On Fair and Efficient Solutions for Budget Apportionment

Extended Abstract

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ABSTRACT

This works deals with an apportionment problem recently introduced in [9]. In this problem involving multiple agents, it is desirable to propose fair and efficient solutions. Several alternative notions of fairness exist but combining efficiency with fairness is often impossible, and a trade-off has to be made. We first study the computation of almost fair and approximately efficient solutions, and we determine when these two goals can be met. Afterwards, we characterize the price of fairness which bounds the loss of efficiency caused by imposing fairness or one of its relaxations.

KEYWORDS

Computational Social Choice; Fairness; Efficiency; Apportionment

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1 INTRODUCTION

The present work deals with BUDGET APPORTIONMENT, a problem which falls into the social choice area that has received considerable attention from the AI community [6, 19, 22]. Suppose that n agents share a common budget B. Every agent submits some demands whose acceptance consumes indivisible portions of the budget. For example, two people named Alice and Bob share a budget of 100, and their demands are $\{15, 41\}$ and $\{37, 85\}$, respectively. A solution is to accept a selection of demands whose total sum does not exceed the budget. Every demand is either accepted or rejected, but fractional decisions are not allowed. We assume that the utility of an agent is equal to the sum of her accorded demands.

Recently introduced in [9], BUDGET APPORTIONMENT models several real-life situations. For example, some people may want to store digital files on a shared memory space (see for example [12] for a similar problem). The budget is the capacity of the memory, and each demand corresponds to a file. The model also applies to big organizations who have to decide which projects of their members should be funded [15].

When allocating resources to a group of agents, two goals are typically pursued: try being fair among the agents and avoid resource wasting [4, 16]. Fairness and efficiency are known to be conflicting objectives [5]. This article explores the possible trade-offs for the BUDGET APPORTIONMENT problem.

Three fundamental notions of fairness are considered: max-min [2, 3], proportionality [23], and envy-freeness [11, 24]. These concepts are borrowed from the fair allocation of indivisible goods and adapted to BUDGET APPORTIONMENT. In particular, since envy-freeness does not suit well with the budget problem (an agent can only benefit from her own demands), we resort to a close notion named *jealousy-freeness* [13]. Existing relaxations of proportionality [1, 19] and envy-freeness [8, 17] ask to satisfy the criterion up to one/any good. Similar relaxations up to one/any demand will be used for the fairness criteria of BUDGET APPORTIONMENT. Regarding efficiency, our primary criterion corresponds to the maximization of the utilitarian social welfare. Our second efficiency criterion is Pareto optimality.

2 BUDGET APPORTIONMENT

In BUDGET APPORTIONMENT, a set N of n agents shares a common budget $B \in \mathbb{N}$. Each agent $j \in N$ has a demand set $D^j = \{d^j_1, \dots, d^j_{m^j}\}$ where $d^j_i \in \mathbb{N}$ is upper bounded by B. We suppose that $\sum_i d^j_i > B/n$ holds for all $j \in N$, otherwise we could accept all the demands of j, and remove j from the instance along with her demands. A feasible solution $S = (S^1, \dots, S^n)$ is a tuple verifying $S^j \subseteq D^j$, $\forall j \in N$, and budget constraint $\sum_{j\in N} \sum_{d\in S^j} d \leq B$.

For a solution $S = (S^1, \dots, S^n)$, the value of S^j is $s^j := \sum_{d\in S^j} d$, and the value of S is defined by $\sum_{j\in N} s^j$. The utility of agent j for S is s^j, corresponding to the proportion of the common budget that is dedicated to her accepted demands.

3 COMBINING FAIRNESS AND EFFICIENCY

In this section we define fairness and efficiency criteria, and study how they can be combined.

Fairness criteria and relaxations. Solution S is:

- ***max-min*** if no solution T satisfies $\min_{j\in N} t^j > \min_{j\in N} s^j$;
- Proportional (***PROP***), if each agent's utility is at least B/n;
- Proportional up to one demand (***PROP1***) (resp., proportional up to any demand (***PROPX***)) if for all $j \in N$, either $D^j = S^j$ or $s^j + d \geq B/n$ holds for some $d \in D^j \setminus S^j$ (resp., for any $d \in D^j \setminus S^j$);
- Jealousy-free (**JF**) if $s^j \geq s^i$ holds for any pair $(i, j) \in N$;
- Jealousy-free up to one demand (***JF1***) (resp., ***JFX***) if for any pair of agents $(i, j) \in N$ such that $s^j < s^i$, $s^j \geq s^i - d$ holds for at least one $d \in S^i$ (resp., for all $d \in S^i$);

Efficiency criteria. Solution S is:

- *Utilitarian optimal* (***UO***) if no solution T satisfies $\sum_{j\in N} t^j > \sum_{j\in N} s^j$;

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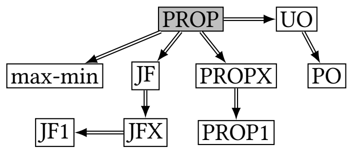

Figure 1: Implications between the criteria.

• *Pareto optimal* (***PO***) if we cannot strictly increase the utility of an agent, without decreasing the utility of another one.

The above criteria are logically connected and Figure 1 summarizes our results on the implication between the different notions. An implication $A \Rightarrow B$ between criteria means that if a solution satisfies A for a given instance, then the same solution also satisfies B for the same instance. PROP is central but it is shaded on the picture because, as opposed to the other criteria, the existence of a PROP solution is not guaranteed (see the example with Alice and Bob in the introduction).

Every instance of BUDGET APPORTIONMENT admits a PROPX solution and we proposed a polynomial algorithm to build such a solution. A JF (and thus JF1 and JFX) solution always exists: $S^j = \emptyset$ for all j. The construction of a max-min (resp., PO) solution is NP-hard but we have shown that one can always compute an ε-approximation[1] of a max-min (resp., PO) solution, for all $\varepsilon > 0$.

Now, given a notion of fairness and an efficiency criterion, we are interested in knowing whether every instance of BUDGET APPORTIONMENT admits a feasible solution that is fair and efficient at the same time. The situation is summarized in Table 1.

	max-min	PROP	PROPX	JF1
UO	✗	✗	✗	✗
PO	✓	✗	✓	✗

Table 1: Guaranteed existence (✓) or not (✗) of a solution combining a notion of efficiency and a concept of fairness.

The good news is that whatever the instance, PO can always be combined with max-min or PROPX. From a computational point of view, this cannot be done in polynomial time (unless P=NP) but an approximation can be computed in polynomial time. Indeed, a solution that is an ε-approximation of a PO solution and PROPX at the same time can be computed in time polynomial in n and $\frac{1}{\varepsilon}$. The bad news is that UO cannot be combined with the given fairness criteria. However, we shall see in the next section how close to UO a fair solution can be.

4 PRICE OF FAIRNESS

One can be interested in the existence of fair solutions that offer an approximation of the maximum utilitarian social welfare. This question is known as the *price of fairness* (PoF in short) [7]. Given an instance I and a fairness concept, the PoF of I is the ratio between the maximum utilitarian social welfare achieved by a feasible solution and the largest utilitarian social welfare of a fair solution. The PoF of a problem is the largest value taken by the previous ratio over all instances. The PoF is a measure of adequacy between a fairness concept and efficiency; the lower the better.

[1] T is an ε-approximation of S if $s^j \leq (1+\varepsilon)t^j$ holds for all $j \in N$.

max-min	PROP	PROPX	PROP1	JF	JFX or JF1
∞	1	∞	n	∞	2
$\frac{n+1}{n}$	1	$\frac{n+1}{n}$	$\frac{n+1}{n}$	∞	$\frac{n+1}{n}$

Table 2: The PoF for several fairness concepts (first/second line: unrestricted/moderate demands). "∞" means unbounded. Grey highlighting boxes indicate that a solution with the corresponding PoF can be built in polynomial time.

The results of this section, summarized in Table 2, deal with BUDGET APPORTIONMENT with $n \geq 2$ agents. The first line of Table 2 gives the PoF when no specific assumption is made on the demands (every demand can be as big as the entire budget B). The second line covers the case where every demand is at most B/n. The motivation for this second scenario comes from the fact that the PoF is often due to extreme instances where the agents have demands of the order of the whole budget. Therefore, it is relevant to study a more likely setting where the demands cannot be too large (here B/n is the fair share of every agent).

Note that when every demand is at most B/n, the PoF of max-min, PROPX, PROP1, JFX and JF1 is $\frac{n+1}{n}$, which is significantly smaller than in the general case. Therefore, even though the existence of an efficient and fair solution is not always guaranteed, there always exists a fair solution that is close to the optimal utilitarian social welfare when every single demand is moderate.

5 CONCLUSION

The central notion of proportionality is a kind of holy grail which, unfortunately, cannot be reached for every instance. However, there are two cases where fairness and efficiency can always be combined: max-min with PO, and PROPX with PO. Even though computing such solutions can be computationally hard, we can still compute ε-approximations of these solutions in polynomial time. Concerning jealousy-freeness, even if we take its weakest relaxation JF1, one cannot guarantee the existence of an efficient and fair solution for every instance. Though JF1 and JFX cannot always be combined with efficiency, we have shown that their PoF is low (namely, 2). In contrast, relaxations of proportionality can be combined with Pareto optimality but their PoF can be high (namely, n or unbounded).

An alternative way to balance fairness and efficiency is to maximize the product of the agents' utilities [8, 10]. This objective, also known as the *Nash product* [14, 20], is reputed to offer a good compromise between fairness and efficiency [16, 18, 21]. Therefore, it would be interesting to study the Nash product of BUDGET APPORTIONMENT in the future. Since this problem is clearly NP-hard, providing an approximation algorithm with a good performance guarantee would be a valuable complement to the present work.

Finally, a natural generalization of BUDGET APPORTIONMENT is when every demand has a size and a profit, and these quantities are not necessarily equal. It would be interesting to extend the results of the present work to this setting.

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