Table 4: Score values of the candidates in Theorem 6.

Candidate	*Score*
For all $x \in X$	$s(\boldsymbol{P'}, x) + s(\boldsymbol{Q}, x)$
$s(\boldsymbol{P'} \cup \boldsymbol{Q}, x)$	$= (\lambda_{P'} + s(\boldsymbol{P'}, x) - \lambda_{P'}) + (\lambda_Q + R_x)$
	$= \lambda - 1.$
For all $y \in Y$	$s(\boldsymbol{P'}, y) + s(\boldsymbol{Q}, y)$
$s(\boldsymbol{P'} \cup \boldsymbol{Q}, y)$	$= (\lambda_{P'} + s(\boldsymbol{P'}, y) - \lambda_{P'}) + (\lambda_Q + R_y)$
	$= \lambda - 1.$
For all $z \in Z$	$s(\boldsymbol{P'}, z) + s(\boldsymbol{Q}, z)$
$s(\boldsymbol{P'} \cup \boldsymbol{Q}, z)$	$= (\lambda_{P'} + s(\boldsymbol{P'}, z) - \lambda_{P'}) + (\lambda_Q + R_z)$
	$= \lambda - 1.$
$s(\boldsymbol{P'} \cup \boldsymbol{Q}, c)$	$s(\boldsymbol{P'}, c) + s(\boldsymbol{Q}, c) = \lambda_{P'} + \lambda_Q = \lambda.$
$s(\boldsymbol{P'} \cup \boldsymbol{Q}, g)$	$s(\boldsymbol{P'}, g) + s(\boldsymbol{Q}, g)$
	$= (\lambda_{P'} + s(\boldsymbol{P'}, g) - \lambda_{P'}) + (\lambda_Q + R_g)$
	$= \lambda + 4q.$
$s(\boldsymbol{P'} \cup \boldsymbol{Q}, d)$	$s(\boldsymbol{P'}, d) + s(\boldsymbol{Q}, d)$
	$= (\lambda_{P'} + s(\boldsymbol{P'}, d) - \lambda_{P'}) + (\lambda_Q + R_d)$
	$= \lambda - q.$
For all $h \in H$,	$s(\boldsymbol{P'}, h) + s(\boldsymbol{Q}, h)$
$s(\boldsymbol{P'} \cup \boldsymbol{Q}, h)$	$= (\lambda_{P'} + s(\boldsymbol{P'}, h) - \lambda_{P'}) + (\lambda_Q + R_h) = \lambda.$
$s(\boldsymbol{P'} \cup \boldsymbol{Q}, w)$	$s(\boldsymbol{P'}, w) + s(\boldsymbol{Q}, w) < \lambda_{P'} + \lambda_Q < \lambda.$

($\Longleftarrow$) Assume that the PW-PC instance is positive. Therefore, there exists a total profile $\boldsymbol{P}^* = \bigcup_{i=1}^{\tau} p_i^*$ such that for all $1 \le i \le \tau$, the vote p_i^* extends p_i and c is a possible winner with score λ. In the following, when one says that a candidate "gains" or "loses" points, it is in relation to the complete profile $\boldsymbol{P'}$. Candidate g must lose at least $4q$ points for c to be a possible winner. Therefore, it must be in a position greater than $3q$ at least q times. Whenever g is in a position greater than $3q$, candidate d gains 1 point. Since d cannot gain more than q points, there are at most q votes where g is in position greater than $3q$. Let these votes be $p^*_{k_1}, \ldots, p^*_{k_q}$ where each $1 \le k_j \le b$ and $K = \{k_j | 1 \le j \le q\}$. Note that candidate g has to lose at least $4q$ points in these q votes. This is possible if and only if it is in position $3q + 4$. Furthermore, whenever g is in position $3q + 4$ in a vote p_i^*, candidates x_{i_1}, y_{i_2} and z_{i_3} gain one point each, for $i \in K$. Since $|X| = |Y| = |Z| = q$, and each $x \in X$, each $y \in Y$, and each $z \in Z$ can gain at most one point each, it must be the case that the element candidates of Y and Z which gained points in the q votes in K are distinct. Since no other candidate can gain any more points, the remaining partial votes in $\boldsymbol{P}$ (p_i for $1 \le i \le \tau$ and $i \notin K$), must have the same completion as in $\boldsymbol{P'}$. Therefore, the set $\{x_{i_1}, y_{i_2}, z_{i_3} | i \in K\}$ must form a cover for $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$.
($\Longrightarrow$) Assume that there exists a cover $\mathcal{S}'$ of the 3DM instance. Let $\boldsymbol{P}^* = \bigcup_{i=1}^{\tau} p_i^*$ where each p_i is extended as given below.

$$p_i^* = \overrightarrow{C_i'} \succ d \succ x_{i_1} \succ y_{i_2} \succ z_{i_3} \succ g \text{ if } S_i \in \mathcal{S}'$$
$$p_i^* = \overrightarrow{C_i'} \succ g \succ d \succ x_{i_1} \succ y_{i_2} \succ z_{i_3} \text{ if } S_i \notin \mathcal{S}'$$

All the candidates have score λ in $\boldsymbol{P}^* \cup \boldsymbol{Q}$. Therefore, candidate c is a possible winner. □

In what follows, we prove an observation analogous to Proposition 1 and then prove hardness for arbitrary unbounded scoring rules. As noted earlier, unbounded scoring rules may have score values that repeat in blocks. Moreover, unlike Borda count, the score values can be non-uniformly decreasing. Recall, that for a scoring vector of length m, with m' distinct score values, the function $\ell(m, j)$ returns the number of times the distinct score value a_j repeats in a block, for $1 \le j \le m'$. Schematically, such a scoring vector can be represented as $(\underbrace{a_1, \ldots, a_1}_{\ell(m,1)}, \underbrace{a_2, \ldots, a_2}_{\ell(m,2)}, \ldots, \underbrace{a_{m'}, \ldots, a_{m'}}_{\ell(m,m')})$.

Next, we prove a basic property of scoring vectors of all unbounded rules.

Proposition 2. *Let r be a positional scoring rule and let γ and β be two positive integers greater than* 1. *Consider the scoring vector $\boldsymbol{s}_m$ of r with length $m = \gamma\beta$. Then either $\boldsymbol{s}_m$ contains at least β distinct values or there exists $1 \le u \le \gamma\beta$ such that $\ell(\gamma\beta, u) \ge \gamma$.*

Theorem 7. *Let r be an unbounded scoring rule.* PW-PC *w.r.t. r is* NP*-complete.*

Proof. (Outline) Assume that $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{S})$ is a 3DM-instance in which $\mathcal{S} = \{S_1, \ldots S_\tau\} \subseteq \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ and $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, for $1 \le i \le \tau$. Let $\boldsymbol{s}_m$ be the scoring vector of length $m = (3q + 4)(3q)$. By Proposition 2, we need to consider the following two cases.
Case 1. There exists a u such that $\ell(m, u) = 3q$.
Case 2. There are $m' = 3q + 4$ distinct values.
For Case 1, the reduction mimics the one in Theorem 5 to create a PW-PC instance. For Case 2, the reduction proceeds as follows.

Let $a_1 > a_2 > \ldots > a_{m'}$ be the m' distinct values. We define $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{m'-1})$, where $\delta_j = a_j - a_{j+1}$, for $1 \le j < m'$. The set of candidates is $C = X \cup Y \cup Z \cup \{c, g, d, w\} \cup H$ where X, Y, and Z contains candidates corresponding to the elements in $\mathcal{X}, \mathcal{Y},$ and $\mathcal{Z}$ respectively. These candidates are called *elements candidates*. The set H contains dummy candidates such that $|H| = m - m'$.

We construct the partial profile $\boldsymbol{P}$ as follows. Let the set H be partitioned into $H_1, \ldots, H_{m'}$, such that $|H_j| = \ell(m, j) - 1$, for $1 \le j \le m'$. For each $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, let $C_i' = C \setminus (\{x_{i_1}, y_{i_2}, z_{i_3}\} \cup \{g, d\} \cup \bigcup_{j=m'-4}^{m'} H_j)$ and $\overrightarrow{C_i'}$ be such that the dummy candidates in H_j are in a position with score value a_j, for $1 \le j \le m' - 3$ and candidate c is ranked lower than candidate w. Define the total orders p_i' and the partial chains p_i, where

$$\begin{aligned} p_i' = \overrightarrow{C_i'} &\succ g \succ \overrightarrow{H}_{m'-4} \succ d \succ \overrightarrow{H}_{m'-3} \\ &\succ x_{i_1} \succ \overrightarrow{H}_{m'-2} \succ y_{i_2} \succ \overrightarrow{H}_{m'-1} \succ z_{i_3} \succ \overrightarrow{H}_{m'} \\ p_i = \overrightarrow{C_i'} &\succ \overrightarrow{H}_{m'-4} \succ d \succ \overrightarrow{H}_{m'-3} \\ &\succ x_{i_1} \succ \overrightarrow{H}_{m'-2} \succ y_{i_2} \succ \overrightarrow{H}_{m'-1} \succ z_{i_3} \succ \overrightarrow{H}_{m'} \end{aligned}$$

Let $\boldsymbol{P} = \bigcup_{i=1}^{\tau} p_i$ and $\boldsymbol{P'} = \bigcup_{i=1}^{\tau} p_i'$. Observe that each p_i' extends p_i. Let $s(\boldsymbol{P'}, c) = \lambda_{P'}$. Moreover, $s(\boldsymbol{P'}, w) < \lambda_{P'}$ since w is in a position greater than c in all $\overrightarrow{C_i'}$, for $1 \le i \le \tau$.

Consider $C = X \cup Y \cup Z \cup \{c, g, d\} \cup \{w\}$. By Lemma 1, there exists a $\lambda_Q \in \mathbb{N}$ and a total profile $\boldsymbol{Q}$ which can be constructed in time polynomial in m' such that the scores of the candidates in the profile $\boldsymbol{P'} \cup \boldsymbol{Q}$ are as in Table 5. We let C, the profile $\boldsymbol{P} \cup \boldsymbol{Q},$ and c be the input to the PW-PC problem.

□

Table 5: Score values of the candidates in Theorem 7.

Candidate	Score
For all $x \in X$,	$s(\boldsymbol{P}', x) + s(\boldsymbol{Q}, x)$
$s(\boldsymbol{P}' \cup \boldsymbol{Q}, x)$	$= (\lambda_{\boldsymbol{P}'} + s(\boldsymbol{P}', x) - \lambda_{\boldsymbol{P}'}) + (\lambda_{\boldsymbol{Q}} + R_x)$
	$= \lambda - \delta_{m'-3}$.
For all $y \in Y$,	$s(\boldsymbol{P}', y) + s(\boldsymbol{Q}, y)$
$s(\boldsymbol{P}' \cup \boldsymbol{Q}, y)$	$= (\lambda_{\boldsymbol{P}'} + s(\boldsymbol{P}', y) - \lambda_{\boldsymbol{P}'}) + (\lambda_{\boldsymbol{Q}} + R_y)$
	$= \lambda - \delta_{m'-2}$.
For all $z \in Z$,	$s(\boldsymbol{P}', z) + s(\boldsymbol{Q}, z)$
$s(\boldsymbol{P}' \cup \boldsymbol{Q}, z)$	$= (\lambda_{\boldsymbol{P}'} + s(\boldsymbol{P}', z) - \lambda_{\boldsymbol{P}'}) + (\lambda_{\boldsymbol{Q}} + R_z)$
	$= \lambda - \delta_{m'-1}$.
$s(\boldsymbol{P}' \cup \boldsymbol{Q}, c)$	$s(\boldsymbol{P}', c) + s(\boldsymbol{Q}, c) = \lambda_{\boldsymbol{P}'} + \lambda_{\boldsymbol{Q}} = \lambda$.
$s(\boldsymbol{P}' \cup \boldsymbol{Q}, g)$	$s(\boldsymbol{P}', g) + s(\boldsymbol{Q}, g)$
	$= (\lambda_{\boldsymbol{P}'} + s(\boldsymbol{P}', g) - \lambda_{\boldsymbol{P}'}) + (\lambda_{\boldsymbol{Q}} + R_g)$
	$= \lambda + q \sum_{j=1}^{4} \delta_{m-j}$.
$s(\boldsymbol{P}' \cup \boldsymbol{Q}, d)$	$s(\boldsymbol{P}', d) + s(\boldsymbol{Q}, d)$
	$= (\lambda_{\boldsymbol{P}'} + s(\boldsymbol{P}', d) - \lambda_{\boldsymbol{P}'}) + (\lambda_{\boldsymbol{Q}} + R_d)$
	$= \lambda - q(\delta_{m'-4})$.
For all $h \in H$,	$s(\boldsymbol{P}', h) + s(\boldsymbol{Q}, h)$
$s(\boldsymbol{P}' \cup \boldsymbol{Q}, h)$	$= (\lambda_{\boldsymbol{P}'} + s(\boldsymbol{P}', h) - \lambda_{\boldsymbol{P}'}) + (\lambda_{\boldsymbol{Q}} + R_h) = \lambda$.
$s(\boldsymbol{P}' \cup \boldsymbol{Q}, w)$	$s(\boldsymbol{P}', w) + s(\boldsymbol{Q}, w) < \lambda_{\boldsymbol{P}'} + \lambda_{\boldsymbol{Q}} < \lambda$.

4 PARTIAL CHAINS AND NEW CANDIDATES

Chevaleyre et al. [8] investigated the Possible co-Winner with New Candidates (PcWNA) problem, which arises in the following natural scenario: for a given set of candidates, the voters have completely ranked them; new candidates join the election *after* the voters have ranked all the initial candidates. In PcWNA, one asks: is a candidate from amongst the initial set of candidates a possible winner? As we shall see next, the PcWNA problem can be viewed as a special case of the PW problem on partial chains.

Observe that, in the PcWNA problem, the rankings of the voters are total for the initial set of candidates. When both the initial candidates and those who joined late are considered, then we have a collection of partial chains that have a special structure: all of them are total orders on the *same* subset of candidates, namely, the set of initial candidates.

Definition 4. Let C be a set of candidates and $\boldsymbol{P} = (P_1, \ldots, P_n)$ be a collection of partial chains on C. We say that $\boldsymbol{P}$ is *uniform* if there exists a set $C' \subseteq C$ such that each $P_i \in \boldsymbol{P}$ consists of a total order on the set C' (no candidates outside C' are comparable).

Definition 5. The PW-UPC problem asks: given a set of candidates C, a uniform collection $\mathbf{P} = (\mathbf{P_1}, \ldots, \mathbf{P_n})$ of partial chains in which every P_i, $1 \leq i \leq n$, consists of a total order on the same set $C' \subseteq C$, and a distinguished candidate $c \in C'$, is $c \in \mathrm{PW}(r, \mathbf{P})$?

In PW-UPC, a candidate $c \in (C \setminus C')$ is trivially a possible winner (each voter ranks c in position one). So the only interesting case is when $c \in C'$. Thus, the PcWNA problem coincides with the PW-UPC problem, which is a special case of the PW-PC problem

The complexity of the PcWNA problem (PW-UPCin the above terminology) has been investigated in [2, 8, 9]. In these papers, it has been shown that the complexity of the PW-UPC problem w.r.t. 2-approval drops from NP-complete to P. Interestingly, this problem

Table 6: Computational complexity of restrictions of PW.

Scoring Rule	PW	PW-PC	PW-UPC
Plurality & Veto	P	P	P
Non-decreasing rate rules	NP-c	**NP-c**	P
2-approval	NP-c	**NP-c**	P
$t \geq 3$-approval	NP-c	NP-c	NP-c
All other 2-valued rules	NP-c	**NP-c**	?
$(a_1, a_2, 1, 0, \ldots, 0)$ s.t. $a_1 > a_2 > 1$	NP-c	**NP-c**	NP-c
All remaining rules	NP-c	**NP-c**	?

continues to be NP-complete w.r.t. t-approval ($t > 2$), unlike the PW problem on other restricted partial orders mentioned earlier, such as doubly-truncated partial orders. The complexity of the PW-UPC w.r.t. Borda count also drops to P; in fact, it drops to P w.r.t. every rule of non-decreasing rate, where a scoring rule r with scoring vector $\boldsymbol{s_m}$ is of *non-decreasing rate* if for all $1 \leq i < m$, we have $s_i - s_{i+1} \leq s_{i+1} - s_{i+2}$. Finally, the PW-UPC problem is NP-complete w.r.t. the rule $(a_1, a_2, 1, 0, \ldots, 0)$, where $a_1 > a_2 > 1$.

Table 6 depicts the above results and compares them with the results obtained here.

5 CONCLUDING REMARKS

In this paper, we completely classified the complexity of the PW problem on partial chains w.r.t. to all pure positional scoring rules. This classification yields the earlier classification of the PW problem on arbitrary partial orders as a corollary, but it cannot be derived from that earlier classification.

A complete classification of the complexity of the PW problem on uniform collections of partial chains (equivalently, of the Possible Co-Winner with New Candidates problem) remains an open problem that is worth pursuing.

Our NP-hardness results for the PW problem on partial chains made use of "long" chains, i.e., chains that contained all but a fixed number of candidates. This type of partial chain arises in settings where a new candidate or a small number of new candidates enter the race late and, at that time, the voters do not know how to rank these new candidates. We are currently investigating the exact impact of the length of the chain on the complexity of the PW problem on partial chains. In particular, we are investigating whether, for each positional scoring rule other than plurality and veto, there is a *threshold* on the length of the chain below which the PW problem is in P, while above it becomes NP-complete.

In a different direction, there is a rich body of work on algorithmic problems about manipulation in voting (PW is a special case of one of these problems), where computational hardness is regarded as a feature because it provides an obstacle to such manipulation (see [10] for a survey). Work in this area includes the study of manipulation in voting with incomplete information [15, 21]; in particular, [21] considers such manipulation for top-truncated partial orders. It would be natural to investigate manipulation in voting with partial chains.

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