# **Three-Dimensional Popular Matching with Cyclic Preferences**

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# ABSTRACT

Two actively researched problem settings in matchings under preferences are popular matchings and the three-dimensional stable matching problem with cyclic preferences. In this paper, we apply the optimality notion of the first topic to the input characteristics of the second one. We investigate the connection between stability, popularity, and their strict variants, strong stability and strong popularity in three-dimensional instances with cyclic preferences. Furthermore, we also derive results on the complexity of these problems when the preferences are derived from master lists.

# **KEYWORDS**

popular matching; three-dimensional stable matching; cyclic preferences; complexity; Condorcet paradox

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# **1** INTRODUCTION

Partitioning agents into desirable groups is one of the core problems of algorithmic game theory. However, the lines between tractability and intractability are often very thin; introducing ties, incomplete lists or slight variations to the preference or group structures can make a previously tractable problem intractable. In this work, we aim to further draw this line by studying popularity in threedimensional matching instances equipped with cyclic preferences.

# 1.1 **Problem Setting**

In a *three-dimensional (3D) matching instance*, we are given three sets of agents *A*, *B*, and *C*, representing for example users, data sources, and servers [13] or as it is commonly referred to in the literature [32, 36], men, women, and dogs. Each agent in *A*, *B*, and *C* declares a subset of the agents in *B*, *C*, and *A*, respectively, acceptable. A matching *M* consists of  $(a, b, c) \in A \times B \times C$  triples such that *a* finds *b* acceptable, *b* finds *c* acceptable, and *c* finds *a* acceptable; furthermore, each agent appears in at most one triple in *M*.

In the problem variant we study, each agent possesses a strictly ordered preference list. *Cyclic preferences* mean that agents in A have preferences over the acceptable agents in B, agents in B have preferences over the acceptable agents in C, and finally, agents in C have preferences over the acceptable agents in A. The standard problem is to decide whether such an instance admits a stable

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matching. Two intuitive stability notions have been investigated in the literature: a *weakly stable matching* does not admit a triple so that all three agents would improve, while according to *strong stability*, a triple already blocks if at least one of its agents improves, and the others in the triple remain equally satisfied.

The optimality criterion we study in this paper is *popularity*, which is a well-studied concept in the context of two-sided matching markets. Given two matchings M and M', matching M is more popular than M' if the number of agents preferring M to M' is larger than the number of agents preferring M' to M. A matching M is called *popular* if there is no matching M' that is more popular than M. Colloquially speaking, a popular matching is a matching that would not lose a head-to-head election against any other matching if the agents were allowed to vote between the matchings.

# 1.2 Related Work

We first review existing work on matchings under preferences in the three-dimensional setting, and then highlight the most important improvements on popular matchings.

Stability in 3 Dimensions. After the introduction of stable matchings by Gale and Shapley [18] and their celebrated algorithm to solve the problem in bipartite graphs, the study of threedimensional stable matchings was initiated by Knuth [29], who asked about a generalization of stable matchings to triples. Subsequently, Ng and Hirschberg [36] studied a stable matching variant with three genders, where agents of one gender have a preference list over pairs of the other two genders. The goal in this model is to find a set of disjoint triples that is not blocked by any triple outside of it. Ng and Hirschberg [36] and independently Subramanian [42] were able to prove that it is NP-complete to decide whether such a three-dimensional stable matching exists. Their result was then generalized by Huang [22], who incorporated ties and stronger notions of stability, as well as restricted preference structures in this model. He showed that all these variants stay NP-complete as well. Danilov [14] identified an even further restricted preference structure that allows for a polynomial-time algorithm for the existence problem. Finally, McKay and Manlove [34] studied the generalization of the three-dimensional stable roommates problem to additive preferences.

3D-Stable Matchings with Cyclic Preferences. One direction proposed by Ng and Hirschberg [36] was to generalize their work to cyclic preferences. This question lead to a family of papers. Biró and McDermid [6] showed that deciding whether a weakly stable matching exists is NP-complete if preference lists can be incomplete, and that the same complexity result holds for strong stability

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even with complete lists. However, the combination of complete lists and weak stability proved to be extremely challenging to solve.

For this setting, Boros et al. [8] proved that each instance admits a weakly stable matching for  $n \leq 3$ , where *n* is the size of each agent set in the tripartition. Eriksson et al. [15] later extended this result to  $n \leq 4$ . Additionally, Pashkovich and Poirrier [38] further proved that not only one, but at least two stable matchings exist for each instance with n = 5. By this time, the conjecture on the guaranteed existence of a weakly stable matching in 3D instances with complete cyclic preferences became one of the most riveting open questions in the matching under preferences literature [29, 32, 43]. Surprisingly, Lam and Plaxton [30] recently disproved this conjecture by showing that weakly stable matchings need not exist for an arbitrary *n*, moreover, it is NP-complete to determine whether a given instance with complete lists admits a weakly stable matching.

The problem is relevant to applications as well, as shown by the papers of Cui and Jia [13], Raveendran et al. [40], and Ma et al. [31], who all studied 3D-cyclic stable matchings in the context of computer networks, as well as by the work of Bloch et al. [7], who applied it to a Paris apartment assignment problem. Additionally, Escamocher and O'Sullivan [16] and Cseh et al. [11] set up constraint programming models for the problem. They discussed instances where agents of the same class have identical preference lists. This type of preference structure is also called a *master list*. Besides them, Bredereck et al. [10] also investigated master lists in the context of 3D stable matchings, and there is a large set of results on 2D stable or popular matchings with master lists in the input [24, 25, 27, 35].

*Popular Matchings.* The concept of a popular matching corresponds to the notion of a weak Condorcet winner in voting. In the context of matchings it was first introduced by Gärdenfors [19] for matching markets with two-sided preferences, and then studied by Abraham et al. [1] in the house allocation problem. Polynomial time algorithms to find a popular matching were given in both settings. These papers inspired a plethora of work on popularity in the house allocation problem. Most importantly, Sng and Manlove [41] extended the model of Abraham et al. [1] with capacities on the houses, while McDermid and Irving [33] studied a weighted variant.

In the classic two-sided preferences model, it was already noticed by Gärdenfors [19] that all stable matchings are popular, which implies that in bipartite stable matching instances, popular matchings always exist. In fact, stable matchings are the smallest size popular matchings, as shown by Biró et al. [4], while maximum size popular matchings can be found in polynomial time as well [23, 26].

Only recently, Faenza et al. [17] and Gupta et al. [21] resolved the long-standing open question that it is NP-complete to find a popular matching in a roommates instance.

Strongly Popular Matchings. A further concept we study is that of a strongly popular matching, corresponding to a strong Condorcet winner, i.e., a matching that wins every head-to-head election. This concept was introduced by Biró et al. [4], who showed that a strongly popular matching in roommates instances exists if and only if it is the unique stable matching. The open question whether a strongly popular matching in a roommates instance with ties can be found in polynomial time was recently answered positively by Brandt and Bullinger [9], who observed that a strongly popular matching must be the unique mixed popular matching. Strong popularity was very recently extended to *b*-matchings as well by Király and Mészáros-Karkus [28].

*Popularity in 3 Dimensions.* Brandt and Bullinger [9] showed that it is hard to find a popular partition into sets of at most size three, even if the ranking of all sets by all agents is the same. This however is different from the 3D-cyclic model in both the structure of the preferences, since the agents in their model have a preference list over subsets of size 2 or 3, as well as in the structure of the solution, since they allow sets of size 2 and 3. Both Brandt and Bullinger [9] and Lam and Plaxton [30] mentioned the 3D-cyclic popular matching problem as an interesting future research direction.

#### 2 PRELIMINARIES

We now define the notation we use and the problems we investigate.

#### 2.1 Input and Output Formats

Input and Notation. We are given three sets of agents A, B, and C. We denote by  $V = (A \cup B) \cup C$  the set of all agents and call A, B, and C the agent classes of our instance. Further we assume that |A| = |B| = |C| = n. Each agent in A has a strict preference list over a subset of agents in B, each agent in B has a strict preference list over a subset of agents in C, and finally, each agent in C has a strict preference list over a subset of agents in A. These preference lists define for each agent x a strict order  $>_x$ , which we call the preference list of x and say that x finds the agents in  $>_x$  acceptable. For two agents y, z such that  $y >_x z$ , we say that x prefers y to z.

*Master Lists.* When defining master lists, we use the terminology from the book of Manlove [32]. We say that the preferences of agents in  $X \subseteq V$  are *derived from a master list* if there is a master preference list from which the preferences of each  $x \in X$  can be obtained by deleting some agents. This means that the preferences might be incomplete, but the relative preferences between acceptable agents are the same in each  $>_x$ , where  $x \in X$ . We say that an instance is derived from a *k*-master list for  $k \in \{1, 2, 3\}$  if the preferences of *k* of the agent classes of our instance are derived from a master list.

*Matchings.* A *matching* M is a subset of  $A \times B \times C$ , such that each agent appears in at most one triple and for each  $(a, b, c) \in M$ , a finds b acceptable, b finds c acceptable, and c finds a acceptable. If  $(a, b, c) \in M$ , we also write M(a) = b, M(b) = c, and M(c) = a. If an agent x does not appear in any triple in M, we write M(x) = x and say that the agent x is *unmatched*. For convenience in notation we assume that for any agent x, x itself appears at the end of  $>_x$ , and prefers being matched to any agent she finds acceptable to staying unmatched. The preference relation  $>_x$  naturally extends to the comparison of two triples by a x if she appears in both triples.

#### 2.2 Optimality Concepts

Weak and Strong Stability. A triple t = (a, b, c) is said to be a strongly blocking triple to matching M if each of a, b, and c prefer t to their respective triples (or to their singleton if they are unmatched) in M. Practically, this means that a, b, and c could abandon their triples to form triple t on their own, and each of them would be strictly better off in t than in M. If a matching M does not admit any

strongly blocking triple, then M is called a *weakly stable* matching. Similarly, a triple t = (a, b, c) is called a *weakly blocking triple* if at least two agents in the triple prefer t to their triple (or to their singleton if they are unmatched) in M, while the third agent does not prefer her triple in M to t. This means that at least two agents in the triple can improve their situation by switching to t, while the third agent does not mind the change. A matching that does not admit any weakly blocking triple is referred as *strongly stable*. By definition, strongly stable matchings are also weakly stable, but not the other way round. Observe that it is impossible to construct a triple t that keeps exactly two agents of a triple equally satisfied, while making the third agent happier, since the earlier two agents need to keep their partners to reach this, which then already defines the triple as one already in M.

Weak and Strong Popularity. Given an agent x and two matchings M and M', we define

$$\operatorname{vote}_{x}(M', M) = \begin{cases} 1, & \text{if } M'(x) >_{x} M(x) \\ 0, & \text{if } M'(x) = M(x) \\ -1, & \text{if } M(x) >_{x} M'(x) \end{cases}$$

i.e.,  $vote_x(M', M)$  represents whether the agent *x* would prefer to be in *M'* or in *M*. We call *M'* more popular than *M* if

$$\Delta(M',M) \coloneqq \sum_{x \in V} \operatorname{vote}_x(M',M) \ge$$

i.e., if M' would win against M in a head-to-head election. Matching M is called *popular* if no matching is more popular than M. Using this we can now define the popular matching problem in 3 dimensions.

3D-CYCLIC POPULAR MATCHING WITH INCOMPLETE LISTS (3DPMI)

**Input:** Sets *A*, *B*, *C* with a cyclic preference structure.

Question: Does a popular matching exist?

Further we also study the corresponding verification problem. 3D-CYCLIC POPULAR MATCHING VERIFICATION WITH INCOMPLETE LISTS (3DPMVI)

**Input:** Sets *A*, *B*, *C* with a cyclic preference structure and a matching *M*.

Question: Is M popular?

The notion of popularity can be strengthened even further to what is commonly referred to as a strongly popular matching. A matching M is *strongly popular* if it is more popular than all other matchings M'. It is easy to see that each instance can admit at most one strongly popular matching. Now we can define the problems of existence and verification for a strongly popular matching.

3D-CYCLIC STRONGLY POPULAR MATCHING WITH INCOMPLETE LISTS (3DSPMI)

**Input:** Sets *A*, *B*, *C* with a cyclic preference structure. **Question:** Does a strongly popular matching exist?

3D-CYCLIC STRONGLY POPULAR MATCHING VERIFICATION WITH INCOMPLETE LISTS (3DSPMVI)

**Input:** Sets *A*, *B*, *C* with a cyclic preference structure and a matching *M*.

Question: Is M strongly popular?

If we want to indicate that the preference lists are complete, i.e., every agent in *A* ranks all agents in *B*, every agent in *B* ranks all

	Existence		Verification	
	incomplete	complete	incomplete	complete
Popularity	NP-h. Thm 4.1	?	NP-c. Thm 4.2	?
Strong Popularity	NP-h. Thm 4.3	?	NP-c. Thm 4.4	?
$(A \cup B)$ -Popularity	NP-h. Thm 4.5	$\in P$ Thm 4.6	?	$\in P$ Thm 4.6

Table 1: Overview of the complexity results shown in Section 4. The columns refer to the cases with incomplete and complete lists, respectively. Question marks denote open problems—these are briefly discussed in Section 6.

agents in C, and every agent in C ranks all agents in A, we omit the I from the end of the problem name.

 $(A \cup B)$ -Popularity. Our last optimality concept relies on a recent real application, described by Bloch et al. [7] who analyzed the Paris public housing market. In their work, A consists of various housing institutions such as the Ministry of Housing, B is the set of households looking for an apartment, and finally, C contains the social housing apartments that are to be assigned to these households. Institutions have preferences over household-apartment pairs, and households rank the available apartments in their order of preference. One characteristic feature of this application is that apartments are treated as objects without preferences, because they should be matched through the institutions.

Here we will study a restricted variant, listed as one of the three most typical interpretations of the institutions' preferences by Bloch et al. [7]: institutions have preferences directly over the households, no matter which apartment they are matched to. This problem setting translates into a 3-dimensional matching instance, where agents in *C* only have the constraint to be matched to an acceptable agent from *A*, while classes *A* and *B* submit preferences over acceptable agents in classes *B* and *C*, respectively. While Bloch et al.[7] focused on the existence of a Pareto optimal solution, here we define popularity for such instances.

Matching *M'* is  $(A \cup B)$ -more popular than matching *M* if

$$\sum_{\boldsymbol{x}\in(A\cup B)}\operatorname{vote}_{\boldsymbol{x}}(M',M)\geq 1,$$

i.e., if M' would win against M in a head-to-head election where only agents in  $(A \cup B)$  are allowed to vote. Analogously, we call a matching M  $(A \cup B)$ -popular if there is no matching that is  $(A \cup B)$ -more popular than M. This definition tallies the votes of each household and institution, but treats apartments as objects. To overcome the technical difficulty of one institution handling more than one apartment and to give a vote to the institution in the decision over each apartment, we can simply clone the institutions as many times as many apartments they oversee.

#### 2.3 Our Contribution

We provide structural results and a complexity analysis of the aforementioned popular matching problems. First we show in Section 3 that several implications from the 2-sided matching world do not hold. In 3 dimensions, stable matchings are not necessarily popular and strongly popular matchings are not necessarily stable.

Then in Section 4 we turn to the complexity of verifying and computing a popular, strongly popular, or  $(A \cup B)$ -popular matching when lists are complete, and show that the defined verification

Main Track

$a_1: \underline{b_1}, \underline{b_2}, b_3$	$a_2$ : $b_3$ , $\underline{b_2}$ , $b_1$	$a_3: \boldsymbol{b_1}, \underline{b_3}, b_2$
$b_1$ : $c_2$ , $\underline{c_1}$ , $c_3$	$b_2$ : $c_3$ , $\underline{c_2}$ , $c_1$	$b_3: \underline{c_3}, c_2, c_1$
$c_1$ : $a_2$ , $\underline{a_1}$ , $a_3$	$c_2: \underline{a_2}, a_1, a_3$	$c_3: a_1, \underline{a_3}, a_2$

Figure 1: Compact representation of the preferences in Lemma 3.1. Agent  $a_1$  has the preference list  $b_1 > a_1$   $b_2 > a_1$   $b_3$ . The triples in M are underlined. Bold font denotes the more popular matching M'.

and search problems for all variants except  $(A \cup B)$ -popularity verification are NP-hard. We complement these results with positive ones for  $(A \cup B)$ -popularity with complete lists. Table 1 summarizes our results.

Following this we investigate instances derived from master lists in Section 5, and show that in general for 3-master lists and 2-master lists popular matchings do not exist. Finally, in Section 6 we list some interesting problems that are left open by this work. Our hardness proofs have been delegated to the Appendix.

#### **3 STRUCTURAL RESULTS**

As a first step, we investigate the relations between stability and popularity. In the traditional stable marriage and roommates problems, stable matchings form a subset of popular matchings [19]. Moreover, if a strongly popular matching exists, then it must be the unique popular matching and also the unique stable matching [4].

First we show that in 3 dimensions, neither kind of stability implies popularity by presenting an instance with a strongly stable matching that is not popular.

LEMMA 3.1. There is an instance I of 3DPMI with a matching M such that M is strongly stable but not popular.

**PROOF.** Consider the preference profiles depicted in Figure 1. First we prove that the matching  $M = \{(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)\}$  is strongly stable. As we observed in Section 2.2, at least two agents in a weakly blocking triple must improve their match. There are only 6 possible improvements to M:  $b_1$  switches to  $c_2$ ,  $c_1$  switches to  $a_2$ ,  $a_2$ switches to  $b_3$ ,  $b_2$  switches to  $c_3$ ,  $a_3$  switches to  $b_1$ , and finally,  $c_3$ switches to  $a_1$ . It is easy to check that no two of these will keep the third agent involved at least as satisfied as she is in M.

However, matching  $M' = \{(a_1, b_2, c_3), (a_2, b_3, c_1), (a_3, b_1, c_2)\}$  is more popular, since all agents except  $\{a_1, b_3, c_2\}$  prefer it to M.  $\Box$ 

Secondly we show that for 3-dimensional instances, even strong popularity does not imply weak stability.

LEMMA 3.2. There is an instance I of 3DSPMI with a matching M such that M is strongly popular but not weakly stable.

**PROOF.** Consider the preference profiles depicted in Figure 2 and the matching  $M := \{(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)\}$ . As can be easily seen, M is not weakly stable, since  $(a_1, b_2, c_3)$  is a strongly blocking triple.

Matching *M* is however strongly popular. Assume indirectly that there is a matching *M'* such that *M* is not more popular than *M'*. The only three agents who can possibly improve are  $a_1, b_2$ , and  $c_3$ ,

$a_1: \boldsymbol{b}_2, \underline{b_1}, b_3$	$a_2: \underline{b_2}, b_3, b_1$	$a_3: \underline{b_3}, b_2, b_1$
$b_1: \underline{c_1}, c_2, c_3$	$b_2$ : $c_3$ , $\underline{c_2}$ , $c_1$	$b_3$ : $\underline{c_3}$ , $c_2$ , $c_1$
$c_1: a_1, a_2, a_3$	$c_2: a_2, a_1, a_3$	$c_3: a_1, a_3, a_2$

Figure 2: Representation of the preferences in Lemma 3.2. The triples in M are underlined and the strongly blocking triple is in **bold**.

because the remaining 6 agents are matched to their first choice. If all three of them improve in M', then  $(a_1, b_2, c_3) \in M'$ , and all possible matchings for the remaining 6 agents match at least 4 of them to an agent who is not their top choice. The other possibility is that at least one of  $a_1, b_2$ , and  $c_3$  remains in the same triple in M' as she was in M. Due to symmetry, we can assume without loss of generality that this agent is  $a_1$ , and thus,  $(a_1, b_1, c_1) \in M'$ . The only agent who can improve from this point on is  $b_2$ , and she must switch to  $c_3$ . This M' can be only finished by taking  $(a_2, b_2, c_3), (a_3, b_3, c_2) \in M'$  or by taking  $(a_3, b_2, c_3), (a_2, b_3, c_2) \in M'$  or by taking  $(a_3, b_2, c_3), (a_2, b_3, c_2) \in M'$  and thus were in M. Thus M is strongly popular.

In the appendix, we also give two examples that these results also hold for non-cyclic preferences. Our third result shows that in an instance with complete lists, a strongly popular matching can only be blocked by strongly blocking triples.

LEMMA 3.3. In an instance I of 3DSPM, a strongly popular and weakly stable matching M is also strongly stable.

PROOF. Consider a triple t = (a, b, c) and assume that t weakly blocks M. Since M is weakly stable, one of the three agents needs to have the same partner in t and in M. Without loss of generality we assume that this agent is b, and thus b and c were matched in M as well. Let  $(a, \beta, \gamma), (\alpha, b, c) \in M$  be triples in M. Since (a, b, c)is a weakly blocking triple to M, we know that  $a >_c \alpha$  and  $b >_a \beta$ . The matching  $M' = M \setminus (a, \beta, \gamma) \setminus (\alpha, b, c) \cup (\alpha, \beta, \gamma) \cup (a, b, c)$ leads to at least two agents, a and c, preferring M' to M, and at most two agents,  $\alpha$  and  $\gamma$ , preferring M to M'. This contradicts the assumption that M was strongly popular.

# 4 NP-HARDNESS RESULTS

In this section we prove hardness for all our problems with incomplete lists, except  $(A \cup B)$ -popularity verification. We also show that  $(A \cup B)$ -popularity can be verified and an  $A \cup B$ -popular matching can be found in polynomial time if preference lists are complete. For a structured summary of these results, please consult Table 1.

# 4.1 Popularity

We start by showing that it is NP-hard to determine whether an instance with incomplete lists admits a popular matching. For this we use a restricted, but still NP-complete variant of 3SAT, known as (2,2)-E3-SAT [3].

#### (2,2)-E3-SAT

Input: A set X of variables and a set C of clauses of size exactly 3 such that each variable appears in exactly two clauses in positive form and in exactly two clauses in negative form.

**Question:** Is there a satisfying assignment for *C*?

First, we present an instance of 3DSPM and show that it admits no popular matching. This instance will come handy later, in the proof of Theorem 4.1.

**Observation 4.1.** Any instance *I* of 3DSPMI obtained by removing any positive number of agents from an instance of 3DSPMI, which contains three agents per type and is derived from a 3-master list, admits a popular matching.

Using this we can now show the NP-hardness of deciding whether a popular matching exists.

THEOREM 4.1. It is NP-hard to decide whether a 3DPMI instance admits a popular matching, even if each agent finds four other agents acceptable. This holds even if the preferences are derived from a 2master list.

PROOF. We reduce from (2,2)-E3-SAT.

*Construction.* Let  $X = \{x_1, ..., x_n\}$  be the set of variables and *C* be the set of clauses. We first define the sets of agents A, B, C of our 3DMPI instance. For each clause  $\varphi = \{x_i, x_j, x_k\}$ , where  $x_i$  might be in either positive or negative form, we add nine agents

- three clause agents  $a_i^{\varphi}, a_i^{\varphi}, a_k^{\varphi}$  in A;
- three dummy agents  $b_1^{\varphi}, b_2^{\varphi}, b_3^{\varphi}$  in B; three dummy agents  $c_1^{\varphi}, c_2^{\varphi}, c_3^{\varphi}$  in C.
- For each variable  $x_i \in X$  we include twelve variable agents
- two agents  $a_1^i, a_2^i$  in A;
- six agents b<sup>i</sup><sub>1</sub>, b<sup>i</sup><sub>2</sub>, b<sup>i,+</sup><sub>1</sub>, b<sup>i,-</sup><sub>1</sub>, b<sup>i,+</sup><sub>2</sub>, b<sup>i,-</sup><sub>2</sub> in B;
  four agents c<sup>i,+</sup><sub>1</sub>, c<sup>i,-</sup><sub>1</sub>, c<sup>i,+</sup><sub>2</sub>, c<sup>i,-</sup><sub>2</sub> in C.
- Next we define our preference lists.
- For any clause  $\varphi$  and any clause agent  $a_i^{\varphi}$  such that  $x_i$  appears is positive form in  $\varphi$ , we define the preference list to be  $b_1^{i,+} > b_2^{i,+} > b_1^{\varphi} > b_2^{\varphi} > b_3^{\varphi}$ .
- For any clause  $\varphi$  and any clause agent  $a_i^{\varphi}$  such that  $x_i$  appears is negative form in  $\varphi$ , we define the preference list to be  $b_1^{i,-} > b_2^{i,-} > b_1^{\varphi} > b_2^{\varphi} > b_3^{\varphi}.$ • For any  $b_m^{\varphi}$  with  $m \in \{1, 2, 3\}$  we add the list  $c_1^{\varphi} > c_2^{\varphi} > c_3^{\varphi}.$
- Lastly for any  $c_m^{\varphi}$ , with  $x_i, x_j$ , and  $x_k$  being the variables that appear either in positive or negative form in  $\varphi$  such that i < j < k, we add the list  $a_i^{\varphi} > a_j^{\varphi} > a_k^{\varphi}$ .

Note that this implies that all the clause and dummy agents belonging to one clause form a sub-instance derived from a 3-master list. Thus following Theorem 5.1 and Observation 4.1, in any popular matching at least one of the clause agents needs to be matched to a non-dummy agent.

Next for the variable gadget of any variable  $x_i$  we define the following preference lists.

- Agent a<sup>i</sup><sub>1</sub> receives the preference list b<sup>i</sup><sub>2</sub> > b<sup>i</sup><sub>1</sub>.
  Agent a<sup>i</sup><sub>2</sub> receives the preference list b<sup>i</sup><sub>1</sub> > b<sup>i</sup><sub>2</sub>.





Figure 3: The figure represents the clause gadget in the proof of Theorem 4.1. The vertices denote the agents, with vertices of the same color belonging to the same class. Solid edges leaving a vertex represent the first choice of the corresponding agent, whereas dotted edges represent the second choice.

- Agent b<sup>i</sup><sub>1</sub> receives the preference list c<sup>i,-</sup><sub>2</sub> > c<sup>i,+</sup><sub>2</sub>.
  Agent b<sup>i</sup><sub>2</sub> receives the preference list c<sup>i,+</sup><sub>1</sub> > c<sup>i,-</sup><sub>1</sub>.
  Agent b<sup>i,+</sup><sub>1</sub> receives the preference list c<sup>i,+</sup><sub>1</sub>.

- Agent  $b_1^{i,-}$  receives the preference list  $c_1^{i,-}$ .
- Agent b<sup>i,+</sup><sub>2</sub> receives the preference list c<sup>i,+</sup><sub>2</sub>.
  Agent b<sup>i,-</sup><sub>2</sub> receives the preference list c<sup>i,-</sup><sub>2</sub>.

Furthermore, we call  $\varphi^+, \psi^+$  the clauses where  $x_i$  appears in positive form, and  $\varphi^-, \psi^-$  the clauses where  $x_i$  appears in negative form and turn to the preferences of the variable agents in C.

- For the agent  $c_1^{i,+}$  we add the preference list  $a_i^{\varphi^+} > a_i^{\psi^+} > a_i^i$ .
- For the agent  $c_1^{i,-}$  we add the preference list  $a_i^{\varphi^-} > a_i^{\psi^-} > a_2^i$ .
- For the agent  $c_2^{i,+}$  we add the preference list  $a_i^{\varphi^+} > a_i^{\psi^+} > a_2^i$ .
- For the agent  $c_2^{i,-}$  we add the preference list  $a_i^{\varphi^-} > a_i^{\psi^-} > a_1^i$ .

Note that in this construction the relative order of the preferences in B and C is the same, thus the preferences are subsets of an instance derived from a 2-master list. For a representation of the construction in the variable gadget, see Figure 3.

 $\Rightarrow$ . We first assume that the (2,2)-E3-SAT instance is satisfiable and  $\Phi$  is a satisfying assignment. We now show how to construct a popular matching M.

• For any variable  $x_i$  that is set to true in  $\Phi$  and appears in positive form in the clauses  $\varphi$  and  $\psi$  we include the triples  $(a_2, b_2, c_1^{i,-}), (a_1, b_1, c_2^{i,-}), (a_i^{\varphi}, b_2^{i,+}, c_2^{i,+}), \text{ and } (a_i^{\psi}, b_1^{i,+}, c_1^{i,+}).$ 

- If  $x_i$  is set to false in  $\Phi$  and appears in negative form in the clauses  $\varphi$  and  $\psi$  we include the triples  $(a_2, b_1, c_2^{i,+}), (a_1, b_2, c_1^{i,+}), (a_i^{\varphi}, b_2^{i,-}, c_2^{i,-}), \text{ and } (a_i^{\psi}, b_1^{i,-}, c_1^{i,-}).$ • For any clause  $\varphi$  where two variables  $x_i, x_j$  are unsatisfied,
- we add the triples  $(a_i^{\varphi}, b_1, c_1)$  and  $(a_i^{\varphi}, b_2, c_2)$ .
- For any clause  $\varphi$  with one variable  $x_i$  unsatisfied we add the triple  $(a_i^{\varphi}, b_1, c_1)$ .

Now all we are left to do is to show that M is popular. First, we already know from Lemma 4.1 that each matching in each clause gadget is popular if one does not match an additional  $a_i^{\varphi}$  to some agent in the clause gadget. If  $a_i^{\varphi}$  would get matched to some agent in the clause gadget, the matching will get worse for  $a_i^{arphi}, b_{1/2}^{i,\pm}$ , and  $c_{1/2}^{i,\pm}$  (depending on how the matching is constructed), while at most three agents in the clause gadget can improve. Furthermore any perfect matching M' in the variable gadget is not more popular, since switching  $a_1^i$  or  $b_1^i$  to M' increases the rank of one of them, while it decreases the rank of the other one compared to M. If the matching M' in the variable gadget is not perfect, at least two of the  $a_{1/2}^i$  or  $b_{1/2}^i$  are now unmatched and matching  $c_{1/2}^{i,\pm}$  to any variable gadget would also unmatch two of the variable dummies, leading to M being preferred by at least one more agent over M'. Finally matching agents  $a_i^{\varphi}$  and  $a_i^{\psi}$  to the respective other  $b_{1/2}^{\pm}$  would make two agents happier and two agents unhappier, thus also not leading to a more popular matching. Therefore in any matching M' for any agent who prefers M' to M there is at least one (unique) agent who prefers M to M', which implies that M is popular.

 $\Leftarrow$ . Next assume that we are given a popular matching *M*. Now we make two observations.

- For any clause  $\varphi$ , at least one agent  $a_i^{\varphi}$  has to be matched to her variable gadget, since otherwise following Theorem 5.1 the matching could not be popular, because the agents in the clause gadget are all derived from a master list.
- Now assume that there is a variable gadget where both at least one of  $b_1^{i,-}$  and  $b_2^{i,-}$  as well as at least one of  $b_1^{i,+}$  and  $b_2^{i,+}$ are matched to clause gadgets. Then without loss of generality  $a_1^i$  and  $b_1^i$  are unmatched in M. Since M is popular, it has to be maximal and thus  $c_1^{i,-}$  is matched to some  $a_i^{\varphi}$  and two agents  $b_k^{\varphi}, c_k^{\varphi}$  are unmatched. Taking the triples  $(a_1^i, b_1^i, c_1^{i,-})$ and  $(a_i^{\varphi}, b_k^{\varphi}, c_k^{\varphi})$  improves the matching for  $a_1^i, b_1^i, b_k^{\varphi}, c_k^{\varphi}$  and makes it worse for  $c_1^{i,-}, b_1^{i,-}, a_i^{\varphi}$ . Thus *M* could not have been popular.

Therefore for each variable, M matches either only  $b_1^{i,-}$  and  $b_2^{i,-}$ or  $b_1^{i,+}$  and  $b_2^{i,+}$  to their respective clause gadgets and since each clause gadget has to be matched, this implies that we can construct a satisfying assignment. 

Further, we can show that 3DPMVI is computationally intractable as well.

THEOREM 4.2. It is NP-complete to decide whether a given 3DPMVI instance with a matching M admits a matching that is more popular than M. This holds even if the preferences are derived from a 1-master list.

#### 4.2 Strong Popularity

Next we show that it is also NP-hard to find a strongly popular matching and to verify whether a given matching is strongly popular. For this we reduce from the problem of finding a perfect matching in a 3D-cyclic matching instance without preferences, shown to be NP-complete by Garey and Johnson [20].

PERFECT 3D-CYCLIC MATCHING

Input: Sets A, B, C with cyclic acceptability relations. Question: Does a perfect matching exist?

THEOREM 4.3. It is NP-hard to determine whether a given 3DSPMI instance admits a strongly popular matching. This holds even if the preferences are derived from a 2-master list.

PROOF. We reduce from PERFECT 3D-CYCLIC MATCHING.

Construction. Assume we are given a PERFECT 3D-CYCLIC MATCH-ING instance *I* with sets  $A_0 = \{a_1, ..., a_n\}, B_0 = \{b_1, ..., b_n\}$ , and  $C_0 = \{c_1, \ldots, c_n\}$ . For our 3DSPMI instance, we create a copy of each of the three classes,  $A'_0 = \{a'_1, ..., a'_n\}, B'_0 = \{b'_1, ..., b'_n\}$ , and  $C'_0 = \{c'_1, ..., c'_n\}$  and we set  $A = A_0 \cup A'_0, B = B_0 \cup B'_0$ , and  $C = C_0 \cup C'_0.$ 

Next we turn to the preferences. For each agent  $x \in A_0 \cup B_0 \cup C_0$ let  $n_1^x, \ldots, n_k^x$  be her set of acceptable agents in I in an arbitrary order, such that the relative order between all agents of one class is the same. We take the indices modulo *n* and as such take n + 1 = 1.

- For any  $a_i \in A_0$  we create the preference list  $b'_i >_{a_i} n_1^{a_i} >_{a_i}$
- · · · > $_{a_i} n_k^{a_i}$ . For any  $a_i' \in A_0'$  we create the preference list only consisting
- For any  $b_i \in B_0$  we create the preference list  $n_1^{b_i} >_{a_i} \cdots >_{b_i} n_k^{b_i}$ . For any  $b'_i \in B'_0$  we create the preference list  $c'_i >_{b'_i} c'_{i+1}$ . For any  $c_i \in C_0$  we create the preference list  $n_1^{c_i} >_{c_i} \cdots >_{c_i}$
- $n_k^{c_i}$ . For any  $c_i' \in C_0'$  we create the preference list  $a_i >_{c_i'} a_{i-1}'$ .

 $\Rightarrow$ . First, assume that I admits no perfect matching. We then show that the matching  $M = \{(a_i, b'_i, c'_i) \mid a_i \in A_0\}$  is strongly popular. Let M' be a matching different from M. For any  $i \in [n]$ , we define the vertex set  $X_i = \{a_i, a'_i, M'(a_i), M'(M'(a_i)), b'_i, c'_i\}$  and also  $\operatorname{vote}_i(M') = \sum_{x \in X_i} \operatorname{vote}_x(M', M)$ . By the definition of popularity it holds that  $\operatorname{vote}(M', M) = \sum_{i=1}^n \operatorname{vote}_i(M')$ . Now we can distinguish four cases, based on whether  $a_i$  and  $a'_i$  are unmatched or matched to  $b'_i$ .

- If  $M'(a_i) = b'_i$ , then it holds that  $X_i = \{a_i, a'_i, b'_i, c'_i\}$ . Hence, the matching changes for no member of  $X_i$  and thus  $\operatorname{vote}_i(M') = 0.$
- Similarly, if  $M'(a_i) = a_i$ , then it is easy to see that  $\operatorname{vote}_i(M') \leq -2$ , since  $a_i, b'_i, c'_i$  all get a worse partner (or no partner at all).
- Otherwise if  $M'(a_i) \neq b'_i$  and  $M'(a'_i) = b'_i$ , then  $a'_i, M'(a_i)$ , and  $M'(M'(a_i))$  prefer M' to M, while the rest of  $X_i$  prefers *M* to *M'*, which results in vote<sub>*i*</sub>(*M'*) = 0.
- If  $M'(a_i) \neq b'_i$  and  $M'(a'_i) = a'_i$ , then  $M'(a_i)$  and  $M'(M'(a_i))$  prefer M' to M, while the rest of  $X_i$  prefers *M* to *M'*, which results in vote<sub>*i*</sub>(*M'*)  $\leq -1$ .

Since  $M' \neq M$ ,  $M'(a_i) \neq b'_i$  has to hold for at least one  $a_i \in A_0$ . However, since no perfect matching exists, there has to be an agent  $a_i \in A$  not matched to an agent in  $B_0$ . From this follows that either  $M'(a_i) = a_i$  or  $M(a'_i) = a'_i$  has to hold if this  $a_i$  is matched to  $b'_i$ , which would imply that  $vote_i(M') < 0$ . Therefore, we get that  $\operatorname{vote}(M', M) = \sum_{i=1}^{n} \operatorname{vote}_i(M') < 0$  and thus *M* is strongly popular.

 $\Leftarrow$ . For the other direction, assume that I admits a perfect matching  $M'_0$  and consider the two matchings  $M = \{(a_i, b'_i, c'_i) \mid a_i \in A_0\}$ and  $M' = M'_0 \cup \{a'_i, b'_i, c'_{i+1} \mid a'_i \in A_0\}$ . First, it is easy to see that for any  $i \in [n]$ , all of  $a_i, b'_i, c'_i$  prefer M to M' while  $b_i, c_i$ , and  $a'_i$ prefer M' to M. Thus, neither M nor M' is strongly popular.

Now assume that M' is a matching different from M. Let vote<sub>i</sub> be as in the previous case. If  $M'(a_i) = b'_i$ , then nothing changes for the agents in  $X_i$  and thus vote<sub>i</sub>(M') = 0. If  $M'(a_i) = a_i$ , then we get  $\operatorname{vote}_i(M') \leq -2$  and if  $M'(a_i) \in B_0$ , we get  $\operatorname{vote}_i(M') \leq 0$ . Thus, the matching M' was not strongly popular either and therefore no strongly popular matching exists. 

A slightly modified version of the proof implies that 3DSPMVI is also computationally intractable.

THEOREM 4.4. It is NP-complete to decide whether a given 3DSPMVI instance admits a matching M' such that M is not more popular than M'. This holds even if the preferences are derived from a 2master list

#### **4.3** $(A \cup B)$ -Popularity

Finally we turn to the application-motivated variant of our problem and show that computing a matching that is  $(A \cup B)$ -popular, i.e., it does not lose any head-to-head election where only agents in  $(A \cup B)$  can vote, is NP-hard as well. We reduce from the problem of finding a popular matching in a bipartite graph with one side having strict preferences and the other side either having a tie or strict preferences, shown to be NP-complete by Cseh et al. [12].

POPULAR MATCHING WITH ONE-SIDED TIES

Input: A bipartite graph  $G = (U \cup W, E)$ , for each  $u \in$ U a strict preference list over its neighbors in W, for each  $w \in W$  either a strict preference list or a preference list containing a single tie over its neighbors in U.

**Question:** Does G admit a popular matching?

THEOREM 4.5. It is NP-hard to decide whether a 3DPMI instance admits an  $(A \cup B)$ -popular matching.

Interestingly, the problem becomes easy with complete lists.

**THEOREM 4.6.** Both verifying  $(A \cup B)$ -popularity and computing an  $(A \cup B)$ -popular matching in a 3DPM instance I can be done in linear time.

PROOF. From  $\mathcal{I}$  we construct two house allocation instances with one-sided preferences,  $I_A := (A, B, (\succ_a)_{a \in A})$  and  $I_B :=$  $(B, C, (\succ_b)_{b \in B})$ . We will show that I admits a popular matching if and only if both  $I_A$  and  $I_B$  admit a popular matching.

First assume that I admits a popular matching M. Using this we now construct the two matchings,  $M_A := \{(a, M(a) \mid a \in A\} \text{ in }$  $I_A$  and  $M_B := \{(b, M(b) \mid b \in B\}$  in  $I_B$ . Without loss of generality assume that  $M_A$  is not popular in  $\mathcal{I}_A$  and let  $M'_A$  be the more popular matching. It is easy to see that the matching  $\{a, M'_A(a), M(M'_A(a)) \mid$  $a \in A$  is also more popular than M in I.

If  $I_A$  and  $I_B$  admit popular matchings  $M_A$  and  $M_B$ , respectively, then the matching  $\{a, M_A(a)M_B(M_A(a)) \mid a \in A\}$  is clearly popular in I.

This immediately yields a linear time algorithm for finding an  $(A \cup B)$ -popular matching and verifying whether a matching is  $(A \cup B)$ -popular matching is  $(A \cup B)$ -popular matching and verifying whether a matching is  $(A \cup B)$ -popular matching is B)-popular matching, since popular matchings in house allocation instances can be found and verified in linear time as shown by Abraham et al. [1]. 

We remark that in the housing application [7], set A is the set of housing institutions, and therefore, it is typically of much smaller cardinality than B and C. The agents of A thus need to be copied multiple times to create an instance with |A| = |B| = |C|, which we assume in our model. However, since vertices in C do not vote, their preferences can be set arbitrarily on these copies, while the preferences of each copy are identical to the preferences of the housing institution it represents. Even though the instance created in our hardness reduction does not quite fulfill the property of having large groups of agents in A with identical preference lists, its agents can be grouped into pairs and triples so that the preferences within these smaller groups are similar-they are even identical for exactly half of the list. This observation follows from the preference list structure in the hardness proof of Cseh et al. [12] and the fact that in our reduction the lists of the agents in A are copied from the lists of the agents in U.

# 5 MASTER LISTS

Now we turn to studying popular matchings in instances with preferences derived from master lists. Examples of real-life applications of master lists occur in resident matching programs [5], dormitory room assignments [39], cooperative download applications such as BitTorrent [2], and 3-sided networking services [13]. Even though the presence of a master list usually simplifies stable matching problems and warrants that a solution exists and it is easy to find [24, 37], here we show that for 3-dimensional popular matchings, instances with master lists tend to admit no popular matching at all. This observation is aligned with what has already been shown by the Condorcet paradox, possibly the first example for the non-existence of a weak majority winner.

#### 5.1 3-Master List

First we show that an instance derived from a 3-master list has no popular matching if there are at least 3 agents per class.

THEOREM 5.1. A 3DPM instance derived from a 3-master list has no popular matching if  $n \ge 3$ .

**PROOF.** Let M be a maximal matching (otherwise the matching is trivially not popular) and let  $(a_i, b_i, c_i), (a_j, b_j, c_j), (a_k, b_k, c_k) \in M$ be three disjoint triples. Without loss of generality we can assume that  $a_i >_c a_j >_c a_k$ . We will now distinguish two cases. First assume that one of

- $b_i >_a b_k >_a b_j$ ;
- $b_j >_a b_i >_a b_k;$   $b_k >_a b_j >_a b_i$

holds, i.e., the ranking of the three agents in *B* is 'reversed' compared to the ranking of the agents in *A* they are matched to. Then the matching *M*' resulting from replacing the triples  $\{(a_i, b_i, c_i), (a_j, b_j, c_j), (a_k, b_k, c_k)\}$  by the triples  $\{(a_k, b_i, c_i), (a_i, b_j, c_j), (a_j, b_k, c_k)\}$  in *M* is more popular, since two of  $a_i, a_j, a_k$  (as can be seen by the two  $>_a$ ) and  $c_j, c_k$  prefer *M*', while only two agents are against *M*'.

For the second case, we can assume that one of

- $b_j >_a b_k >_a b_i$ ;
- $b_k >_a b_i >_a b_j;$
- $b_i >_a b_j >_a b_k$

holds, i.e., the ranking of the agents in *B* is cyclically shifted from the ranking of the agents in *A*. Now we construct matching *M'* by replacing the triples  $\{(a_i, b_i, c_i), (a_j, b_j, c_j), (a_k, b_k, c_k)\}$  by the triples  $\{(a_i, b_k, c_j), (a_j, b_i, c_k), (a_k, b_j, c_i)\}$  in *M*. Since *B* is derived from a master list, two agents in *A*,  $c_j$  and  $c_k$ , and the agent in *B* who was previously matched to the worst of the three agents in *C* prefer their partner in *M'* to their partner in *M*. Thus *M'* is more popular than *M*, and therefore no popular matching exists.

For the sake of completeness, we remark that for  $n \leq 2$ , all perfect matchings in a 3DPM instance derived from a 3-master list are trivially popular.

Interestingly, Escamocher and O'Sullivan [16] were able to show that instances derived from a 3-master list have exponentially many stable matchings, so Theorem 5.1 shows a stark contrast between stability and popularity in three-dimensional cyclic matching.

#### 5.2 2-Master List

In the spirit of Theorem 5.1, we can also show that even if only the preferences in *A* and *B* are derived from a master list, no popular matching exists if there are more than four agents in each of the three classes.

THEOREM 5.2. In an instance of 3DPM derived from a 2-master list, no popular matching exists if  $n \ge 5$ .

PROOF. Without loss of generality we can assume that classes A and B each have a master list, and consider a matching M. Let the rankings for B and C be  $b_1 >_a \cdots >_a b_n$  and  $c_1 >_b \cdots >_b c_n$ , respectively. For any  $\gamma \in \{a, b, c\}$  let  $\gamma_i \oplus 1 = \gamma_{(i-1 \mod n)}$ . Intuitively, the  $\oplus$  operation takes one step up on the list of agents in a class, and if it is applied to the first agent in the class, then it jumps to the last agent. Now we compare the matching that consists of triples in the form  $(a_i, M(a_i) \oplus 1, M(M(a_i) \oplus 1) \oplus 1))$ , i.e., we cyclically shift up the agents in B and C. In this operation, all agents in A except the agent matched to  $b_1$ , and all agents in B except the agent matched to  $c_1$  improve. Thus at least 2n - 2 agents improve and at most n + 2 agents receive a worse partner than in M. Therefore if  $n \ge 5$ , then M was not popular.

For the sake of completeness we elaborate on the case of instances with  $n \le 4$ . Firstly, for  $n \le 2$ , it is easy to see that there is at least one popular matching in each 3DPM instance derived from a 2-master list. Instances with n = 3 and n = 4 can be yes- and no-instances as well. Since the input size is constant, even iterating through all matchings and checking each of them for popularity delivers a polynomial-time algorithm to decide whether a given instance admits a popular matching.

# 5.3 1-Master List

A result analogous to Theorems 5.1 and 5.2 is unlikely to exist if only one agent class is equipped with a master list. Instead, we give a characterization for strongly popular matchings in instances derived from a 1-master list with complete lists. This characterization also immediately gives us a linear time algorithm to find and verify a strongly popular matching in these instances. The analogous questions for popularity are discussed as open problems in Section 6.

THEOREM 5.3. In an 3DSPM instance derived from a 1-master list, a matching is strongly popular if and only if all agents without a master list are matched to their top choice.

# **6 OPEN PROBLEMS**

Our work leaves three important questions open. The first, related to our results in Section 4, is the complexity of our problems with regard to complete preference lists. The technique of introducing so-called 'boundary dummy-agents' of Lam and Plaxton [30] for showing hardness with complete lists for the stable matching problem does not seem to be applicable for popularity, since the presence of blocking edges if an agent is matched below her 'boundary' does not restrict the set of popular matchings. Thus, in order to reduce either from the problem with incomplete lists or from a separate problem altogether, a new technique might be needed.

Related to this is also the complexity of verifying whether a matching is  $(A \cup B)$ -popular with incomplete lists. Due to the inherent hardness of computing weight-optimal or even perfect matchings in 3 dimensions, we conjecture that this problem is NP-complete as well. This problem seems to have an inherent relation to the NP-complete Multiple Choice Matching problem, see [20], which might be another indicator for the hardness of the problem.

The third open problem, in case the problem of finding a popular matching with complete lists turns out to be intractable, is that of finding a popular matching in a 3DPM derived from a 1-master list. Here, as opposed to instances derived from 2- and 3-master lists, popular matchings can exist. Interestingly enough the structure of these popular matchings seems to be quite limited, since in any situation the agents with the master list could be 'shifted up' to generate a matching that is more popular for at least n - 1 agents, similarly as we argued in the proof of Theorem 5.2. This might lead to results similar to the classification of popular matchings in house allocation instances by Abraham et al. [1]-for instance, if there is a perfect matching *M* and agents  $b_i, b_k \in B$  and  $c_j, c_l \in C$ with  $l > k \ge j > i$  such that  $c_j >_{b_i} M(b_i)$  and  $c_l >_{b_i} M(b_l)$ , then M is not popular. Other results of this type might pave the path to a full classification of popular matchings in these instances. We believe that this sub-problem might be the best starting point for the exploration of tractable subcases of the three-dimensional matching problem.

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