exist for the given profiles. We now recall the definition of AMRs, which include max-sum and max-num [8, 33].[2]

Definition 3 (Additive majority rule). *A JA rule F is an* ***additive majority rule*** *(AMR) if there exists a non-decreasing gain function* $g : [0, n] \to \mathbb{R}$ *such that* $g(t) < g(t')$ *for* $t < \frac{n}{2} \leqslant t'$*, and for every feasibility constraint* Γ' *and JA profile* $\boldsymbol{J}$*, it holds that:*

$$F(\boldsymbol{J}, \Gamma') = \text{argmax}_{J \in \text{Mod}(\Gamma')} \sum_{\varphi \in J} g(n_{(\boldsymbol{J},\varphi)})$$

We obtain the following simulation result:

Theorem 5. *When restricted to* RANKING*-rational profiles, every JA outcome* $J \in F(\cdot, k\text{-INCOMP-INCOMP})$ *for an AMR F that is based on a gain function g with the property that* $g(t) = g(t')$ *for any two* $t, t' \geqslant \frac{n}{2}$ *corresponds to a* ***weakly Gehrlein-stable committee****, provided such a stable committee exists at all.*

Proof. Take an m-sized set of alternatives X and suppose that a weakly Gehrlein-stable committee $S \subseteq \mathcal{P}_k(X)$ exists. Moreover, to derive a contradiction, suppose that there is some judgment $J^A \in F(\boldsymbol{J}, k\text{-INCOMP-INCOMP})$ that corresponds to a committee $A \subseteq \mathcal{P}_k(X)$ that is not weakly Gehrlein-stable.

Since A is not weakly Gehrlein-stable, there must be some $x \in A$ and some $y \in X \setminus A$ such that $|\{i \in N \mid x >_i y\}| < |\{i \in N \mid y >_i x\}|$. Then the score $\sum_{\varphi \in J^A} g(n_{(\boldsymbol{J},\varphi)})$ achieved by J^A is strictly less than $k(m-k)g_{\max}$, where $g_{\max} = g(\lceil n/2 \rceil) = \cdots = g(n)$.

However, the judgment J^S that corresponds to the committee S achieves the score $\sum_{\varphi \in J^S} g(n_{(\boldsymbol{J},\varphi)}) = k(m-k)g_{\max}$, and thus achieves a strictly higher score than J^A. This is a contradiction with our assumption that $J^A \in F(\boldsymbol{J}, k\text{-INCOMP-INCOMP})$. Therefore we can conclude that all judgments in $F(\boldsymbol{J}, k\text{-INCOMP-INCOMP})$ correspond to a weakly Gehrlein-stable committee. □

This result makes a large selection from the AMR class available to those interested in committee stability with tools to easily define novel Gehrlein-stable rules. In fact, the subclass of AMRs to which Theorem 5 applies includes some rules that correspond to multiwinner voting rules that have been studied in the literature.

For example, consider the AMR based on the gain function g with $g(t) = 1$ when $t \geqslant \frac{n}{2}$ and $g(t) = 0$ otherwise—which coincides with the max-num rule. When restricted to RANKING-rational profiles, using k-INCOMP-INCOMP as the feasibility constraint, this rule simulates the rule known as *Number of External Defeats* [3, 19].

Another example is the AMR based on the gain function g with $g(t) = 0$ when $t \geqslant \frac{n}{2}$ and $g(t) = 2t - n$ otherwise, or put differently, $g(t) = \max\{0, 2t - n\}$. When restricted to RANKING-rational profiles, using k-INCOMP-INCOMP as constraint, this rule simulates the *k-Kemeny* multiwinner voting rule [6, 37]. This correspondence revolves around the fact that for each pair (x, y), if x is selected in the outcome and y is not, a score of $\max\{0, |\{i \in N \mid y >_i x\}| - |\{i \in N \mid x >_i y\}|\}$ is added for $p_{x \succcurlyeq y}$ to the total score.

We now transition to approval-based rules, with a natural starting point being the simple AV rule. As previously mentioned, the rationality constraint for these rules will be INDIFF-INCOMP, which allows agents to have approval ballots of arbitrary size.

[2] We obtain the max-sum rule for $g(t) = t$, while max-num has $g(t) = 1$ when $t \geqslant \frac{n}{2}$ and $g(t) = 0$ otherwise.

Theorem 6. *When restricted to* INDIFF-INCOMP*-rational profiles, the* max-sum$(\cdot, k\text{-INDIFF-INCOMP})$ *rule simulates* ***AV****.*

Proof (sketch). Take J^A from Proposition 2 and the usual agenda decomposition. With INDIFF-INCOMP-rational profiles, each voter sets indifference between her most-preferred alternatives. We fix an approval set P_i for voter i such that $J_i(p_{x \succcurlyeq y}) = 1$ if and only if $x \in P_i$. It is easy to verify through counting relevant agreements that the max-sum rule induced by k-INDIFF-INCOMP on INDIFF-INCOMP-rational profiles is $\text{argmax}_{J^A \text{ s.t. } A \in \mathcal{P}_k(X)} \sum_{i \in N} |\{x \mid x \in A \cap P_i\}|$. The elected committee/s of size k maximise the approval of the committee members which gives us a simulation of AV. □

We now extend the AV simulation to other Thiele rules. Let us adjust max-sum by incorporating a scoring vector. For any scoring vector $\boldsymbol{w}^{(k)}$ and number $\ell \geqslant 0$, let $f_{\boldsymbol{w}^{(k)}}(\ell) = \sum_{i=0}^{\ell} w_i$. This function allows us to refine max-sum to f-max-sum$(\boldsymbol{J}, \Gamma', \boldsymbol{w}^{(k)}) = \text{argmax}_{J \in \text{Mod}(\Gamma')} \sum_{i \in N} f_{\boldsymbol{w}^{(k)}}(|\text{Agr}(J, J_i)|)$. That the f-max-sum rule facilitates the simulation of PAV and α-CC is immediate from its definition and our proof sketch for Theorem 6, so we present the next result without proof.

Proposition 7. *When restricted to* INDIFF-INCOMP*-rational profiles, the f-*max-sum *rule induced by k-*INDIFF-INCOMP *simulates* ***PAV*** *and* ***α-CC*** *when using the scoring vectors* $(1, 1/2, 1/3, \ldots, 1/k)$ *and* $(1, 0, \ldots, 0)$*, respectively.*

3.4 Constraints as Circuits

Worst-case intractability has been shown for many JA rules. Specifically, computing outcomes under max-sum and max-num is Θ_2^{p}-hard [18]. Thus, when simulating multiwinner rules in JA, we encounter the paradox that ordinarily easy-to-compute rules, such as k-Borda and AV [4, 14], now seem computationally difficult to implement. To address this mismatch, we employ the approach proposed by de Haan [26] who showed that JA rules can be used efficiently when the integrity constraint is represented as a circuit in *decomposable negation normal form*, or a DNNF circuit. We begin with the circuit definition given by Darwiche and Marquis [10].

Definition 4 (DNNF circuits). *A Boolean circuit in negation normal form (NNF) is a directed acyclic graph with a single root where each internal node is labelled with* $\vee$ *or* $\wedge$*, and every leaf is labelled with* $\top$*,* $\bot$*,* x *or* $\neg x$ *for a propositional variable* x*. A* ***DNNF circuit*** *is an NNF circuit that satisfies decomposability: for each conjunction in the circuit, no two conjuncts share a propositional variable.*

The results of de Haan [26] cover certain members in the class of scoring rules [11], including max-sum and max-num. Scoring rules select those constraint-satisfying JA outcomes that maximise the score of an associated scoring function. Such a function attaches a score to each issue with respect to an agent's judgment. Before restating the relevant result in Theorem 8, we define the *outcome determination problem* Outcome(F) for a given JA rule F.

Outcome(F)	
Given:	A judgment profile $\boldsymbol{J}$ for an agenda Φ, an integrity constraint Γ', and a partial judgment d on Φ.
Question:	Is there a $J \in F(\boldsymbol{J}, \Gamma')$ that agrees with d?

Theorem 8 (De Haan, 2018). *When the integrity constraint Γ' is represented as a DNNF circuit, then both* Outcome(max-sum) *and* Outcome(max-num) *are polynomial-time solvable.*[3]

3.5 Encoding and Complexity Results

We now show that the k-indiff-incomp constraint can be represented as a DNNF circuit. Recall that this constraint sets indifference amongst the top alternatives and incomparability between those in the bottom set. Observe that for any alternative x in the top set, the proposition $p_{x \succcurlyeq y}$ is true for all $y \in X$. On the other hand, if x is in the bottom set, the proposition $p_{x \succcurlyeq y}$ is false for all $y \in X$.

Theorem 9. *Given a finite set X of alternatives and a corresponding preference agenda $\Phi^X_{\succcurlyeq}$, the k-indiff-incomp constraint can be encoded into a DNNF circuit in polynomial time.*

Proof. Given the set of alternatives $X = \{x_1, \dots, x_m\}$, we construct the circuit according to the (arbitrary) ordering $x_1, \dots, x_m$ of X. An ordering of propositions such as $p_{x \succcurlyeq x}$ for each $x \in X$ is then set. We say x_i is the ith alternative in X while $p_{x_i \succcurlyeq x_i}$ is the corresponding proposition in $\Phi^X_{\succcurlyeq}$. We also use the counting variables i and j during the circuit construction, both starting at 0.

The circuit contains nodes $N_{i,j}$, each of which denoting that we have assessed the propositions in the sequence up to (and including) index $i-1$ with j the current size of the winning set. We set $N_{0,0}$ to be the root of the circuit. If $i = |X|+1$ and $j = k$, then $N_{i,j} = \top$. If $i = |X|+1$ and $j \neq k$, then $N_{i,j} = \bot$. Now if $i < |X|+1$, we either have (i) $p_{x_i \succcurlyeq x_i}$ is true or (ii) $p_{x_i \succcurlyeq x_i}$ is false. We set the node $N_{i,j}$ to be the disjunction $\alpha \vee \beta$, where $\alpha = (N(i+1,\ j+1) \wedge \bigwedge_{y \in X} p_{x_i \succcurlyeq y})$ and $\beta = (N(i+1,\ j) \wedge \bigwedge_{y \in X} \neg p_{x_i \succcurlyeq y})$. We have that every leaf is either $\top$, $\bot$ or $p_{x \succcurlyeq y}$ for some x and y. Thus, we have a NNF circuit. Each proposition appears exactly once in the circuit so we also have that it is decomposable. The circuit is only satisfied by a preference agenda $\Phi^X_{\succcurlyeq}$ if the agenda has a k-sized top set of alternatives with indifference between them while the bottom set's alternatives are incomparable. So we have that the circuit corresponds to our constraint. This circuit can also be constructed in polynomial time as the process terminates once each alternative in X has been assessed exactly once. ◻

Corollary 10. *Given a finite set X of alternatives and a corresponding preference agenda $\Phi^X_{\succcurlyeq}$, the* indiff-incomp *constraint can be encoded into a DNNF circuit in polynomial time.*[4]

These results ensure that for max-sum and max-num when using k-indiff-incomp as the feasibility constraint, such as in our simulations, computing the outcomes can still be done in polynomial time. We continue with this approach to analyse k-indiff-indiff.

Recall that k-indiff-indiff sets indifference within both the k-sized top set and the bottom set. We now show that this constraint cannot be constructed as a DNNF circuit in polynomial time. The claim is that, given a set of alternatives X, computing the max-sum rule induced by k-indiff-indiff with the rationality constraint $\top$, i.e., when the ballots are unconstrained, is a computationally difficult problem. This implies that we cannot construct a DNNF circuit representing k-indiff-indiff in polynomial time (assuming that P ≠ NP). We show that this problem is NP-hard by giving a reduction from the following problem.

NAE-3SAT	
Given:	A formula φ in 3CNF.
Question:	Is there a truth assignment satisfying φ that falsifies at least one literal in each clause of φ?

Theorem 11. *Given a finite set X of alternatives and a corresponding preference agenda $\Phi^X_{\succcurlyeq}$, computing any outcome of the* max-sum *rule induced by k-*indiff-indiff *on $\top$-restricted ballots is NP-hard.*

Proof (sketch). We reduce from NAE-3SAT. Let φ be an arbitrary instance of NAE-3SAT, where $x_1, \dots, x_m$ are the variables in φ and $c_1, \dots, c_u$ are the clauses in φ. For each variable x_i in φ, we create an alternative a_{x_i} for each literal of the variable x_i. This produces $2m$ alternatives. The profile on $\Phi^X_{\succcurlyeq}$, is as follows: for each variable x_i, we add $10u$ agents, each of which has a judgment set that sets every preference issue to true except for $p_{a_{x_i} \succcurlyeq a_{\neg x_i}}$ and $p_{a_{\neg x_i} \succcurlyeq a_{x_i}}$. For each clause c_j, and each pair of literals (ℓ_1, ℓ_2) within c_j, we create an individual that has the judgment set that sets every issue in the agenda to true except for $p_{a_{\ell_1} \succcurlyeq a_{\ell_2}}$ and $p_{a_{\ell_2} \succcurlyeq a_{\ell_1}}$. We claim that there exists a truth assignment that both satisfies and falsifies at least one literal in each clause of φ if and only if the translated outcome, given by max-sum induced by m-indiff-indiff, has a score of at least $(10mu+3u)\cdot(4\binom{m}{2}+m^2)-10mu-4u$. Verifying the claim is straightforward. For space reasons, we omit the details. ◻

So in general, we cannot efficiently use the k-indiff-indiff-induced max-sum rule. However, when used on ranking-restricted profiles, it simulates k-Borda (Proposition 2). So in this particular case, we obtain computational efficiency. This highlights the care required in JA constraint selection.

4 AIMING FOR PROPORTIONAL JA RULES

So far, we have shown simulations of existing multiwinner voting rules using existing JA rules (or, in one case, rules that are very close to existing JA rules). But our embedding approach also suggests itself as a tool for porting ideas from multiwinner voting to JA that have so far not been considered in the latter field. An example is the notion of *proportional representation*, which plays a central role in multiwinner voting [2, 39].[5] While the simulation of PAV and α-CC represents an initial step towards introducing this notion into JA, in this section we explore this direction more systematically.

We start by introducing a proportionality axiom for JA as well as two concrete aggregation rules designed to satisfy this axiom (at least under certain conditions). We then proceed to studying the computational complexity of our rules.

4.1 Proportional JA Rules

While PAV and α-CC ensure some form of proportionality when electing a committee of fixed size, it is much less clear how to design

[3] Rey et al. [38] extend this result to the general class of *additive rules*, which includes both the scoring rules and the AMRs.

[4] The proof of the circuit encoding for indiff-incomp works as in Theorem 9 except the tracking of the winning set's size is omitted.

[5] Recent work by Haret et al. [27] on importing proportionality into belief merging, a formalism that is conceptually similar to JA, has similar motivations and underlines the relevance of this idea. We note that despite this conceptual similarity, the technical results obtained by the authors are technically unrelated to ours.

such a proportional rule for electing variable-sized committees (because the trivial solution of electing *all* alternatives to maximise voter approval is clearly not attractive). But for more general JA applications, a rule that does not force us to accept a fixed number of propositions seems more relevant. Also, such a rule must be measured against some criteria to determine the extent to which its outcomes are proportional. To this end, we adapt the multiwinner axiom of *proportional justified representation* (PJR) [39].

Definition 5 (Sánchez-Fernández et al., 2017). *Given an approval ballot profile $\boldsymbol{A} = (A_1, \ldots, A_n)$ over a set of alternatives X and a fixed committee size $k \leqslant |X|$, a group of voters is ℓ-cohesive for some $\ell \in [k]$, if $|N^*| \geqslant \ell \cdot \frac{n}{k}$ and $|\bigcap_{i \in N^*} A_i| \geqslant \ell$. A committee $C \in \mathcal{P}_k(X)$ satisfies* ***proportional justified representation*** *(PJR) for $\boldsymbol{A}$ and k, if for every $\ell \in [k]$ and every ℓ-cohesive group of voters $N^* \subseteq N$, it is the case that $|C \cap (\bigcup_{i \in N^*} A_i)| \geqslant \ell$.[6] An approval-based voting rule satisfies PJR if for every profile $\boldsymbol{A}$ and every committee size k, it outputs a committee that satisfies PJR for $\boldsymbol{A}$ and k.*

For our JA axiom, we cannot rely on a fixed size k to identify cohesive groups, as a variable number of issues may be accepted by a JA outcome. So we use this number of accepted issues as if it were the committee target size to begin with, which differs from other approaches to variable-sized multiwinner proportionality [22]. Also, to account for logical dependencies between JA issues, we focus on the issues that groups agree on that are, in a sense, logically independent of the agenda. This approach may lead to a weaker notion that is less compatible with restrictive constraints, but it is clear that complex constraints may rule out proportionality.

We provide extra notation for our axiom. For an integrity constraint Γ, an issue φ is logically independent of another issue ψ if both $\Gamma \not\models \psi \rightarrow \varphi$ and $\Gamma \not\models \psi \rightarrow \neg\varphi$ are the case. We say an issue is logically independent of some set S if $\Gamma \not\models (\bigwedge_{\psi \in S} \psi) \rightarrow \varphi$ and $\Gamma \not\models (\bigwedge_{\psi \in S} \psi) \rightarrow \neg\varphi$. The set of issues accepted by a judgment J is denoted as $J^+ = \{\varphi \in \Phi \mid J(\varphi) = 1\}$. For an agent group $N^* \subseteq N$, we define a judgment $U_{\boldsymbol{J}}(N^*)$ such that, for every $\varphi \in \Phi$, it is the case that $U_{\boldsymbol{J}}(N^*)(\varphi) = 1$ if $J_i(\varphi) = 1$ for some agent i in N^*; otherwise, $U_{\boldsymbol{J}}(N^*)(\varphi) = 0$. The judgment $I_{\boldsymbol{J}}(N^*)$ requires every agent i in N^* to be accepting of φ for $I_{\boldsymbol{J}}(N^*)(\varphi) = 1$ to hold; otherwise, $I_{\boldsymbol{J}}(N^*)(\varphi) = 0$. We now define our JA proportionality axiom.

Definition 6 (ℓ-JA-PJR). *Consider some $\ell \in [|J^+|]$ for a judgment J over an agenda Φ accepting $|J^+|$ issues. We say a group of agents $N^* \subseteq N$ is (J, Γ, ℓ)-cohesive if $|N^*| \geqslant \ell \cdot \frac{n}{|J^+|}$ and $|\{\varphi \in I_{\boldsymbol{J}}(N^*)^+ \mid \varphi$ is logically independent of $\Phi \setminus \{\varphi\}\}| \geqslant \ell$. Given a judgment profile $\boldsymbol{J}$ and an integrity constraint Γ, we say that an outcome J provides* ***ℓ-JA proportional justified representation*** *(ℓ-JA-PJR), if for every (J, Γ, ℓ)-cohesive group of agents $N^* \subseteq N$, it is the case that $|\mathrm{Agr}(J, U_{\boldsymbol{J}}(N^*))^+| \geqslant \ell$. We say a JA rule F satisfies ℓ-JA-PJR if every JA outcome $J \in F(\boldsymbol{J}, \Gamma)$ provides ℓ-JA-PJR.*

Next, we are going to propose new JA rules geared towards proportionality. In the variable-sized multiwinner literature, there is a class of rules that take both approvals and disapprovals into account when scoring a committee [9, 20]. We adopt this approval-disapproval dynamic and apply it to a JA outcome's accepted issues.

[6] For $\ell = 1$, this axiom reduces to *justified representation*, as defined by Aziz et al. [2].

Let $f_{\boldsymbol{w}^{(m)}}(\ell) = \sum_{i=0}^{\ell} w_i$ for any given scoring vector $\boldsymbol{w}^{(m)}$. The general form of our rules is the following variant of max-sum, but now using two separate scoring vectors, $\boldsymbol{u}^{(m)}$ and $\boldsymbol{v}^{(m)}$, for approvals and disapprovals, respectively:

$$(a\text{-}d)\text{-max-sum}(\boldsymbol{J}, \Gamma) = \operatorname*{argmax}_{J \in \mathrm{Mod}(\Gamma')} \sum_{i \in N} f_{\boldsymbol{u}^{(m)}}(|\mathrm{Agr}(J, J_i)^+|) - f_{\boldsymbol{v}^{(m)}}(|\mathrm{Dis}(J, J_i)^+|)$$

Through varying the scoring vectors, we now define candidates for new JA rules. The first attaches standard AV scoring to agreed-upon accepted issues, and 'penalises' disagreed-with accepted issues with an 'inverted' PAV scoring.

Definition 7 (PAV-JA). *Given an agenda Φ with m issues and an integrity constraint Γ, the* ***PAV-JA rule*** *is defined as the $(a\text{-}d)$-max-sum rule with the scoring functions induced by following scoring vectors $\boldsymbol{u}^{(m)} = (1, 1, \ldots, 1)$ and $\boldsymbol{v}^{(m)} = (1/m, \ldots, 1/2, 1)$.*

For the second rule, an agent awards points to an outcome as with α-CC scoring, but subtracts a point if the majority threshold, a commonly-used threshold [1, 20, 21], of rejected issues is crossed.

Definition 8 (CC-JA). *Given an agenda Φ with m issues and an integrity constraint Γ, the* ***CC-JA rule*** *is defined as the $(a\text{-}d)$-max-sum rule with the scoring functions induced by the scoring vectors $\boldsymbol{u}^{(m)} = (1, 0, \ldots, 0)$ and $\boldsymbol{v}^{(m)} = (0, \ldots, 0, 1, 0, \ldots, 0)$. In $\boldsymbol{v}^{(m)}$ we set the scoring vector's 'threshold' at position $\lceil m/2 \rceil + 1$.*

Having proposed two JA rules based on well-known proportional approaches, we assess whether these rules satisfy our ℓ-JA-PJR axiom, beginning with PAV-JA.

Theorem 12. *The rule PAV-JA$(\boldsymbol{J}, \Gamma)$ satisfies the ℓ-JA-PJR axiom for every value $\ell \geqslant |J^+| \,/\, (m - |J^+| + 1)$.*

Proof (sketch). Take a $J \in$ PAV-JA$(\boldsymbol{J}, \Gamma)$ and assume there is a (J, Γ, ℓ)-cohesive group N^* such that $|\mathrm{Agr}(J, U_{\boldsymbol{J}}(N^*))^+| < \ell$. We can show that a judgment J' that accepts a currently-rejected issue in $I_{\boldsymbol{J}}(N^*)^+$, all else being equal, yields a strictly higher PAV-JA score. We now detail the change in score which occurs with the acceptance of φ. The group N^* adds at least $\ell \cdot (n/|J^+|)$ to the score. At least one agent is already satisfied by J and this agent deducts at most $1/(m - |J^+| + 1)$ while at most $n - |N^*| - 1$ agents will each deduct at most $1/(m - |J^+|)$. So the score strictly increases when we have: $|N^*| \geqslant \ell \cdot (n/|J^+|) > (n - |N^*| - 1)/(m - |J^+|) + 1/(m - |J^+| + 1) < (n(|J^+| - \ell))/(|J^+|(m - |J^+|))$. To conclude, observe that the score is strictly positive when $\ell \geqslant |J^+|/(m - |J^+| + 1)$. □

Next, we establish that CC-JA generally fails ℓ-JA-PJR; but with a stronger independence assumption, CC-JA satisfies 1-JA-PJR.

Theorem 13. *Assuming logical independence between all agenda items, CC-JA satisfies ℓ-JA-PJR for $\ell = 1$ and fails it for every $\ell > 1$.*

Proof (sketch). Our claim is that, for any $\ell > 1$, we can construct an agenda Φ and a profile $\boldsymbol{J}$ such that, for $\Gamma = \top$, there is a $J \in$ CC-JA$(\boldsymbol{J}, \Gamma)$ that does not provide ℓ-JA-PJR. Consider an arbitrary $\ell > 1$. We choose an agenda with an odd number $m \geqslant 5$ of issues such that $\ell = \lfloor m/2 \rfloor$. We fix a set of agents N with $|N| = m + 1$ and an agent subset $N^\ell \subset N$ such that $|N^\ell| = (|N| \cdot \lfloor m/2 \rfloor)/\lceil m/2 \rceil = |N| - 2$. This ensures the existence of two agents $i, j \notin N^\ell$. Given this agent population, we can define the judgment profile $\boldsymbol{J}$. In this profile, the agents in N^ℓ uniformly accept the same ℓ issues, so we have $|I_{\boldsymbol{J}}(N^\ell)^+| = |U_{\boldsymbol{J}}(N^\ell)^+| = \ell$. For the judgments of agents $i, j \notin N^\ell$, we have two issues $\varphi, \psi \notin I_{\boldsymbol{J}}(N^\ell)^+$, i.e., neither φ nor ψ are accepted by agents in N^ℓ, such that $U_{\boldsymbol{J}}(\{i\})^+ = \{\varphi\}$ and $U_{\boldsymbol{J}}(\{j\})^+ = \{\psi\}$. This completes the construction of the profile. Note that $|U_{\boldsymbol{J}}(N)^+| = \lceil m/2 \rceil + 1$. Now we assess the issues accepted by CC-JA for $\boldsymbol{J}$. Once N^ℓ is represented by $\ell - 1$ issues, observe that (due to $\ell > 1$) accepting the issue φ for agent $i \notin N^\ell$ gives a greater score than accepting an ℓ-th issue in $I_{\boldsymbol{J}}(N^\ell)^+$. The same holds for ψ and agent $j \notin N^\ell$. After φ and ψ are accepted, accepting an ℓ-th issue in $I_{\boldsymbol{J}}(N^\ell)^+$ decreases the score as the majority threshold is crossed. Notice that CC-JA accepts exactly $\lceil m/2 \rceil$ issues and thus, from the definition of N^ℓ's size, it follows that N^ℓ is a (J, Γ, ℓ)-cohesive group. So we have an outcome where a (J, Γ, ℓ)-cohesive group is only represented by $\ell - 1$ issues.

Next, we show that every $J \in$ CC-JA$(\boldsymbol{J}, \Gamma)$ provides ℓ-JA-PJR for $\ell = 1$, assuming logical independence throughout the agenda. The argument is that any rejected issue in $I_{\boldsymbol{J}}(N^*)^+$ for an unrepresented (J, Γ, ℓ)-cohesive group N^* can be 'swapped' for some accepted issue in an outcome J. This only decreases the CC-JA score if every already-accepted issue represents a unique group that is at least as large as N^*. Thus, any such issue would at least match the contribution to the score of an issue accepted by N^*. However, this cannot be the case for all $|J^+|$ issues as this would imply that at least $|J^+| \cdot \frac{n}{|J^+|} = n$ agents have been represented thus far, contradicting our assumption of the existence of this unrepresented group N^*. ◻

Beyond defining ℓ-JA-PJR, we established some values of ℓ for which our new rules satisfy the JA axiom. Hence, despite ℓ-JA-PJR being restrictive, the axiom can be used to study JA rules, as those that fail it outright may be ill-suited for JA proportionality.

4.2 Hardness of JA Rules for Proportionality

We end by showing computational intractability of the PAV-JA and CC-JA rules, beginning with an NP-hardness result for the former.

Theorem 14. Outcome(PAV-JA) *is NP-hard.*

Proof (sketch). We show NP-hardness by reducing from the following problem. Take a positive integer $t \in \mathbb{N}$ and an approval-based multiwinner election. Moreover, we know that there exists some $b \in \mathbb{N}$ such that: (i) the maximum PAV score of any committee of size t equals $b \cdot t$, and this can only be achieved by getting a score of 1 from $b \cdot t$ different voters, (ii) there exists at least one such committee of size t, and (iii) each voter approves of exactly 2 candidates. The problem is to decide, for a given candidate c, whether it is part of a committee of size t with maximum PAV score.

This problem can straightforwardly be shown to be NP-hard, by adapting the proof of a known result for multiwinner PAV voting [4, Theorem 1], which uses a reduction from the classical problem of independent set. By instead considering a suitable variant of independent set—where the size of the maximum independent set is known in advance, and the question is whether there is a maximum independent set that contains a given vertex—this proof directly yields NP-hardness of the problem that we will reduce from.

In the restricted setting where conditions (i)–(iii) hold, the simulation of multiwinner PAV that we used to establish Proposition 7 also works if we consider PAV-JA instead. Therefore, we can use this simulation to construct a reduction to Outcome(PAV-JA). ◻

Moving on to CC-JA, we see that, not only is α-CC NP-hard [31, 36], but computing outcomes for CC-JA is also hard. We give a reduction from a variant of the well-known problem MaxSAT [35]—one can straightforwardly prove that this variant is Θ_2^{p}-complete as well. In this variant of the problem, we are given a set $\mathcal{L}$ of literals, two sets φ_1 and φ_2 of clauses, with clauses in both being of size at most 3, and some variable x^* occurring in φ_2. The question is to decide whether—among the truth assignments that satisfy all clauses in φ_1 and that satisfy a maximum number of clauses in φ_2—there is a truth assignment that sets x^* to true.

Theorem 15. Outcome(CC-JA) *is* Θ_2^{p}*-complete.*

Proof (sketch). We describe Θ_2^{p}-hardness by reducing from the MaxSAT-variant described above. For space reasons, we omit the straightforward proof of membership in Θ_2^{p}.

We introduce an agenda item y_ℓ for each literal ℓ over the variables in φ_1 and φ_2. Hence, we have issues such as y_{x_1} and $y_{\neg x_1}$ in the agenda Φ. For each clause v_i appearing in φ_2, we create a voter i. These voters accept those issues which correspond to literals appearing in their associated clause. We construct an integrity constraint that expresses that one of two cases must hold: (i) at most three issues are set to true, or (ii) exactly one of each pairs of issues, y_{x_i} or $y_{\neg x_i}$, is set to true in a way that satisfies all clauses in φ_1. Finally, we construct a partial ballot d that only sets y_{x^*} to true. We claim that the outcomes of the CC-JA rule over the constructed profile correspond to the truth assignments that satisfy all of φ_1 and that maximise the number of satisfied clauses in φ_2. Therefore, there is an outcome that agrees with the partial ballot d if and only if the original instance is a yes-instance. For reasons of space, we omit a detailed proof of this claim. ◻

5 CONCLUSION

We illustrated how the JA model with rationality and feasibility constraints enables us to simulation important multiwinner voting rules in JA. We subsequently showed, by encoding the constraints as DNNF circuits, that some of these simulations retain the computational efficiency of their multiwinner counterparts. On the other hand, for one specific feasibility constraint, this efficiency cannot be retained in general. Also, we demonstrated how a class of JA rules can help produce Gehrlein-stable multiwinner voting rules. Finally, we suggested an axiom and two aggregation rules for JA intended to reflect and satisfy as suitable notion of proportionality, and we briefly analysed the complexity of using these rules.

For future research, the JA simulation of more sophisticated multiwinner voting rules, such as sequential rules, should be explored. Our brief excursion into JA proportionality suggests multiple research paths, such as adapting ℓ-JA-PJR to handle complex logical constraints. And for the newly-proposed JA rules, the development of approximate versions of the rules can be pursued. Also, JA rules could aid the enrichment of multiwinner voting with new rules satisfying notions other than committee stability.

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