

# Corruption in Auctions: Social Welfare Loss in Hybrid Multi-Unit Auctions

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## ABSTRACT

We initiate the study of the social welfare loss caused by corrupt auctioneers, both in single-item and multi-unit auctions. In our model, the auctioneer may collude with the winning bidders by letting them lower their bids in exchange for a (possibly bidder-dependent) fraction  $\gamma$  of the surplus. We consider different corruption schemes. In the most basic one, all winning bidders lower their bid to the highest losing bid. We show that this setting is equivalent to a  $\gamma$ -hybrid auction in which the payments are a convex combination of first-price and the second-price payments. More generally, we consider corruption schemes that can be related to  $\gamma$ -approximate first-price auctions ( $\gamma$ -FPA), where the payments recover at least a  $\gamma$ -fraction of the first-price payments. Our goal is to obtain a precise understanding of the robust price of anarchy (POA) of such auctions. If no restrictions are imposed on the bids, we prove a bound on the robust POA of  $\gamma$ -FPA which is tight (over the entire range of  $\gamma$ ) for the single-item and the multi-unit auction setting. On the other hand, if the bids satisfy the no-overbidding assumption a more fine-grained landscape of the price of anarchy emerges, depending on the auction setting and the equilibrium notion. Albeit being more challenging, we derive (almost) tight bounds for both auction settings and several equilibrium notions, basically leaving open some (small) gaps for the coarse-correlated price of anarchy only.

## KEYWORDS

Mechanism Design; Auction Theory; Price of Anarchy

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## 1 INTRODUCTION

*Motivation and Background.* We consider auction settings where a seller wants to sell some items and for this purpose recruits an auctioneer to organize an auction on their behalf.<sup>1</sup> Such settings are widely prevalent in practice as they emerge naturally whenever the seller lacks the expertise (or facilities, time, etc.) to host the auction themselves. For example, individual sellers usually involve

<sup>1</sup>Throughout this paper, we use “they” as the gender-neutral form for third-person singular pronouns.

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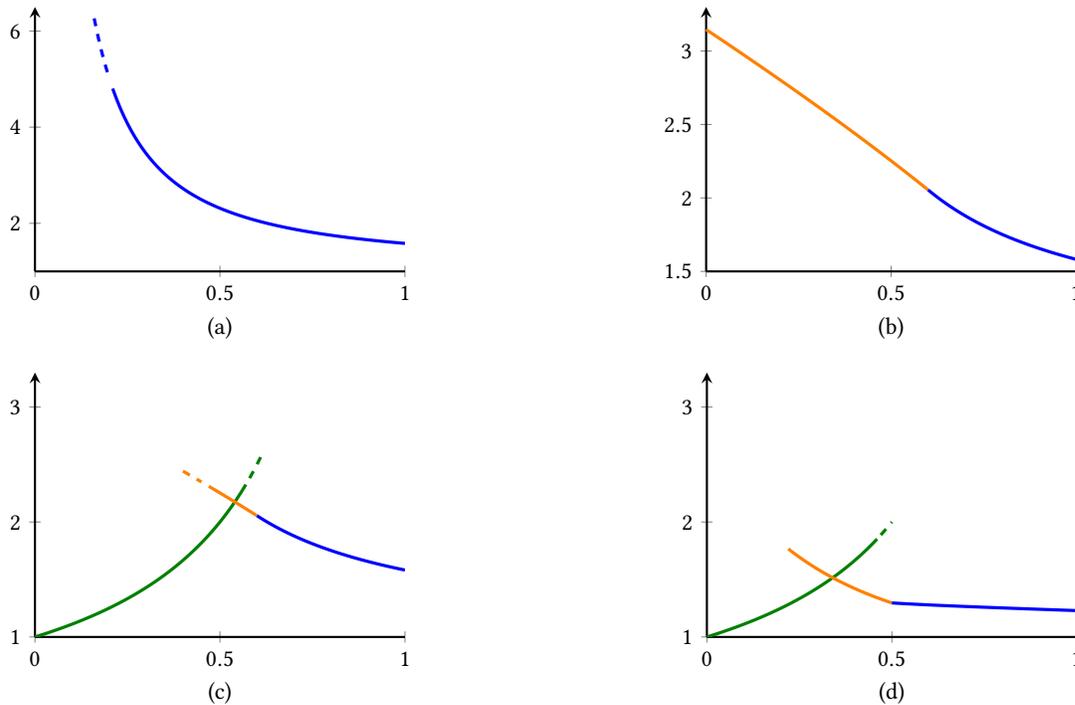
dedicated auctioneers or auction houses when they want to sell particular objects (such as real estate, cars, artwork, etc.). In private companies, the responsible finance officers are typically in charge of handling the procurement auctions. Similarly, government procurement is usually executed by some entity that acts on behalf of the government. The dilemma in such settings is that the incentives of the seller and the auctioneer are rather diverse in general: while the seller is interested in extracting the highest payments for the objects (or getting service at the lowest cost), the agent primarily cares about maximizing their own gains from hosting the auction. Although undesirably, this misalignment leads (unavoidably) to fraudulent schemes which might be used by the auctioneer to manipulate the auction to their own benefit.

Corruption in auctions, where an auctioneer engages in bid rigging with one (or several) of the bidders, occurs rather frequently in practice, especially in the public sector (e.g., in construction and procurement auctions). For example, in 1999 the procurement auction for the construction of the new Berlin Brandenburg airport had to be rerun after investigations revealed that the initial winner was able to change the bid after they had illegally acquired information about the application of one of their main competitors (see [20]). As another example, in 1993 the New York City School Construction Authority caused a scandal when investigation revealed that they used a simple (but effective) bid-rigging scheme in a procurement auction setting (see [15]):

*“In what one investigator described as a nifty scheme, that worker would unseal envelopes at a public bid opening, saving for last the bid submitted by the contractor who had paid him off. At that point, knowing the previous bids, the authority worker would misstate the contractor’s bid, insuring that it was low enough to secure the contract but as close as possible to the next highest bid so that the contractor would get the largest possible price.”*

This kind of bid rigging, where the winning bid “magically” aligns with the highest losing bid, is also known as *magic number cheating* (see [8]). We refer the reader to [10, 14] (and the references therein) for several other bid rigging examples. Despite the fact that this form of corruption occurs frequently in practice, its negative impact is still poorly understood theoretically and only a few studies exist (mostly in the economics literature, see the related work section).

Our goal is to initiate the study of the social welfare loss caused by corrupt auctioneers in fundamental auction settings. We focus on a basic model that captures the magic number cheating mentioned above and generalizations thereof. Clearly, more sophisticated bid



**Figure 1:** Overview of our upper bounds on the POA ( $y$ -axis) for  $\gamma$ -FPA and  $\gamma$ -HYA, respectively, as a function of  $\gamma$  ( $x$ -axis). (a) CCE-POA for multi-unit  $\gamma$ -FPA with overbidding (Theorem 4). (b) CCE-POA for multi-unit  $\gamma$ -FPA without overbidding (Theorems 4 & 5). (c) CCE-POA for single-item  $\gamma$ -HYA without overbidding (Theorems 4, 5 & 7). (d) CCE-POA for single-item  $\gamma$ -HYA without overbidding and  $n = 2$  bidders (Theorems 7, 8 & 9).

rigging models are conceivable and we hope that our work will trigger some future work along these lines.

*Capturing Corruption with Hybrid Auctions.* Consider the single-item auction setting and suppose the auctioneer runs a sealed bid first-price auction. After receipt of all bids, the auctioneer approaches the highest bidder with the offer that they can lower their bid to the second highest bid in exchange for a bribe. If the highest bidder agrees, they win the auction and pay the second-highest bid for the items plus the corresponding bribe to the auctioneer. If the highest bidder disagrees, they still win the auction but pay their bid for the item according to the first-price auction format. We assume that the bribe to be paid to the auctioneer is a pre-determined fraction  $\gamma \in [0, 1]$  of the savings of the highest bidder, i.e., the auctioneer’s bribe amounts to  $\gamma$  times the difference between the highest and second highest bid. In case of the multi-unit auction setting, the procedure described above is adapted accordingly by offering the winning bidders to lower their bids to the highest losing bid.

Observe that the payment scheme described above essentially reduces to the winning bidders paying a convex combination of  $\gamma$  times their bids and  $(1 - \gamma)$  times the highest losing bid. As we will argue below, this setting is tantamount to studying a *hybrid auction* ( $\gamma$ -HYA), where the items are assigned to the highest bidders (according to the respective single-item or multi-unit auction scheme) and the payments are a convex combination of the first-price and the second-price payments. By varying the parameter  $\gamma \in [0, 1]$ ,

$\gamma$ -HYA thus interpolates between the respective second-price auction ( $\gamma = 0$ ) and the first-price auction ( $\gamma = 1$ ) schemes.

More elaborate corruption schemes are of course conceivable. For example, the auctioneer might ask for a fixed amount rather than a fraction of the gains. Or, to avoid setting all bids to the magic number, the auctioneer may want to announce different (bribed) bids for every winning bidder. To capture more general corruption schemes, we also study what we term  *$\gamma$ -approximate first-price auctions* ( $\gamma$ -FPA) in this paper. Basically, these auctions implement a payment scheme that recovers at least a fraction of  $\gamma \in [0, 1]$  of the first-price payment rule (formalized below). The  $\gamma$ -HYA also belongs to this class. Not only does this capture more elaborate bribing schemes, it also handles the situation where some bidders have moral objections against partaking in such a scheme and do not accept the bribe. Additionally, this also enables us to capture corruption schemes with *heterogeneous bidders*, i.e., where the auctioneer handles a different  $\gamma_i$  for each bidder  $i$ .

In our view, the corruption settings described above serve as suitable motivations to analyze the resulting auctions  $\gamma$ -HYA and  $\gamma$ -FPA. But, at the same time, we feel that the study of such *hybrid* auction formats is interesting in its own right, purely from an auction design perspective. For example, tight bounds on the price of anarchy (as a function of  $\gamma$ ) provide insights on which payment rule should ideally be used to reduce the inefficiency.

*Our Contributions.* We study the inefficiency of equilibria of  $\gamma$ -FPA and  $\gamma$ -HYA, both in the single-item and the multi-unit auction setting. More specifically, our goal is to obtain a precise understanding of the (*robust*) price of anarchy (POA) [9, 16, 19]. We opt for the price of anarchy notion here because it is one of the most appealing and widely accepted measures to assess the efficiency of equilibria, especially in the context of social welfare analysis. We focus on the analysis of the robust price of anarchy under the complete information setting, incorporating equilibrium notions ranging from pure Nash equilibria (PNE) to coarse correlated equilibria (CCE).<sup>2</sup> Moreover, we analyze the price of anarchy distinguishing between the case when bidders can overbid and when they cannot overbid their actual valuations for the items.

The main results that we obtain in this paper are summarized below (see Figure 1 for an overview). Without any restrictions on the bids, we obtain the following result:

- (1) We prove an upper bound of  $(1/\gamma) \cdot e^{1/\gamma}/(e^{1/\gamma} - 1)$  on the coarse correlated POA (CCE-POA) of any  $\gamma$ -FPA in the multi-unit auction setting when bidders can overbid; see Figure 1(a). Our upper bound follows from a suitable adaptation of the smoothness technique for multi-unit auctions [3, 19]. Further, by means of a single-item  $\gamma$ -HYA, we prove a matching lower bound over the entire range  $\gamma \in [0, 1]$ . As a result, our bound settles the CCE-POA of  $\gamma$ -FPA exactly for both the single-item and multi-unit auction setting over the entire range of  $\gamma \in [0, 1]$ .

A standard assumption that often needs to be made to derive meaningful bounds on the POA is that the bidders cannot overbid (see also related work section). Under the no-overbidding assumption, a more fine-grained landscape of the price of anarchy emerges:

- (2) We show that the pure POA (PNE-POA) of  $\gamma$ -HYA in the multi-unit auction setting is 1 for  $\gamma \in (0, 1)$ . This result is complemented by PNE-POA = 2.1885 for  $\gamma = 0$  [1] and PNE-POA = 1 for  $\gamma = 1$  [3]. Note that this reveals an interesting transition at  $\gamma = 0$ .
- (3) We prove that the CCE-POA of any  $\gamma$ -FPA in the multi-unit auction setting is upper bounded by

$$-(1 - \gamma)\mathcal{W}_{-1}\left(-\frac{1}{e^{(2-\gamma)/(1-\gamma)}}\right),$$

for  $\gamma \lesssim 0.607$  where  $\mathcal{W}$  is the Lambert- $\mathcal{W}$  function. Combined with our upper bound (first contribution above) for  $\gamma > 0.607$  (i.e. with overbidding), we obtain the combined bound depicted in Figure 1(b).

- (4) We prove that the correlated POA (CE-POA) of  $\gamma$ -HYA in the single-item auction setting is 1 for every  $\gamma \in (0, 1)$ . This result together with CE-POA = 1 for  $\gamma = 1$  [5] and our next result, shows that CE-POA = 1 for the entire range  $\gamma \in [0, 1]$ .
- (5) We show that the CCE-POA of  $\gamma$ -HYA in the single-item auction setting with  $n$  bidders is bounded as indicated in Figure 1(c). Concretely, we prove an upper bound of  $1/(1-\gamma)$  and combine it with the multi-unit bounds from Figure 1(b).
- (6) We show that the CCE-POA of  $\gamma$ -HYA in the single-item auction setting with  $n = 2$  bidders is bounded as indicated

in Figure 1(d). This bound is derived by combining three different upper bounds, one of which the  $1/(1-\gamma)$  bound from Figure 1(c). Technically, this is the most challenging part of the paper as we use the cumulative distribution functions (CDF) of equilibrium bids directly to derive these bounds.

*Implications.* Altogether, our bounds provide a rather complete picture of the POA of  $\gamma$ -FPA and, in particular,  $\gamma$ -HYA, for different equilibrium notions both in the single-item and the multi-unit auction setting and with and without overbidding. If the bidders can overbid then our (tight) bound on the CCE-POA (Figure 1(a)) shows that the POA increases from a small constant  $e/(e-1)$  to infinity as  $\gamma$  decreases from 1 to 0. Intuitively this makes sense: As  $\gamma$  decreases from 1 to 0, the auctioneer withholds a smaller fraction of the surplus which incentivizes the bidders to exploit the corruption.

Our bounds reveal that there is a substantial difference in the POA depending on whether or not bidders can overbid; e.g., compare the bounds depicted in (a) and (b) (multi-unit setting), or (a) and (c) (single-item setting) in Figure 1. In general, it is not well-understood how the no-overbidding assumption influences the POA of auctions; this question also relates to the *price of undominated anarchy* studied by Feldman et al. [5] (see related work below). Our bounds shed some light on this question for  $\gamma$ -FPA.

*Technical Merits.* Our upper bounds for  $\gamma$ -FPA are based on an adapted smoothness notion which relates directly to the highest marginal winning bids (i.e., first-price payments). In particular, our smoothness argument does *not* exploit the second-price payments of  $\gamma$ -HYA at all. As it turns out, this allows us to derive *tight* bounds for  $\gamma$ -HYA and, more generally, for  $\gamma$ -FPA when bidders can overbid. On a high level, our results thus reveal that the (approximate) first-price payments are the determining component of such composed payment schemes. This triggers some interesting questions for future research.

In contrast, when overbidding is not allowed it becomes crucial to exploit the second-price payments of  $\gamma$ -HYA to obtain improved bounds. The price of anarchy of both the first-price auction and the second-price auctions is well understood in the single-item setting. However, it is not straightforward to extend these bounds to the combined payment scheme of  $\gamma$ -HYA. In fact, to prove our bounds in Theorem 8 and Theorem 9, we exploit constraints on the CDF of the first-price payments which are imposed by the CCE conditions; but, additionally, we have to get a grip on the CDF of the second-price payments. We need several new insights (and a somewhat involved numerical analysis) to derive these bounds.

*Extensions.* Although we focus on the complete information setting in this paper, most of our bounds can be lifted to the incomplete information setting as introduced by Harsanyi [7], where players have private valuation functions drawn from a common prior. More specifically, all bounds displayed in Figure 1(a–c) remain valid for Bayes-Nash equilibria as well. We defer further details to the full version of the paper.

*Related Work.* There is a large body of research in economics studying collusion among bidders in auctions (see, e.g., [6, 13] for some standard references). Collusion between the auctioneer and the bidders in the form of bid rigging (as considered in this paper) has also been studied in the literature, but less intensively. Most

<sup>2</sup>Several bounds are based on an adapted smoothness approach and extend to the incomplete information setting; see the extensions section below for more details.

existing works study certain aspects of equilibrium outcomes (e.g., equilibrium structure, auctioneer surplus, seller revenue, optimal bribe schemes, etc.); for an overview of the existing works along these lines, see [10, 11, 14] and the references therein.

The specific bid rigging model that we consider here was first studied by Menezes and Monteiro [14] and a slight generalization thereof by Lengwiler and Wolfstetter [11], both for the single-item auction setting. These works consider a Bayesian setting where the valuations are independent draws from a common distribution function. Menezes and Monteiro [14] prove the existence of symmetric equilibrium bidding strategies and derive an optimal bribe function for the auctioneer. The authors also study a *fixed-price* bribe scheme, where the auctioneer charges a fixed amount that is independent of the gained surplus.

Subsequently, Lengwiler and Wolfstetter [10] study a more complex bid rigging scheme for the single-item auction setting, where the auctioneer additionally offers the second highest bidder to increase their bid. To the best of our knowledge, none of the existing works studied the price of anarchy of corrupt auctions.

Studying the price of anarchy in auctions has recently received a lot of attention; we refer to the survey paper by Roughgarden et al. [17] for an overview. A lot of work has gone into deriving bounds on the price of anarchy for various auction formats, both in the complete and incomplete information setting. The *smoothness notion*, originally introduced by Roughgarden in [16] to analyze the robust price of anarchy of strategic games, turned out to be very useful in an auction context as well. Syrgkanis and Tardos [19] build upon this notion and provide a powerful (smoothness-based) toolbox for the analysis of a broad range of auctions that fall into their composition framework.

With respect to the multi-unit auction setting, de Keijzer et al. [3] use an adapted smoothness approach to derive bounds on the POA of Bayes-Nash equilibria for the first-price and the second-price multi-unit auction (mostly focussing on the setting with no overbidding). Our bounds coincide with theirs for the extreme points  $\gamma = 0$  and  $\gamma = 1$ . For the more general class of subadditive valuations, the POA of Bayes-Nash equilibria for the first-price multi-unit auction is 2, which follows from [3] and [2]. Birmpas et al. [1] recently settled the PNE-POA of the second-price multi-unit auction and show that it is 2.1885.

Our bounds on the CCE-POA are also based on a smoothness approach. We use an adapted smoothness notion (inspired by [3, 19]) to derive our bounds, both in the overbidding and the no-overbidding setting. Interestingly, our smoothness proofs crucially exploit that the payments recover at least a fraction of  $\gamma$  of the first-price payments (but never exceed them). As a side result, Syrgkanis and Tardos [19] also derive a first bound on the CE-POA for  $\gamma$ -HYA in the single-item auction setting; our bound (significantly) improves on theirs and exploits some additional ideas.

The POA of the first-price and second-price auction has been investigated intensively for both the single-item and the multi-unit auction setting. An assumption that often needs to be made to derive meaningful bounds is that the bidders cannot overbid. For example, it is folklore that the PNE-POA of the second-price single-item auction is unbounded if the bidders can overbid. On the other hand, it is one if bidders cannot overbid. In the second-price single-item auction, overbidding is a dominated strategy for each bidder

and the no-overbidding assumption thus emerges naturally. But this might not be true in general. For example, for the second-price multi-unit auction, this analogy breaks already. We refer to [4] for a more general discussion of the no-overbidding assumption.

In general, the impact that the no-overbidding assumption has on the price of anarchy is not well-understood. This aspect also relates to the *price of undominated anarchy* studied by Feldman et al. [5]. The authors prove a clear separation for the POA in single-item first-price auctions: While the CE-POA is 1 (even with overbidding), the CCE-POA increases to 1.229 (without overbidding) and  $e/(e-1)$  (with overbidding). A similar separation holds for the multi-unit auction setting and the uniform price auction, where the PNE-POA is  $(e-1)/e$  (without overbidding) [12] and 2.1885 (with overbidding) [1]. Our results contribute to this line of research also because we show that the POA might improve significantly under the no-overbidding assumption.

## 2 PRELIMINARIES

*Standard Auction Formats.* We focus on the description of the multi-unit auction setting; the single-item auction setting follows as a special case (choosing  $k = 1$  below). In the multi-unit auction setting, there are  $k \geq 1$  identical items (or goods) that we want to sell to  $n \geq 2$  bidders (or players). We identify the set of bidders  $N$  with  $[n] = \{1, \dots, n\}$ . Each bidder  $i$  has a non-negative and non-decreasing valuation function  $v_i : \{0, \dots, k\} \rightarrow \mathbb{R}_{\geq 0}$  with  $v_i(0) = 0$ , where  $v_i(j)$  specifies  $i$ 's valuation for receiving  $j$  items. We assume that for each bidder  $i \in N$  the valuation function  $v_i$  is *submodular* or, equivalently, that the marginal valuations are *non-increasing*, i.e., for every  $j \in [k-1]$ ,  $v_i(j) - v_i(j-1) \geq v_i(j+1) - v_i(j)$ . The valuation function  $v_i$  is assumed to be private information, i.e., it is only known to bidder  $i$  themselves. We use  $\mathbf{v} = (v_1, \dots, v_n)$  to denote the profile (or vector) of the valuation functions of the bidders. We assume that the bidders submit their bids according to the following *standard format*: Each bidder  $i$  submits a bid vector  $\mathbf{b}_i = (b_i(1), \dots, b_i(k))$  of  $k$  non-negative and non-increasing *marginal bids*, i.e.,  $b_i(j)$  specifies the additional amount  $i$  is willing to pay for receiving  $j$  instead of  $j-1$  items. The overall amount that  $i$  bids for receiving  $q$  items is thus  $\sum_{j=1}^q b_i(j)$ . For  $k = 1$  we write  $\mathbf{b}_i = b_i(1)$ .

Consider a multi-unit auction setting and suppose the auctioneer uses an *auction mechanism*  $\mathcal{M}$  to determine an assignment of the items and the respective payments of the bidders. Each bidder submits their bid vector  $\mathbf{b}_i$  to the mechanism. Based on the bidding profile  $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ , the mechanism  $\mathcal{M}$  orders the submitted marginal bids non-increasingly (breaking ties in an arbitrary but consistent way) and assigns the  $k$  items to the bidders who submitted the  $k$  highest marginal bids (according to this order). We use  $\beta_j(\mathbf{b})$  to refer to the  $j$ -th lowest winning (marginal) bid in  $\mathbf{b}$ , i.e.,  $\beta_k(\mathbf{b}) \geq \dots \geq \beta_1(\mathbf{b})$ . We use  $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), \dots, x_n(\mathbf{b}))$  to refer to the resulting allocation, where  $x_i(\mathbf{b})$  specifies the number of items that bidder  $i$  receives;  $x_i(\mathbf{b}) = 0$  if  $i$  does not receive any item. Each bidder  $i$  who receives at least one item is called a *winner*.

There are two standard payment schemes that determine for each winner  $i$  the respective payment  $p_i(\mathbf{b})$ ; we adopt the convention that  $p_i(\mathbf{b}) = 0$  for each bidder  $i$  who is not a winner.

- *First-price payment scheme*: Every bidder  $i$  pays their bid for the received items, i.e.,  $p_i(\mathbf{b}) = \sum_{j=1}^{x_i(\mathbf{b})} b_i(j)$
- *Second-price payment scheme*: Every bidder  $i$  pays the highest losing bid  $\bar{p}(\mathbf{b})$  for each received item, i.e.,  $p_i(\mathbf{b}) = x_i(\mathbf{b})\bar{p}(\mathbf{b})$

Suppose we fix the payment scheme of mechanism  $\mathcal{M}$  according to one of these schemes. We refer to mechanism  $\mathcal{M}$  with the first-price payment or the second-price payment scheme, respectively, as *FP-AUCTION* or *SP-AUCTION*.<sup>3</sup>

The utility  $u_i^{v_i}(\mathbf{b})$  of bidder  $i$  is defined as the total valuation minus the payment for receiving  $x_i(\mathbf{b})$  items, i.e.,  $u_i^{v_i}(\mathbf{b}) = v_i(x_i(\mathbf{b})) - p_i(\mathbf{b})$ ; note that  $u_i^{v_i}(\mathbf{b}) = 0$  by definition if bidder  $i$  is not a winner. Whenever  $v_i$  is clear from the context, we simply denote the utility of bidder  $i$  by  $u_i(\mathbf{b})$ . We assume that each bidder strives to maximize their utility.

Finally, we introduce some standard assumptions that we use throughout this paper; we adopt the convention that the first two must always be satisfied by a mechanism.

- (1) *No positive transfers (NPT)*: The payment of each bidder  $i$  is non-negative, i.e.,  $p_i(\mathbf{b}) \geq 0$ .
- (2) *Individual rationality (IR)*: The payment of each bidder  $i$  does not exceed their bid, i.e.,  $p_i(\mathbf{b}) \leq \sum_{j=1}^{x_i(\mathbf{b})} b_i(j)$ .
- (3) *No overbidding (NOB)*: The bid vector of each bidder  $i$  does not exceed their valuations, i.e., for every  $q \in [k]$ ,  $\sum_{j=1}^q b_i(j) \leq v_i(q)$ .

*Approximate First-Price Auctions.* In this paper, we also consider auctions with first-price approximate payment schemes. The allocation is still determined as above, but the payment scheme is relaxed as follows: We say that a mechanism  $\mathcal{M}$  with payment rule  $\mathbf{p} = (p_1(\mathbf{b}), \dots, p_n(\mathbf{b}))$  is a  $\gamma$ -*approximate first-price auction* ( $\gamma$ -*FPA*) for some  $\gamma \in [0, 1]$  if it always recovers at least a fraction of  $\gamma$  of the first-price payments, i.e., for every bidding profile  $\mathbf{b}$ ,  $\sum_{i \in N} p_i(\mathbf{b}) \geq \gamma \sum_{j=1}^k \beta_j(\mathbf{b})$ . Further, if for every bidding profile  $\mathbf{b}$  it holds that  $\sum_{i \in N} p_i(\mathbf{b}) \leq \sum_{j=1}^k \beta_j(\mathbf{b})$  then we call the mechanism *first-price dominated*. Note that every mechanism that satisfies individual rationality must be first-price dominated.

*Equilibrium Notions and the Price of Anarchy.* We focus on the complete information setting here. Below, we briefly review the different equilibrium notions used in this paper. A bidding profile  $\mathbf{b} = (b_1, \dots, b_n)$  is a *pure Nash equilibrium (PNE)* if no bidder has an incentive to deviate unilaterally; more formally,  $\mathbf{b}$  is a PNE if for every bidder  $i$  and every bidding profile  $\mathbf{b}'_i$  of  $i$  it holds that  $u_i(\mathbf{b}) \geq u_i(\mathbf{b}'_i, \mathbf{b}_{-i})$ . Here we use the standard notation  $\mathbf{b}_{-i}$  to refer to the bid vector  $\mathbf{b}$  with the  $i$ th component being removed;  $(\mathbf{b}'_i, \mathbf{b}_{-i})$  then refers to the bid vector  $\mathbf{b}$  with the  $i$ th component being replaced by  $\mathbf{b}'_i$ .

We also consider randomized bid vectors. Suppose bidder  $i$  chooses their bid vectors randomly according to a probability distribution  $\sigma_i$ , independently of the other bidders. Let  $\sigma = \prod_{i \in N} \sigma_i$  be the respective product distribution. Then  $\sigma$  is a *mixed Nash equilibrium (MNE)* if for every bidder  $i$  and every bid vector  $\mathbf{b}'_i$  it holds that  $\mathbb{E}_{\mathbf{b} \sim \sigma} [u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \sigma} [u_i(\mathbf{b}'_i, \mathbf{b}_{-i})]$ . We may also allow correlation

<sup>3</sup>We remark that in the multi-unit auction setting these auctions are usually referred to as *discriminatory price auction* and *uniform price auction*; however, here we stick to the given naming convention to align it with the common terminology of the single-item auction setting.

among the bidders. Let  $\sigma$  be a joint distribution over bidding profiles of the bidders. Then  $\sigma$  is a *correlated equilibrium (CE)* if for every bidder  $i \in N$  and for every deviation function  $m_i(\mathbf{b}_i)$  it holds that  $\mathbb{E}_{\mathbf{b} \sim \sigma} [u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \sigma} [u_i(m_i(\mathbf{b}_i), \mathbf{b}_{-i})]$ . Intuitively, conditional on bid vector  $\mathbf{b}_i$  being realized,  $i$  has no incentive to deviate to any other bid vector  $m_i(\mathbf{b}_i)$ . The most general equilibrium notion that we consider in this paper is defined as follows: Let  $\sigma$  be a joint distribution over bidding profiles of the bidders. Then  $\sigma$  is a *coarse correlated equilibrium (CCE)* if for every bidder  $i$  and every bid vector  $\mathbf{b}'_i$  it holds that  $\mathbb{E}_{\mathbf{b} \sim \sigma} [u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \sigma} [u_i(\mathbf{b}'_i, \mathbf{b}_{-i})]$ . Below, we also use PNE( $v$ ), MNE( $v$ ), CE( $v$ ) and CCE( $v$ ) to refer to the sets of pure, mixed, correlated and coarse correlated equilibria with respect to a valuation profile  $v = (v_1, \dots, v_n)$ , respectively.

We define the *social welfare* of a bidding profile  $\mathbf{b} = (b_1, \dots, b_n)$  as the overall valuation obtained by the bidders, i.e.,  $SW(\mathbf{b}) = \sum_{i \in N} v_i(x_i(\mathbf{b}))$ . Note that although social welfare is defined independently of the payments, we can equivalently write  $SW(\mathbf{b}) = \sum_{i \in N} u_i(\mathbf{b}) + p_i(\mathbf{b})$ . The expected social welfare of a joint distribution  $\sigma$  over bidding profiles is then defined as  $\mathbb{E}[SW(\sigma)] = \mathbb{E}_{\mathbf{b} \sim \sigma} [SW(\mathbf{b})]$ . We use  $\mathbf{x}^*(v)$  to refer to an assignment that maximizes the social welfare with respect to the valuation functions  $v = (v_1, \dots, v_n)$ ; i.e.,  $SW(\mathbf{x}^*(v)) = \sum_{i \in N} v_i(x_i(v))$  is the maximum social welfare achievable for the bidders. The assignment  $\mathbf{x}^*(v)$  is also called a *social optimum*.

The *price of anarchy* is defined as the maximum ratio of the social welfare of the social optimum and the (expected) social welfare of an equilibrium. Let  $X$  be a placeholder that refers to one of the equilibrium notions above, i.e.,  $X \in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$ . More formally, given a valuation profile  $v = (v_1, \dots, v_n)$ , the *price of anarchy* with respect to  $X$  (or  $X$ -POA for short) is defined as  $X\text{-POA}(v) = \sup_{\sigma \in X(v)} SW(\mathbf{x}^*(v)) / \mathbb{E}[SW(\sigma)]$ . The price of anarchy of an auction format then refers to the worst-case price of anarchy over all possible valuation profiles, i.e.,  $X\text{-POA} = \sup_v X\text{-POA}(v)$ . We use PNE-POA, MNE-POA, CE-POA and CCE-POA to refer to the respective price of anarchy notions.

Due to page limitations, some proofs are sketched or omitted from this extended abstract; all missing details can be found in the full version of this paper.

### 3 CAPTURING CORRUPTION WITH $\gamma$ -FPA

We give a formal description of the model that we consider and elaborate on its relation to the  $\gamma$ -hybrid auction. We also introduce the adapted smoothness approach.

*Corruption in Auctions.* Suppose the bidders submit their bid vectors  $\mathbf{b} = (b_1, \dots, b_n)$  in a “sealed manner”, i.e., at first only the auctioneer sees the bidding profile  $\mathbf{b}$ .<sup>4</sup> After receipt of the bidding profile  $\mathbf{b}$ , the auctioneer runs a first-price multi-unit auction (see Section 2) to obtain the respective assignment  $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), \dots, x_n(\mathbf{b}))$  and payments  $\mathbf{p}(\mathbf{b}) = (p_1(\mathbf{b}), \dots, p_n(\mathbf{b}))$  but does not reveal this outcome yet. The auctioneer then approaches each winning bidder  $i$  individually with the offer that they can lower all their  $x_i(\mathbf{b})$  winning bids to the highest losing bid  $\bar{p}(\mathbf{b})$  (while receiving the same number of items), in exchange for a fixed

<sup>4</sup>It is important to realize though that the final bids, which might not necessarily correspond to the submitted ones, might have to be revealed eventually because the bidders might want to verify the “soundness” of the outcome of the auction.

fraction  $\gamma \in [0, 1]$  of the surplus gained by  $i$ . The bidder can either reject or accept this offer. If bidder  $i$  rejects the offer, the allocation  $x_i(\mathbf{b})$  and respective payment  $p_i(\mathbf{b})$  remain unmodified. If bidder  $i$  accepts the offer, they receive the  $x_i(\mathbf{b})$  items at a reduced price of  $\bar{p}(\mathbf{b})$  each, but additionally pay a fee  $f_i^\gamma$  of  $\gamma$  times the surplus to the auctioneer; more formally, the total payment of a winning bidder  $i$  who accepts the offer is

$$p_i^\gamma(\mathbf{b}) = x_i(\mathbf{b})\bar{p}(\mathbf{b}) + f_i^\gamma(\mathbf{b}) \quad \text{where} \quad f_i^\gamma(\mathbf{b}) = \gamma \sum_{j=1}^{x_i(\mathbf{b})} (b_i(j) - \bar{p}(\mathbf{b})).$$

We also refer to this setting as the  $\gamma$ -corrupt auction.<sup>5</sup>

Note that the change in the bid vector of player  $i$  conforms to the imposed bidding format, i.e., the modified marginal bids of bidder  $i$  are still non-negative and non-increasing. It is not hard to show that it is a dominant strategy for every winning bidder to accept the offer of the auctioneer, independently of the parameter  $\gamma$ . Subsequently, we assume that each winning bidder always accepts the offer.

*Hybrid Auction Scheme.* We introduce our novel hybrid auction scheme, which we term  $\gamma$ -hybrid auction (or  $\gamma$ -HYA for short):  $\gamma$ -HYA uses the same allocation rule as in the multi-unit auction setting (see Section 2), but uses a convex combination of the first-price and second-price payment scheme (parameterized by  $\gamma$ ), i.e.,

$$p_i^\gamma(\mathbf{b}) = \gamma \sum_{j=1}^{x_i(\mathbf{b})} b_i(j) + (1 - \gamma)x_i(\mathbf{b})\bar{p}(\mathbf{b}). \quad (1)$$

Said differently,  $\gamma$ -HYA interpolates between SP-AUCTION ( $\gamma = 0$ ) and FP-AUCTION ( $\gamma = 1$ ) as  $\gamma$  varies from 0 to 1. It is immediate that every  $\gamma$ -HYA is a  $\gamma$ -FPA. We also use  $p^\gamma(\mathbf{b})$  to refer to the above payment in the single-item auction setting.

The following proposition follows immediately from the discussion above and allows us to focus on the POA of  $\gamma$ -HYA to study  $\gamma$ -corrupt auctions.

**PROPOSITION 1.** *Fix some  $\gamma \in [0, 1]$ . Then the  $\gamma$ -corrupt auction and  $\gamma$ -HYA admit the same set of equilibria and have identical social welfare objectives. Therefore, the price of anarchy for both these settings is the same.*

*Other Corruption Models.* In our basic bid rigging model introduced above all winning bidders lower their bids to the highest losing bid. While this magic number bidding phenomenon has been observed in real-life for single-item auctions (as mentioned in the introduction), it might seem somewhat awkward in the multi-unit auction setting.<sup>6</sup> We therefore consider more general corruption schemes that also capture non-uniform bid rigging. More precisely, most of our upper bounds hold for the more general class of  $\gamma$ -FPA introduced above. These auctions capture several additional corruption settings. For example, suppose some bidders never accept the offer of the auctioneer (say due to moral objections) and their payments thus remains the first-price payment. While this setting is

<sup>5</sup>As the final payments are dependent on  $\gamma$ , we (implicitly) assume that the bidders are aware of this parameter when considering the complete information setting here (much alike it is assumed that the bidders know the used payment scheme in other auction formats).

<sup>6</sup>We refer to the full version for further discussion on how the auctioneer could “camouflage” the magic number bidding in this case.

not covered by  $\gamma$ -HYA, it is covered by  $\gamma$ -FPA. As another example, if the auctioneer handles a different fraction  $\gamma_i$  for each bidder  $i$ , the resulting auction is  $\gamma$ -FPA with  $\gamma = \min_{i \in N} \gamma_i$ .

*Adapted Smoothness Notion.* We introduce our adapted smoothness notion to derive upper bounds on the coarse correlated price of anarchy of  $\gamma$ -FPA.<sup>7</sup> Recall that, given a bidding profile  $\mathbf{b}$ , we use  $\beta_j(\mathbf{b})$  to refer to the  $j$ th lowest winning bid under  $\mathbf{b}$ .

**Definition 2.** A mechanism  $\mathcal{M}$  for the multi-unit auction setting is  $(\lambda, \mu)$ -smooth for some  $\lambda > 0$  and  $\mu \geq 0$  if for every valuation profile  $\mathbf{v}$  and for each bidder  $i \in N$  there exists a (possibly randomized) deviation  $\sigma'_i$  such that for every bidding profile  $\mathbf{b}$  we have

$$\sum_{i \in N} \mathbb{E}_{\mathbf{b}'_i \sim \sigma'_i} [u_i(\mathbf{b}'_i, \mathbf{b}_{-i})] \geq \lambda \text{SW}(\mathbf{x}^*(\mathbf{v})) - \mu \sum_{j=1}^k \beta_j(\mathbf{b}).$$

By using the above smoothness definition together with the properties that the payments in  $\gamma$ -FPA are first-price dominated and  $\gamma$ -approximate, we obtain the following theorem.

**THEOREM 3.** *Let  $\alpha > 0$  be fixed arbitrarily. The coarse correlated price of anarchy of any  $\gamma$ -FPA is*

$$\text{CCE-POA} \leq \frac{\max\{1, 1 + \alpha - \gamma\}}{\alpha(1 - e^{-1/\alpha})}, \quad (2)$$

where we need that the no-overbidding assumption holds if  $\alpha > \gamma$ .

## 4 OVERBIDDING

We derive a tight bound on the coarse correlated price of anarchy of  $\gamma$ -FPA for  $\gamma > 0$  in the multi-unit auction setting when bidders can overbid. Interestingly, tightness is already achieved by a single-item  $\gamma$ -HYA. It is known that the price of anarchy is unbounded for SP-AUCTION ( $\gamma = 0$ ). The bound is displayed in Figure 1(a). We give a sketch of the proof of Theorem 4 below.

**THEOREM 4.** *Consider a multi-unit  $\gamma$ -FPA and suppose that bidders can overbid. For  $\gamma \in (0, 1]$ , the coarse correlated price of anarchy is  $\text{CCE-POA} \leq \frac{1}{\gamma(1 - e^{-1/\gamma})}$ . Further, this bound is tight, even for single-item  $\gamma$ -HYA.*

**PROOF SKETCH.** The upper bound is based on Theorem 3. Recall that for the second term in  $\max\{1, 1 + \alpha - \gamma\}$  from equation (2), we need the no-overbidding assumption to hold. Since bidders are allowed to overbid in this section, we restrict to  $\alpha \leq \gamma$ . The corresponding bound is minimized by setting  $\alpha = \gamma$  for any  $\gamma \in (0, 1]$ .

This bound can be proven to be tight for all  $\gamma \in (0, 1]$  by generalizing an example used by Syrgkanis [18] to provide a lower bound on the CCE-POA for the first-price single-item auction: We consider a single-item auction with two bidders and using the  $\gamma$ -hybrid pricing rule as defined above. We have  $v_1 = v$  for some  $v > 0$  and  $v_2 = 0$ . If both bidders bid 0, the tie is broken in favor of bidder 2, whereas bidder 1 wins the auction if bidders tie with any positive bid. Let  $t$  be a random variable with support  $[0, (1 - e^{-1/\gamma})v]$  whose cumulative distribution function  $F$  is given by  $F(t) = (1 - \gamma) + v/(v - t)\gamma e^{-1/\gamma}$ .

<sup>7</sup>Our new notion comes close to the *weak smoothness* definition in [19], but relates more directly to the winning bids in the multi-unit setting. A similar definition is also used in [3], but there it is imposed on a per-player basis and used only for first-price dominated payment schemes.

Then, the bidding profile  $\sigma = (t, t)$  is a CCE for which the welfare loss matches the upper bound.  $\square$

## 5 NO OVERBIDDING

### 5.1 Multi-Unit Auction

In the previous section, we have completely settled the coarse correlated price of anarchy of  $\gamma$ -FPA when overbidding is allowed. We see that especially when  $\gamma$  gets small this has an extremely negative effect on the price of anarchy. In this section, we will investigate how these bounds improve under the no-overbidding assumption (NOB as defined above). It is a standard assumption to make and we will see that it leads to a significant improvement of the price of anarchy bounds, most notably for lower values of  $\gamma$ .

We can show that pure Nash equilibria of  $\gamma$ -HYA without overbidding are always efficient for all  $\gamma \in (0, 1]$  (see the full version for details). For coarse correlated equilibria, we can significantly improve the upper bound derived in Theorem 4 for  $\gamma \lesssim 0.607$ .

**THEOREM 5.** *Consider a multi-unit  $\gamma$ -FPA and suppose that bidders cannot overbid. For  $\gamma \lesssim 0.607$ , the coarse correlated price of anarchy is*

$$\text{CCE-POA} \leq -(1-\gamma)\mathcal{W}_{-1}\left(-\frac{1}{e^{(2-\gamma)/(1-\gamma)}}\right). \quad (3)$$

**PROOF SKETCH.** Similar to the upper bound in Theorem 4, this result is based on Theorem 3. However, since bidders cannot overbid, we now also allow for  $\alpha > \gamma$  in the minimization of the bound. This makes the optimization somewhat more involved, but leads to a significantly better bound for  $\gamma \lesssim 0.607$ .  $\square$

Combining the improved bound of Theorem 5 with the bound of Theorem 4 yields the upper bound displayed in Figure 1(b) for all  $\gamma \in [0, 1]$ . In particular, we obtain  $\text{CCE-POA} \leq -\mathcal{W}_{-1}(-e^{-2}) \approx 3.146$  for  $\gamma = 0$  and  $\text{CCE-POA} \leq e/(e-1) \approx 1.582$  for  $\gamma = 1$ .

### 5.2 Single-Item $\gamma$ -HYA

We can further improve the price of anarchy bounds for single-item  $\gamma$ -HYA. It allows us to make more direct use of the payments giving us more control. We start with the general  $n$ -player setting, for which we show that the single-item  $\gamma$ -HYA is fully efficient up to correlated equilibria. For coarse correlated equilibria, we then derive a strong bound for low values of  $\gamma$ , namely  $\text{CCE-POA} \leq 1/(1-\gamma)$ . This bound can in turn be complemented by the bound we derived for multi-unit auctions. Finally, to improve upon this multi-unit bound for the higher range of  $\gamma$ , we derive two technically more involved bounds that work specifically in a two-player setting.

We need some more notation. Given a bid vector  $\mathbf{b}$ , let  $HB(\mathbf{b}) = \max_i b_i$  and  $SB(\mathbf{b})$  denote the highest and second highest bid in  $\mathbf{b}$ , respectively, and let  $HB_{-i}(\mathbf{b}) = \max_{j \neq i} b_j$  be the highest bid excluding bid  $b_i$ . For a randomized bid vector  $\sigma$ , let  $HB(\sigma)$  be the random variable equal to the highest bid when the bids are distributed according to  $\sigma$ . We sometimes write  $\mathbb{E}[HB(\sigma)]$  for  $\mathbb{E}_{\mathbf{b} \sim \sigma}[HB(\mathbf{b})]$  (similarly for  $SB(\sigma)$  and  $HB_{-i}(\sigma)$ ).

*Correlated Price of Anarchy.* We prove that  $\gamma$ -HYA is fully efficient for all  $\gamma \in (0, 1]$  up to correlated equilibria. We extend a result in [5], which only does it for  $\gamma = 1$ . Below we show that for  $\gamma = 0$  even

coarse correlated equilibria are always efficient, so that Theorem 6 in fact holds for all  $\gamma \in [0, 1]$ .

**THEOREM 6.** *Consider a single-item  $\gamma$ -HYA and suppose that bidders cannot overbid. Then, the correlated price of anarchy of  $\gamma$ -HYA is 1 for all  $\gamma \in (0, 1]$ .*

**PROOF SKETCH.** Without loss of generality assume that player 1 has the highest valuation  $v_1$ . Assume towards contradiction that the CE-POA is not 1. Then, there must be a player  $i$  with  $v_i < v_1$  who has a positive probability of winning. Let  $b^* = \inf\{b \mid \mathbb{P}[HB(\sigma) < b] > 0\}$ . Since we assume that players cannot overbid, we would need  $b^* \leq v_i$ . However, we can show that  $b^* \leq v_i$  always leads to a contradiction with the CE conditions.  $\square$

*Coarse Correlated Price of Anarchy.* It is known that the coarse correlated price of anarchy for the first-price auction is approximately 1.229 [5], which implies that the result of Theorem 6 does not extend to coarse correlated equilibria. We derive the following bound which is good for small values of  $\gamma$ .

**THEOREM 7.** *Consider a single-item  $\gamma$ -HYA and suppose that bidders cannot overbid. Then, the coarse correlated price of anarchy of  $\gamma$ -HYA is at most  $1/(1-\gamma)$  for all  $\gamma \in [0, 1]$ .*

Any upper bound for the multi-unit auction setting of course also holds for the single-item setting. By combining the bounds of Theorem 4, Theorem 5 and Theorem 7, we obtain the upper bound displayed in Figure 1(c) for the coarse correlated price of anarchy in the single-item auction setting.

*Coarse Correlated Price of Anarchy for 2-player Auctions.* We now present a more fine-grained picture for the coarse correlated price of anarchy for the 2-player setting. Ultimately, the upper bound for CCE-POA for two players becomes a combination of three upper bounds, as represented by the three colors in Figure 1(d). We already derived the bound we use for small values of  $\gamma$  in Theorem 7, corresponding to the green graph in the figure. To derive the two remaining bounds, we use an approach inspired by [5]. The extra difficulty we have is bounding the second-price component. The first-price has a direct relation with winning the auction and so we can use the CCE conditions to bound it while the second-price component is more difficult to get a grip on. These bounds significantly improve on the bounds of Theorem 4 and Theorem 5.

First we tackle the interval  $\gamma \in [\frac{1}{2}, 1]$ . Note that for  $\gamma = 1$  this bound coincides with the (tight) bound in [5].

**THEOREM 8.** *Consider a 2-player single-item  $\gamma$ -HYA and suppose that bidders cannot overbid. For  $\gamma \in [\frac{1}{2}, 1]$ , the coarse correlated price of anarchy of  $\gamma$ -HYA is upper bounded by the blue graph in Figure 1(d) (with  $\text{CCE-POA} \leq 1.295\dots$  for  $\gamma = 0.5$  and  $\text{CCE-POA} \leq 1.229\dots$  for  $\gamma = 1$ ).*

**PROOF SKETCH.** Without loss of generality we assume that player 1 has a valuation of 1 and player 2 has a valuation of  $v \leq 1$ . Fix  $\gamma$  and consider some coarse correlated equilibrium  $\sigma$ . Let  $\alpha = \mathbb{E}[u_1(\sigma)]$  be the utility of player 1 and  $\beta = \mathbb{E}[u_2(\sigma)]$  be the utility of player 2 in  $\sigma$ . The maximum social welfare is clearly 1, namely when player 1 wins all the time. Lower bounding the expected welfare of an

arbitrary  $\sigma$  translates into an upper bound on the price of anarchy. We have

$$\mathbb{E}[\text{SW}(\sigma)] \geq \alpha + \beta + \mathbb{E}[p^Y(\sigma)] = \alpha + \beta + \gamma \mathbb{E}[\text{HB}(\sigma)] + (1 - \gamma) \mathbb{E}[\text{SB}(\sigma)].$$

We try to find the  $v$ ,  $\alpha$  and  $\beta$  that minimize (a lower bound on) this expression and this will then give a lower bound on the expected social welfare. Let  $F_X$  be the cumulative distribution function of the random variable  $X$  where  $X \in \{\text{HB}, \text{HB}_{-1}, \text{HB}_{-2}, \text{SB}\}$ . Then by the CCE conditions, and the fact that a CDF is always bounded by 1, we know that

$$F_{\text{HB}_{-1}(\sigma)}(x) \leq \min\left\{\frac{\alpha}{1-x}, 1\right\}, \quad F_{\text{HB}_{-2}(\sigma)}(x) \leq \min\left\{\frac{\beta}{v-x}, 1\right\} \quad (4)$$

$$F_{\text{HB}(\sigma)}(x) \leq \min\left\{\frac{\alpha}{1-x}, \frac{\beta}{v-x}, 1\right\}. \quad (5)$$

For example, if  $F_{\text{HB}_{-1}(\sigma)}(x) > \frac{\alpha}{1-x}$  and player 1 changes their bid to  $x$  their utility will be strictly greater than  $\frac{\alpha}{1-x} \cdot (1-x) = \alpha$  which is more than their current utility, contradicting the CCE conditions.

Also note that  $\alpha \geq 1 - v$  because player 1 bidding  $v + \epsilon$  will yield a utility of at least  $1 - v - \epsilon$  for any positive  $\epsilon$ . The other player is not allowed to bid above  $v$ , thus player 1 always wins when bidding  $v + \epsilon$ .

For  $n = 2$  players it holds that

$$\begin{aligned} F_{\text{SB}(\sigma)}(x) &= \mathbb{P}[\text{SB}(\sigma) \leq x] \\ &= F_{\text{HB}_{-1}(\sigma)}(x) + F_{\text{HB}_{-2}(\sigma)}(x) - F_{\text{HB}(\sigma)}(x). \end{aligned} \quad (6)$$

Let us get a more explicit expression for the expected payment using (6)

$$\begin{aligned} \mathbb{E}[p^Y(\sigma)] &= \gamma \mathbb{E}[\text{HB}(\sigma)] + (1 - \gamma) \mathbb{E}[\text{SB}(\sigma)] \\ &= (2\gamma - 1) \int_0^1 1 - F_{\text{HB}(\sigma)}(x) dx \\ &\quad + (1 - \gamma) \sum_{i=1}^2 \int_0^1 1 - F_{\text{HB}_{-i}(\sigma)}(x) dx. \end{aligned}$$

Using the two bounds in (4), we can lower bound the two integrals in the summation

$$\begin{aligned} \int_0^1 1 - F_{\text{HB}_{-1}(\sigma)}(x) dx &\geq 1 - \alpha + \alpha \ln(\alpha) \\ \int_0^1 1 - F_{\text{HB}_{-2}(\sigma)}(x) dx &\geq v - \beta + \beta \ln(\beta/v). \end{aligned}$$

If  $\gamma \geq \frac{1}{2}$  then  $2\gamma - 1 \geq 0$  and so we can use (5) to lower bound the integral on the left by

$$\int_0^1 1 - F_{\text{HB}(\sigma)}(x) dx \geq \int_0^1 1 - \min\left\{\frac{\alpha}{1-x}, \frac{\beta}{v-x}, 1\right\} dx.$$

We do a case distinction on  $\beta \geq v\alpha$  and  $\beta < v\alpha$ . In both cases we can analytically reason that the social welfare is bounded below by

$$\begin{aligned} \mathbb{E}[\text{SW}(\sigma)] &\geq 1 + \gamma\beta + (2\gamma - 1)(\alpha - \beta) \ln(\alpha - \beta) + (2 - 3\gamma)\alpha \ln(\alpha) \\ &\quad + (2\gamma - 1)\beta \ln(\alpha) + \gamma\beta \ln(\beta) - \gamma\beta \ln(1 - \alpha), \end{aligned} \quad (7)$$

where  $\beta \leq \alpha(1 - \alpha)$ .

The derivative of (7) with respect to  $\beta$  becomes 0 when

$$\frac{\beta^\gamma}{(\alpha - \beta)^{2\gamma - 1}} - \frac{(1 - \alpha)^\gamma}{e\alpha^{2\gamma - 1}} = 0.$$

For fixed  $\alpha$  the expression on the left hand side is negative for  $\beta$  close to 0, and positive for  $\beta$  close to  $\alpha$ . Also the second derivative with respect to  $\beta$  is always positive on  $[0, \beta]$ . Thus we can use binary search to quickly find  $\beta$  satisfying the equality; call this  $\beta_\alpha$ .<sup>8</sup>

<sup>8</sup> $\beta_\alpha$  may violate  $\beta \leq \alpha(1 - \alpha)$ , but removing this restriction can only decrease the minimum value of the expected social welfare.

Then we have

$$\begin{aligned} \mathbb{E}[\text{SW}(\sigma)] &\geq 1 + \gamma\beta_\alpha + (2\gamma - 1)(\alpha - \beta_\alpha) \ln(\alpha - \beta_\alpha) \\ &\quad + (2 - 3\gamma)\alpha \ln(\alpha) + (2\gamma - 1)\beta_\alpha \ln(\alpha) \\ &\quad + \gamma\beta_\alpha \ln(\beta_\alpha) - \gamma\beta_\alpha \ln(1 - \alpha). \end{aligned}$$

For  $\gamma = \frac{1}{2}$  we compute  $\beta_\alpha = \frac{1 - \alpha}{e^\alpha}$  and then the social welfare is minimized for  $\alpha = e^{-1 - 1/e^2} \approx 0.3213\dots$  with value 0.7716\dots While for  $\gamma = 1$  we have  $\beta_\alpha = \frac{\alpha(1 - \alpha)}{1 - \alpha + e^\alpha}$  where the social welfare is minimized for  $\alpha \approx 0.2743\dots$  with value 0.8135\dots In both these cases (7) becomes a unimodal function. Plotting (7) for various values of  $\alpha$ , when doing binary search to find  $\beta_\alpha$  as a subroutine, suggests that this is the case for all  $\gamma$ . Making this assumption we can use a ternary search on  $\alpha$  with a binary search to find  $\beta_\alpha$  as a subroutine to quickly find the minimum. Finally, taking 1 over this value gives us an upper bound on the price of anarchy, presented as the blue graph in Figure 1(d).  $\square$

The previous theorem holds for  $\gamma \in [\frac{1}{2}, 1]$ . With a similar proof template, making use of an upper bound on the highest bid, we can derive an upper bound on the coarse correlated price of anarchy for the lower to mid range of  $\gamma$

**THEOREM 9.** Consider a 2-player single-item  $\gamma$ -HYA and suppose that bidders cannot overbid. For  $\gamma \in (0.217\dots, \frac{1}{2}]$ , the coarse correlated price of anarchy of  $\gamma$ -HYA is upper bounded by the orange graph in Figure 1(d) (with CCE-POA  $\leq 1.515\dots$  for intersection point  $\gamma = 0.339\dots$  and CCE-POA  $\leq 1.295\dots$  for  $\gamma = 0.5$ ).

## 6 CONCLUSION AND FUTURE WORK

Our bound on the CCE-POA of  $\gamma$ -FPA is tight over the entire range of  $\gamma \in [0, 1]$  if players can overbid, both in the single-item and multi-unit auction setting. Despite the fact that our bounds on the CCE-POA are rather low already if players cannot overbid, further improvements might still be possible. We consider this a challenging open problem for future work.

On a more conceptual level, in this paper we considered a basic bid rigging model where the auctioneer colludes with the winning bidders only. It will be very interesting to study the price of anarchy of more complex bid rigging models; for example, the model introduced in [10] (ideally generalized to the multi-unit auction setting) might be a natural next step.

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## REFERENCES

- [1] Georgios Birmpas, Evangelos Markakis, Orestis Telelis, and Artem Tsirikidis. 2019. Tight Welfare Guarantees for Pure Nash Equilibria of the Uniform Price Auction. *Theory of Computing Systems* 63, 7 (2019), 1451–1469. <https://doi.org/10.1007/s00224-018-9889-7>
- [2] George Christodoulou, Annamária Kovács, Alkmini Sgouritsa, and Bo Tang. 2016. Tight Bounds for the Price of Anarchy of Simultaneous First-Price Auctions. *ACM Transactions on Economics and Computation* 4, 2 (2016). <https://doi.org/10.1145/2847520>
- [3] Bart De Keijzer, Evangelos Markakis, Guido Schäfer, and Orestis Telelis. 2013. Inefficiency of Standard Multi-unit Auctions. In *Proceedings of the European Symposium on Algorithms (ESA)*. Springer, 385–396.

- [4] Michal Feldman, Hu Fu, Nick Gravin, and Brendan Lucier. 2013. Simultaneous auctions are (almost) efficient. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*. 201–210.
- [5] Michal Feldman, Brendan Lucier, and Noam Nisan. 2016. Correlated and coarse equilibria of single-item auctions. In *Proceedings of the International Conference on Web and Internet Economics (WINE)*. Springer, Montreal, Canada, 131–144.
- [6] Daniel A. Graham and Robert C. Marshall. 1987. Collusive Bidder Behavior at Single-Object Second-Price and English Auctions. *Journal of Political Economy* 95, 6 (1987), 1217–1239.
- [7] John C. Harsanyi. 1967. Games with Incomplete Information Played by “Bayesian” Players, Parts I, II and III. *Management Science* 14, 3 (1967), 159–182, 320–334, and 486–502.
- [8] Allan Ingraham. 2005. A Test for Collusion between a Bidder and an Auctioneer in Sealed-Bid Auctions. *Contributions in Economic Analysis & Policy* 4 (2005). <https://doi.org/10.2202/1538-0645.1448>
- [9] Elias Koutsoupias and Christos Papadimitriou. 1999. Worst-Case Equilibria. In *Proceedings of the Symposium on Theoretical Aspects of Computer Science (STACS)*. Springer, 404–413.
- [10] Yvan Lengwiler and Elmar Wolfstetter. 2010. Auctions and corruption: An analysis of bid rigging by a corrupt auctioneer. *Journal of Economic Dynamics and Control* 34, 10 (2010), 1872–1892. <https://doi.org/10.1016/j.jedc.2010.03.002>
- [11] Yvan Lengwiler and Elmar G. Wolfstetter. 2000. *Auctions and Corruption*. CESifo Working Paper Series 401. CESifo. [https://ideas.repec.org/p/ces/ceswps/\\_401.html](https://ideas.repec.org/p/ces/ceswps/_401.html)
- [12] Evangelos Markakis and Orestis Telelis. 2015. Uniform Price Auctions: Equilibria and Efficiency. *Theory of Computing Systems* 57, 3 (2015), 549–575. <https://doi.org/10.1007/s00224-014-9537-9>
- [13] Randolph McAfee and John McMillan. 1992. Bidding Rings. *American Economic Review* 82, 3 (1992), 579–99.
- [14] Flavio M. Menezes and Paulo Klinger Monteiro. 2006. Corruption and auctions. *Journal of Mathematical Economics* 42, 1 (2006), 97–108. <https://doi.org/10.1016/j.jmateco.2005.04.002>
- [15] Larry Olmstead. April 21, 1993. 2 Managers Held in Bidding Scheme at School Agency. *The New York Times* (April 21, 1993). <https://www.nytimes.com/1993/04/21/nyregion/2-managers-held-in-bidding-scheme-at-school-agency.html> accessed May 11, 2021.
- [16] Tim Roughgarden. 2015. Intrinsic Robustness of the Price of Anarchy. *J. ACM* 62, 5, Article 32 (2015), 42 pages. <https://doi.org/10.1145/2806883>
- [17] Tim Roughgarden, Vasilis Syrgkanis, and Éva Tardos. 2017. The Price of Anarchy in Auctions. *Journal on Artificial Intelligence Research* 59, 1 (2017), 59–101.
- [18] Vasileios Syrgkanis. 2014. *Efficiency of mechanisms in complex markets*. Ph.D. Dissertation. Cornell University.
- [19] Vasilis Syrgkanis and Eva Tardos. 2013. Composable and efficient mechanisms. In *Proceedings of the ACM Symposium on Theory of Computing (STOC)*. 211–220.
- [20] The Wall Street Journal. August 19, 1999. State Governments Consider Reopening Bidding for Berlin Airport. *The Wall Street Journal* (August 19, 1999). <https://www.wsj.com/articles/SB935028690392278172> accessed May 11, 2021.