SIMPLE APPROXIMATION ALGORITHM

A good approximation result, as for example a constant-factor approximation, is unlikely for the Max-Shapley-Group problem. We however obtain a positive result for small values of k.

Approximating ϕ^{Sh} through RR sets. By sampling a sufficient number of RR sets, we can give a set function $\hat{\phi}^{\text{Sh}}$ that with high probability approximates ϕ^{Sh} to within a factor of $1 \pm \epsilon$. The idea is to approximate the expected value in Lemma 2.1 via a Chernoff bound. This is captured by the following lemma.

Lemma 4.1. *Let $\epsilon \in (0,1)$ and let $R_1, \ldots, R_t$ be a sequence of t RR sets. For a value of t that is polynomial in n and ϵ^{-1}, w.h.pr.,*

$$\hat{\phi}^{\text{Sh}}(S) := \frac{n}{t}\sum_{i=1}^{t}\frac{\mathbb{1}_{R_i\cap S\neq\emptyset}}{|R_i \setminus S| + 1}$$

is a $(1 \pm \epsilon)$-approximation of ϕ^{Sh}.

The Harmonic-Max-Hitting-Set problem. Lemma 4.1 suggests to sample a near-linear number t of RR sets and compute a set of nodes S that maximizes $\hat{\phi}^{\text{Sh}}(S)$. We call the resulting problem the Harmonic-Max-Hitting-Set problem: the input consists of a set $X = \{x_1, \ldots, x_n\}$, a set $Z = \{Z_1, \ldots, Z_m\}$ of subsets of X, and an integer k. The task is to find a subset $S \subseteq X$ s.t. $|S| \leq k$ maximizing

$$f_Z(S) := \sum_{i=1}^{m}\frac{\mathbb{1}_{Z_i\cap S\neq\emptyset}}{|Z_i \setminus S| + 1}.$$

This is a non-linear variant of the well-known Max-Hitting-Set problem (itself equivalent to the Max-Set-Cover problem [6]) in which the objective function is $\sum_{i=1}^{m}\mathbb{1}_{Z_i\cap S\neq\emptyset}$. The problem of maximizing $\hat{\phi}^{\text{Sh}}$ can be stated as a Harmonic-Max-Hitting-Set problem by letting $X = V$ and Z be the set of generated RR sets.

Approximation Algorithm. Consider an instance (X, Z, k) and define the following set function $h_Z(S) := \sum_{i=1}^{m}\mathbb{1}_{Z_i\cap S\neq\emptyset}/|Z_i|$. Note the similarity between h_Z and f_Z. The approximation algorithm that we propose is to greedily maximize h_Z instead of f_Z. Why would this be a good idea? (1) The set function h_Z is monotone and submodular; thus the greedy algorithm will yield a $1 - 1/e$ approximation to maximizing h_Z. (2) Given a set $S \subseteq X$ with $|S| \leq k$, it holds that $f_Z(S) \geq h_Z(S) \geq f_Z(S)/k$, that is, the error when passing from h_Z to f_Z is at most k. Hence, if we denote by S^*_f (resp. S^*_h) an optimal solution of size k for maximizing f_Z (resp. h_Z), we have that $h_Z(S^*_h) \geq h_Z(S^*_f) \geq f_Z(S^*_f)/k$. Now let S be the solution of size k returned by the greedy algorithm. Then, S is a $(1 - 1/e)/k$ approximation to maximizing f_Z as

$$f_Z(S) \geq h_Z(S) \geq \left(1 - \frac{1}{e}\right) \cdot h_Z(S^*_h) \geq \frac{1 - 1/e}{k} \cdot f_Z(S^*_f).$$

Theorem 4.2. *Let $\epsilon \in (0,1)$. There is an algorithm with running time polynomial in n and ϵ^{-1} that, with high probability, returns a $\frac{1-1/e}{k} - \epsilon$-approximation for the Max-Shapley-Group problem.*

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