

On Achieving Leximin Fairness and Stability in Many-to-One Matchings

Extended Abstract

Shivika Narang
Indian Institute of Science
Bengaluru, Karnataka, India
shivika@iisc.ac.in

Arpita Biswas
Harvard University
Boston, MA, USA
arpitabiswas@seas.harvard.edu

Yadati Narahari
Indian Institute of Science
Bengaluru, Karnataka, India
narahari@iisc.ac.in

ABSTRACT

The past few years have seen a surge of work on fairness in allocation problems where items must be fairly divided among agents having individual preferences. In comparison, fairness in matching settings with preferences on both sides, that is, where agents have to be matched to other agents, has received much less attention. Moreover, the two-sided matching literature has largely focused on ordinal preferences. We study leximin optimality over stable many-to-one matchings under cardinal preferences. We first investigate matching problems with *ranked valuations* for which we give efficient algorithms to find the leximin optimal matching over the space of stable matchings. We complement these results by showing that relaxing the ranked valuations condition in any way, makes finding the leximin optimal stable matching intractable.

KEYWORDS

Many-to-One Matchings; Fairness, Stability; Leximin Optimality

ACM Reference Format:

Shivika Narang, Arpita Biswas, and Yadati Narahari. 2022. On Achieving Leximin Fairness and Stability in Many-to-One Matchings: Extended Abstract. In *Proc. of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), Online, May 9–13, 2022*, IFAAMAS, 3 pages.

1 INTRODUCTION

In the past decade, the computational problem of achieving fairness has been receiving intense attention [1, 2, 8, 9, 11, 12, 22]. Several fairness notions have been studied for fair allocation problems. These have been mostly unexplored for two-sided (bipartite) matching problems [3, 10, 15, 17, 18, 21, 23–25], with the exception of some very recent work [6, 13, 16]. Also, the two-sided matching literature has largely assumed ordinal preferences. This paper initiates the study of achieving fairness in stable, two-sided matchings under cardinal valuations. We are motivated by several real-world situations to explore leximin optimality as the notion of fairness and focus on stable many-to-one matchings. We derive several novel algorithmic and complexity results for various special cases of many-to-one matchings. We use the setting of college admissions to capture a general many-to-one matching setting.

In this paper, we seek to find a leximin optimal stable many-to-one matching with cardinal valuations. Previous work has studied leximin for fair allocation [4, 5, 11, 14, 22] and for the special case

of dichotomous valuations [7, 19]. Table 1 summarises our contributions. Here, m and n are the number of colleges and students. Strict preferences mean that no agent is indifferent about any agent on the other side (each agent’s valuations imply a strict linear order over agents on the other side). Weak preferences allow agents to be indifferent between two agents (their valuations imply a partial order on the agents on the other side). Ranked valuations imply that each agent has the same preference ordering over the other side. Isometric valuations imply that the value a student has for a college is the same as the value that the college has for that student. The complete proofs of all results can be found in the full version of the paper [20].

| Nature of Valuations | Strict Preferences | Weak Preferences |
|----------------------|----------------------|------------------|
| General | NP-Hard (Thm 4) | APX-Hard (Thm 5) |
| Ranked Valuations | $O(m^2 n^2)$ (Thm 2) | APX-Hard (Thm 5) |
| Ranked + Isometric | $O(mn)$ (Thm 1) | NP-Hard (Thm 3) |

Table 1: Summary of Results

2 PRELIMINARIES AND NOTATIONS

Let $\mathcal{S} = \{s_1, \dots, s_n\}$ and $\mathcal{C} = \{c_1, \dots, c_m\}$ be the non-empty, finite, and ordered sets of students and colleges, respectively. We assume that there are at least as many students as colleges, that is, $n \geq m$. We shall assume that each college can be matched to as many students at a time as needed. Let $u_i(\cdot)$ and $v_j(\cdot)$ be the valuation functions of student s_i , and college c_j , $i \in [n]$, $j \in [m]$. This paper studies only non-negative and additive (for colleges) valuation functions. We define $U = (u_1, \dots, u_n)$ and $V = (v_1, \dots, v_m)$. Hence, an instance of stable many-to-one matchings (SMO) is captured by the tuple $I = \langle \mathcal{S}, \mathcal{C}, U, V \rangle$. Rankings or ranked valuations imply that $u_i(c_1) > u_i(c_2) > \dots > u_i(c_m)$ for all $i \in [n]$ and $v_j(s_1) > v_j(s_2) > \dots > v_j(s_n)$ for all $j \in [m]$. Weak rankings shall mean that the $>$ relations are replaced by \geq .

We also look at isometric valuations; the details are deferred to the full version of our paper [20]. Our goal is to find a many-to-one matching μ of the bipartite graph $G = (\mathcal{S}, \mathcal{C}, \mathcal{S} \times \mathcal{C})$ such that μ satisfies stability as well as fairness properties. A (many-to-one) matching $\mu \subseteq \mathcal{S} \times \mathcal{C}$ is such that each student has at most one incident edge present in the matching. We shall use $\mu(a)$ to denote the set of agents matched to $a \in \mathcal{S} \cup \mathcal{C}$ under μ . The valuation of a student s_i under μ is $u_i(\mu) = u_i(\mu(s_i))$. For a college c_j the valuation under μ is $v_j(\mu) = \sum_{s_i \in \mu(c_j)} v_j(s_i) \geq 0$.

Definition 1 (Blocking Pair). Given a matching μ , (s_i, c_j) are called a *blocking pair* if $s_i \notin \mu(c_j)$ and there exists $s_{i'} \in \mu(c_j)$, $c_{j'} = \mu(s_i)$ such that $v_j(s_i) > v_j(s_{i'})$ and $u_i(c_j) > u_i(c_{j'})$.

Proc. of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), P. Faliszewski, V. Mascardi, C. Pelachaud, M.E. Taylor (eds.), May 9–13, 2022, Online. © 2022 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org). All rights reserved.

Definition 2 (Stable Matching). A matching μ of instance $I = \langle S, C, V \rangle$ is said to be stable if no (s_i, c_j) is a *blocking pair* for μ .

Our work aims to find the leximin optimal matching over the space of stable matchings. The leximin tuple of any matching is simply the tuple containing the valuations of all the agents (students and colleges) under this matching, listed in non-decreasing order. Hence, the position of an agent’s valuation in the leximin tuple may change under different matchings. The leximin tuple of a matching μ will be denoted by \mathcal{L}_μ . The t^{th} index of \mathcal{L}_μ is denoted by $\mathcal{L}_\mu[t]$.

Definition 3 (Leximin Domination). We say that matching μ_1 leximin dominates μ_2 if there exists a valid index k such that $\mathcal{L}_{\mu_1}[k'] = \mathcal{L}_{\mu_2}[k']$ for all $k' < k$ and $\mathcal{L}_{\mu_1}[k] > \mathcal{L}_{\mu_2}[k]$.

Definition 4 (Leximin Optimal Matching). A leximin optimal matching μ^* is one that is not leximin dominated by any other matching.

3 MAIN RESULTS

Finding a leximin optimal matching is intractable in general, so we first look at ranked valuations, where the space of stable matchings has an appealing structure that we exploit.

Lemma 1. Given an instance of ranked valuations, a matching μ is stable, if and only if, for all $j \in [m]$, $\mu(c_j) = \{s_{w_j+1}, \dots, s_{w_j+k_j}\}$ where $k_j = |\mu(c_j)|$ and $w_j = \sum_{t=1}^{j-1} k_t$.

Lemma 1 ensures that a stable solution for a ranked instance would necessarily match a contiguous set of students to each c_j . We exploit this to give the following algorithmic result.

Theorem 1. A leximin optimal stable matching for ranked isometric valuations can be found in time $O(mn)$.

The proof is constructive and building on it, we give an algorithm for general ranked valuations FaSt-Gen (Algorithm 1) which runs in time $O(m^2n^2)$. We prove its correctness in the following theorem.

Theorem 2. FaSt-Gen (Algorithm 1) finds a leximin optimal stable matching given an instance of general ranked valuations.

FaSt-Gen starts with the student optimal stable matching. In each iteration, we increase the number of students matched to the leftmost unfixed college by 1 if it increases the leximin value. If not, the algorithm fixes the upper limit of this college and the lower limit of the next highest ranked one. In order to do this, we decrease one student from a higher ranked college using the demote procedure (details in [20]). The choice of this higher ranked college is c_1 initially, till it can no longer give out any more students, then we consider the next college whose lower limit is not fixed. Thus, the algorithm gradually fixes the upper and lower limits of the students matched to each college. There is one more subtlety to note. When increasing the number of students of a particular college a leximin decrease may happen due to a higher ranked student-college pair. That is, a student which is being moved, but not to the lowest unfixed college and may cause a leximin decrease. In such cases, we “soft fix” the upper limits of some colleges and then unfix them, once the college which caused the leximin decrease is, starts giving out students currently matched to it. The details of all routines called by FaSt-Gen are in [20]. Unfortunately, efficient algorithms for more general settings are not possible if $P \neq NP$.

Algorithm 1: FaSt-Gen

Input: Instance of general ranked valuations $\langle S, C, U, V \rangle$
Output: μ

```

1  $\mu(c_1) \leftarrow \{s_1, \dots, s_{n-m+1}\}$  and  $\mu(c_j) \leftarrow \{s_{n-(m-j)}\}$  for  $j \geq 2$ ;
2  $UpperFix \leftarrow \{c_1\}$ ,  $LowerFix \leftarrow \{c_m\}$ ;
3  $SoftFix \leftarrow \emptyset$ ,  $Unfixed \leftarrow UpperFix^c$ ;
4 while  $|UpperFix \setminus LowerFix| + |LowerFix| < m$  do
5    $up \leftarrow \min_{j \notin LowerFix} j$ ;
6    $down \leftarrow \arg \min_{j \in Unfixed} v_j(\mu)$ ;
7    $SoftFix \leftarrow SoftFix \setminus \{(j, j') \mid j' \leq up < j\}$ ;
8   if  $|\mu(c_{up})| = 1$  OR  $v_{up}(\mu) \leq v_{down}(\mu)$  then
9      $LowerFix \leftarrow LowerFix \cup \{c_{up}\}$ ;
10  else
11     $\mu' \leftarrow Demote(\mu, down, up)$ ;
12    if  $\mathcal{L}_{\mu'} \geq \mathcal{L}_\mu$  then
13       $\mu \leftarrow \mu'$ ;
14    else
15      if  $sourceDec(\mu', \mu) = c_{up}$  then
16         $LowerFix \leftarrow LowerFix \cup \{c_{up}\}$ ;
17         $UpperFix \leftarrow UpperFix \cup \{c_{up+1}\}$ ;
18      else
19        if  $sourceDec(\mu', \mu) \in S$  then
20           $c_t \leftarrow \mu(sourceDec(\mu', \mu))$ ;
21           $LowerFix \leftarrow LowerFix \cup \{c_t\}$ ;
22           $UpperFix \leftarrow UpperFix \cup \{c_{t+1}\}$ ;
23           $A \leftarrow \{j \mid j > t+1, j \in Unfixed\}$ ;
24           $SoftFix \leftarrow SoftFix \cup (A \times \{t+1\})$ ;
25        else
26           $(\mu, LowerFix, UpperFix, SoftFix) \leftarrow$ 
            $LookAheadRoutine(\mu, down, LowerFix,$ 
            $UpperFix, SoftFix)$ ;
27   $Unfixed \leftarrow \{j \mid j \notin UpperFix \text{ or } (j, j') \notin SoftFix \text{ for a } j' > j\}$ ;

```

Theorem 3. It is NP-Hard to find the leximin optimal stable matching under isometric valuations with weak rankings with $m = 2$ and strongly NP-Hard with $n = 3m$.

Theorem 4. It is NP-Hard to find a leximin optimal stable matching under strict preferences, even in the absence of rankings.

Theorem 5. Unless $P=NP$, for any $\delta > 0$, $c \in \mathbb{Z}^+$ there is no $1/cn^\delta$ -approximation algorithm to find a leximin optimal stable matching under unconstrained additive valuations.

4 DIRECTIONS FOR FUTURE WORK

This paper provides a relatively comprehensive report of when it is possible to find a leximin optimal stable matching without any restriction on the number of students or colleges. One open problem is whether for a constant number of colleges with strict preferences, an exact algorithm is possible. Another potential direction is to find, an a more general subclass of matching instances, to find a fair or approximately-fair and stable matching. While our results rule out approximations for additive valuations in general, under isometric valuations or general strict preferences, approximations may exist.

ACKNOWLEDGMENTS

S.N. is supported by a research fellowship from the Tata Consultancy Services. A.B. is supported by the Harvard Center for Research on Computation and Society (CRCS).

REFERENCES

- [1] Siddharth Barman, Ganesh Ghalmé, Shweta Jain, Pooja Kulkarni, and Shivika Narang. 2019. Fair Division of Indivisible Goods Among Strategic Agents. *Autonomous Agents and Multi-Agent Systems* (2019), 1811–1813.
- [2] Siddharth Barman and Sanath Kumar Krishnamurthy. 2020. Approximation algorithms for maximin fair division. *ACM Transactions on Economics and Computation (TEAC)* 8, 1 (2020), 1–28.
- [3] Surender Baswana, Partha Pratim Chakrabarti, Sharat Chandran, Yashodhan Kanoria, and Utkarsh Patange. 2019. Centralized Admissions for Engineering Colleges in India. *Economics and Computation* (2019), 323–324.
- [4] Nawal Benabbou, Mithun Chakraborty, Ayumi Igarashi, and Yair Zick. 2020. Finding Fair and Efficient Allocations When Valuations Don't Add Up. *Symposium on Algorithmic Game Theory* (2020), 32–46.
- [5] Ivona Bezáková and Varsha Dani. 2005. Allocating indivisible goods. *ACM SIGecom Exchanges* 5, 3 (2005), 11–18.
- [6] Arpita Biswas, Gourab K. Patro, Niloy Ganguly, Krishna P. Gummadi, and Abhijnan Chakraborty. 2021. Toward Fair Recommendation in Two-Sided Platforms. *ACM Trans. Web* 16, 2, Article 8 (dec 2021), 34 pages. <https://doi.org/10.1145/3503624>
- [7] Anna Bogomolnaia and Hervé Moulin. 2004. Random matching under dichotomous preferences. *Econometrica* 72, 1 (2004), 257–279.
- [8] Sylvain Bouveret and Michel Lemaître. 2016. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Autonomous Agents and Multi-Agent Systems* 30, 2 (2016), 259–290.
- [9] Eric Budish. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy* 119, 6 (2011), 1061–1103.
- [10] Ioannis Caragiannis, Aris Filos-Ratsikas, Panagiotis Kanellopoulos, and Rohit Vaish. 2019. Stable fractional matchings. *Economics and Computation* (2019), 21–39.
- [11] Xingyu Chen and Zijie Liu. 2020. The Fairness of Leximin in Allocation of Indivisible Chores. *arXiv preprint arXiv:2005.04864* (2020).
- [12] Bailey Flanigan, Paul Gözl, Anupam Gupta, Brett Hennig, and Ariel D Procaccia. 2021. Fair algorithms for selecting citizens' assemblies. *Nature* 596, 7873 (2021), 548–552.
- [13] Rupert Freeman, Evi Micha, and Nisarg Shah. 2021. Two-Sided Matching Meets Fair Division. *IJCAI* (2021).
- [14] Rupert Freeman, Sujoy Sikdar, Rohit Vaish, and Lirong Xia. 2019. Equitable allocations of indivisible goods. *arXiv preprint arXiv:1905.10656* (2019).
- [15] David Gale and Lloyd S Shapley. 1962. College admissions and the stability of marriage. *The American Mathematical Monthly* 69, 1 (1962), 9–15.
- [16] Sreenivas Gollapudi, Kostas Kollias, and Benjamin Plaut. 2020. Almost Envy-free Repeated Matching in Two-sided Markets. *International Conference on Web and Internet Economics* (2020), 3–16.
- [17] Yannai A Gonczarowski, Lior Kovalio, Noam Nisan, and Assaf Romm. 2019. Matching for the Israeli "Mechinot" Gap-Year Programs: Handling Rich Diversity Requirements. *Proceedings of the 2019 ACM Conference on Economics and Computation* (2019).
- [18] Bettina Klaus and Flip Klijn. 2006. Procedurally fair and stable matching. *Economic Theory* 27, 2 (2006), 431–447.
- [19] David Kurokawa, Ariel D Procaccia, and Nisarg Shah. 2015. Leximin allocations in the real world. *Economics and Computation* (2015), 345–362.
- [20] Shivika Narang, Arpita Biswas, and Y Narahari. 2020. On Achieving Fairness and Stability in Many-to-One Matchings. *arXiv preprint arXiv:2009.05823* (2020).
- [21] Shivika Narang and Yadati Narahari. 2020. A Study of Incentive Compatibility and Stability Issues in Fractional Matchings. *19th International Conference on Autonomous Agents and MultiAgent Systems* (2020), 1951–1953.
- [22] Benjamin Plaut and Tim Roughgarden. 2020. Almost envy-freeness with general valuations. *SIAM Journal on Discrete Mathematics* 34, 2 (2020), 1039–1068.
- [23] Alvin E Roth. 1982. The economics of matching: Stability and incentives. *Mathematics of operations research* 7, 4 (1982), 617–628.
- [24] Alvin E Roth, Uriel G Rothblum, and John H Vande Vate. 1993. Stable matchings, optimal assignments, and linear programming. *Mathematics of Operations Research* 18, 4 (1993), 803–828.
- [25] Chung-Piaw Teo and Jay Sethuraman. 1998. The geometry of fractional stable matchings and its applications. *Mathematics of Operations Research* 23, 4 (1998), 874–891.