Proof. We obtain $\rho_2(\alpha) \geq \frac{k-1+2\alpha}{2k}$ if $\frac{1}{2(k+1)} < \alpha \leq \frac{k^2+1}{2(k^3+k+1)}$, and $\rho_2(\alpha) \geq \frac{k(1-k\alpha)}{k+1}$ if $\frac{k^2+1}{2(k^3+k+1)} < \alpha \leq \frac{1}{2k}$ from (15) with the help of $\rho_2(\alpha)B = \gamma_2(d^*, B)$, $B' = B/2$, $d^* = \alpha B$, and the definition of γ_1 given in (6). Conversely, we know that $\rho_2(\alpha) \leq \frac{k-1+2\alpha}{2k}$ if $\frac{1}{2(k+1)} < \alpha \leq \frac{k^2+1}{2(k^3+k+1)}$, and $\rho_2(\alpha) \leq \frac{k(1-k\alpha)}{k+1}$ if $\frac{k^2+1}{2(k^3+k+1)} < \alpha \leq \frac{1}{2k}$ from Proposition 1 where $n = 2$. □

Thus, Corollary 4 gives a characterization of $\rho_2(\alpha)$ when $\alpha \in]0, B/6]$ as claimed in (4), and Algorithm 6 produces a solution where the utility of the worst-off agent is $\rho_2(\alpha)B$ when $\alpha \in]0, B/6]$.

4.2.2 When $d^* \in]B/6, B/4]$. Now we consider the case where $k = 2$, $d^* \in \mathcal{I}_2^1(B')$, which corresponds to $d^* \in]B/6, B/4]$. The bounds are obtained with the solution built by Algorithm 6.

Lemma 14. *For $d^* \in \mathcal{I}_2^1(B')$, it holds that:*

$$\gamma_2(d^*, B) \geq \begin{cases} \gamma_1(d^*, B'), & \text{for } d^* \leq \frac{11}{25}B' \\ \gamma_1(d^*, B' - \Delta), & \text{for } \frac{11}{25}B' < d^* \leq \frac{8}{17}B' \\ \frac{4}{3}(B' - d^*), & \text{for } \frac{8}{17}B' < d^* \end{cases}$$

and

$$\Delta := \begin{cases} 0, & \text{for } \frac{1}{3}B' < d^* \leq \frac{11}{25}B' \\ \frac{25d^*-11B'}{13}, & \text{for } \frac{11}{25}B' < d^* \leq \frac{8}{17}B' \\ B' - 2d^*, & \text{for } \frac{8}{17}B' < d^* \end{cases}$$

Corollary 5. *It holds that:*

$$\rho_2(\alpha) = \begin{cases} 1/4 + \alpha/2, & \text{if } 1/6 < \alpha \leq 11/50 \\ \frac{6}{13}(1-\alpha), & \text{if } 11/50 < \alpha \leq 4/17 \\ 2/3 - 4\alpha/3, & \text{if } 4/17 < \alpha \leq 1/4 \end{cases}$$

Proof. The lower bounds on $\rho_2(\alpha)$ are immediate from Lemma 14 where $B' = B/2$, $d^* = \alpha B$, $\gamma_2(d^*, B) = \rho_2(\alpha)B$, and the definition of γ_1.

These lower bounds can be paired with matching upper bounds as follows. Use Proposition 1 with $n = k = 2$ to get that $\rho_2(\alpha) \leq 1/4 + \alpha/2$ if $1/6 < \alpha \leq 5/22$, and $\rho_2(\alpha) \leq 2/3 - 4\alpha/3$ if $5/22 < \alpha \leq 1/4$. Since $11/50 < 5/22 < 4/17$, it follows that

$$\rho_2(\alpha) \leq \begin{cases} 1/4 + \alpha/2, & \text{if } 1/6 < \alpha \leq 11/50 \\ 2/3 - 4\alpha/3, & \text{if } 4/17 < \alpha \leq 1/4 \end{cases}$$

The remaining part is obtained with the following instance.

Instance 2. *Suppose $B = 17 - \delta$ with $\delta \in [0, 1/3]$. Agent 1 has 4 demands of values $4-\delta, 2+\epsilon, 2+\epsilon,$ and 2, where $1 \gg \epsilon > 0$. Agent 2 has 3 demands, each of value 3. Thus, $\alpha = \frac{4-\delta}{17-\delta}$ and $\alpha \in [11/50, 4/17]$.*

If we accept 1, 2 or 3 demands of agent 2 then she gets 3, 6, or 9, respectively. An exhaustive search of all the possible subsets of demands for agent 1 provides the following positive amounts: 2, $2 + \epsilon$, $4 - \delta$, $4 + \epsilon$, $4 + 2\epsilon$, $6 - \delta$, $6 - \delta + \epsilon$, $6 + 2\epsilon$, $8 - \delta + \epsilon$, $8 - \delta + 2\epsilon$, and $10 - \delta + 2\epsilon$. If we accept the 3 demands of agent 2, then the remaining budget is $8 - \delta$. In that case we can give at most $6 + 2\epsilon$ to agent 1. If we accept two demands of agent 2 then she gets 6. The best guarantee for this instance when $\epsilon \to 0$ is $\frac{6}{17-\delta}$. Since $\frac{6}{17-\delta} = \frac{6}{13}(1 - \alpha)$ when $\alpha = \frac{4-\delta}{17-\delta}$, the best guarantee for this instance is $\frac{6}{13}(1 - \alpha)$ when $\alpha \in [\frac{11}{50}, \frac{4}{17}]$. Thus, we can conclude that $\rho_2(\alpha) \leq \frac{6}{13}(1 - \alpha)$ if $\frac{11}{50} < \alpha \leq \frac{4}{17}$. □

Therefore, Corollary 5 gives the characterization of $\rho_2(\alpha)$ when $\alpha \in]1/6, 1/4]$ as claimed in (4), and Algorithm 6 produces a solution where the utility of the worst-off agent is $\rho_2(\alpha)B$ when $\alpha \in]B/6, B/4]$.

5 ACCURACY

For n agents, let $\mathrm{Acc}(n)$ be the *accuracy* of our bounds on ρ_n. It is defined as the largest ratio between the lower and the upper bound.

$$\mathrm{Acc}(n) = \sup_{\alpha \in]0,1/n]} \frac{\text{best known lower bound on } \rho_n(\alpha)}{\text{best known upper bound on } \rho_n(\alpha)}.$$

We exclude $\alpha \in]\frac{1}{n}, 1] = \mathcal{I}_0^n$ in the definition of $\mathrm{Acc}(n)$ because $\rho_n(\alpha) = 0$ on $\mathcal{I}_0^n$ (see Observation 1). The accuracy is between 0 and 1. The closer to 1 the better the accuracy. Having $\mathrm{Acc}(n) = 1$ would mean that we have a characterization of ρ_n.

Proposition 2. *$\mathrm{Acc}(n)$ restricted to $\mathcal{I}_k^n$ is $\frac{(n-1)(k+1)+k^3}{(n-1)(k+1)+k^3+1}$.*

Since a characterization of ρ_n is known for $n = 1, 2$, the worst accuracy is for $n > 2$. According to Proposition 2, the worst case occurs when $k = 2$ and $n = 3$, namely $\mathrm{Acc}(n) \geq 14/15$.

6 FUTURE WORK

An immediate future work would be to characterize ρ_n for any number of agents, or to provide an approximation with a ratio better than 14/15. The present work proposed a common worst-case bound and it depends on α defined as the largest demand (over all agents) divided by B. However, as done by Markakis and Psomas [14], each agent j may have her own worst-case guarantee as a function of α_j where α_j is agent j's largest demand divided by B.

The common budget problem is cast as a multi-agent subset sum problem. The utility for a demand corresponds to its value. However, studying a multi-agent knapsack problem where the utility for an object is not necessarily aligned with its size would be interesting [2]. In that case, one needs significant parameters (analogous to α) before determining worst-case bounds on the value of an agent. In the same vein, one can think of a bi-dimensional version of the problem studied in this article. Suppose there is a common piece of land where multiple agents want to locate private facilities (see for example [19] for a recent work on the fair division of land). The problem can be to place some geometric shapes (the facilities) on a given area (the piece of land) in such a way that the shapes fit in the area and do not overlap. Then, which surface an agent is guaranteed to cover?

Finally, the strategic aspect of the common budget problem deserves attention. If the agents submit their demands to a central authority, then the selection of accepted demands should incentivize the agents to report the true values of their demands. At first glance, the fact that we accept all the demands of an agent if their sum is at most B/n, does not promote truthfulness. Indeed, every agent is tempted to submit her largest subset of demands which is below B/n. However, the agents are not necessarily aware of B and n when they communicate their demands.

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