(Initialization) computing an element of S_i, and
(Membership) deciding whether a given size vector is in S_i.

Also, for a given vector x in S_1 or S_2, there is a polynomial time algorithm that finds an allocation whose size vector is equal to x; indeed, we can find such an allocation by computing a clean utilitarian optimal allocation for the profile $P' = (v'_1, \dots, v'_n)$ such that $v'_i(X) = \min\{v_i(X), x_i\}$ for each $i \in N$ and $X \subseteq N$.

Frank and Murota [23, Theorem 5.7] proved that the set of *increasingly maximal elements*[3] of an M-convex set is a matroidal M-convex set. Further, they showed that, in the matroidal M-convex set, an increasingly maximal element that minimizes a linear function can be found in polynomial time if (Initialization) and (Membership) for the M-convex set can be solved in polynomial time. By combining this with the facts that S_2 is a M-convex set and an increasingly maximal element corresponds to a clean Lorenz dominating allocation, we obtain the following lemma.[4]

Lemma 3.5. *The set of size vectors corresponding to clean Lorenz dominating allocations $S^* := \{\mathrm{sv}(A) \mid A \in \mathbf{cLD}\}$ is a matroidal M-convex set. Additionally, for a given weight $w \in \mathbb{R}^N$ a minimum-weight clean Lorenz dominating allocation $\arg\min_{s \in S^*} \sum_{i\in N} w_i s_i$ can be found in polynomial time.*

Since S^* is a matroidal M-convex set, the difference between values of the best and the worst clean Lorenz dominating allocations for each agent is at most one.

Proposition 3.6. $\max_{B,C\in\mathbf{cLD}}(|B_i|-|C_i|) \in \{0,1\}$ *for any $i \in N$.*

The resulting utilities under the SE mechanism are thus invariant under permutations of agent names. Formally, we say that a mechanism is *weakly anonymous* if for any permutation $\sigma\colon [n] \to [n]$, $v_i(A_i) + p_i = v_{\sigma(i)}(A'_{\sigma(i)}) + p'_{\sigma(i)}$ where (A, p) and (A', p') are the outcomes of the mechanism when applied to $P = (v_i)_{i\in[n]}$ and $P_\sigma = (v_{\sigma(i)})_{i\in[n]}$ respectively. Note that this property is weak in the sense that the valuations of the allocated bundles may change.

Theorem 3.7. *The SE mechanism is weakly anonymous.*

Note that $\min_{B\in\mathbf{cLD}} |B_i|$ can be computed in polynomial time for each i using Lemma 3.5, e.g., by setting the weight w as $w_i = 0$ and $w_j = 1$ for all $j \in N \setminus \{i\}$. Hence, the outcome of the SE mechanism can be computed in polynomial time.

Lemma 3.8. *The SE mechanism is polynomial-time implementable.*

Furthermore, the M-convex structure of **cLD** leads to some properties that are useful to prove the truthfulness of the SE mechanism. Due to space limitation, we defer the details to Section 3.1 of the full version [25].

3.2 Envy-freeness of the SE mechanism

Here, we prove that the SE mechanism is envy-free. We remark that the matroidal M-convex structure of **cLD** is not sufficient to prove it because the structure gives no information about the value of a bundle received by anyone other than oneself. Instead, we obtain envy-freeness of the SE mechanism by exploiting Lemma 3.3.

Lemma 3.9. *The SE mechanism is envy-free.*

Proof. Let (A, p) be the pair of clean allocation and subsidy vector returned by the SE mechanism. To obtain a contradiction, suppose that i envies j, i.e., $v_i(A_i)+p_i < v_i(A_j)+p_j$. We separately consider the following three cases: $v_i(A_i) > v_i(A_j)$, $v_i(A_i) < v_i(A_j)$, and $v_i(A_i) = v_i(A_j)$.

Case 1. Suppose that $v_i(A_i) > v_i(A_j)$. This case is impossible since $v_i(A_i) + p_i < v_i(A_j) + p_j$ and $p_i, p_j \in \{0, 1\}$.

Case 2. Suppose that $v_i(A_i) < v_i(A_j)$. As A is a clean allocation, $|A_i|$ must be strictly smaller than $|A_j|$. By the matroid augmentation property, there exists an item $e \in A_j$ such that $v_i(A_i \cup \{e\}) = v_i(A_i)+1$. Let B be the allocation that is obtained from A by moving item e from j's bundle to i's bundle. As $|A_i| < |A_j|$ and A Lorenz dominates B, we have that $|B_i| = |A_i|+1 = |A_j| = |B_j|+1$. Hence, B is also a clean Lorenz dominating allocation. Thus, $\max_{C\in\mathbf{cLD}} |C_i| = |A_i| + 1$ and $\min_{C\in\mathbf{cLD}} |C_j| = |A_j| - 1$, which implies $p_i = 1$ and $p_j = 0$ by Proposition **??**. This contradicts the assumption that i envies j because $v_i(A_i) + p_i = |A_i| + 1 = |A_j| = v_i(A_j) + p_j$.

Case 3. Suppose that $v_i(A_i) = v_i(A_j)$. Note that $|A_i| = v_i(A_i) = v_i(A_j) \le |A_j|$. As $v_i(A_i)+p_i < v_i(A_j)+p_j$, it must be that $p_i = 0$ and $p_j = 1$. Then $|A_j| = \min_{A'\in\mathbf{cLD}} |A'_j| < \max_{k\in N} |A_k|$ because j gets subsidized. Also, $\min_{A'\in\mathbf{cLD}} |A'_i| = |A_i| - 1$ or $|A_i| = \max_{k\in N} |A_k|$ because i gets no subsidy. We observe that $\min_{A'\in\mathbf{cLD}} |A'_i| = |A_i|-1$, because otherwise $\min_{A'\in\mathbf{cLD}} |A'_i| = |A_i| = \max_{k\in N} |A_k| > |A_j|$, and hence $v_i(A_i) = |A_i| > |A_j| \ge v_i(A_j)$, which is a contradiction. As $\{\mathrm{sv}(A') \mid A' \in \mathbf{cLD}\}$ is an M-convex set, there is a clean Lorenz dominating allocation B such that $\mathrm{sv}(B) = \mathrm{sv}(A) - \chi_i + \chi_k$ for some $k \in N$. As A and B are both in **cLD** and hence $\mathrm{sv}^\uparrow(A) = \mathrm{sv}^\uparrow(B)$, we have that $|B_i| + 1 = |A_i| = |B_k| = |A_k| + 1$. Note that $k \ne j$ because $|A_i| \le |A_j|$ by $v_i(A_i) = v_i(A_j)$.

By applying Lemma 3.3 to B and A (note that the roles are interchanged), we obtain a sequence of clean allocations $C^0, C^1, \dots, C^r$ with $k^0, k^1, \dots, k^r$ and $e^1, \dots, e^r$ where $C^0 = A$, $k^0 = k$, $k^r = i$, and $\mathrm{sv}(C^r) = \mathrm{sv}(C^0) + \chi_{k^0} - \chi_{k^r} = \mathrm{sv}(B)$. If $k^t = j$ for some t, then $\mathrm{sv}(C^t) = \mathrm{sv}(A)+\chi_k-\chi_j$ and $|A_k|+1 = |A_i| \le |A_j|$, and hence C^t is a clean Lorenz dominating allocation with $|C^t_j| < |A_j|$. This implies $p_j = 0$, which is a contradiction. Otherwise (i.e., $k^t \ne j$ for all t), we have $C^r_j = A_j$. Then, there exists an element $e \in C^r_j$ such that $v_i(C^r_i \cup \{e\}) = |A_i|$ by $v_i(C^r_i) = |A_i|-1 < |A_i| = v_i(C^r_j)$ and the matroid augmentation property. Thus, the allocation that is obtained from C^r by transferring e from j to i is a clean Lorenz dominating allocation. This also implies that $p_j = 0$, a contradiction. □

3.3 Truthfulness of the SE mechanism

Finally, we prove that the SE mechanism is truthful. In a setting without money, Babaioff et al. [6] proved that a mechanism is truthful if it satisfies *strong faithfulness* and *monotonicity*. We introduce two similar properties that can be applied to a setting with subsidies: subsidized-monotone and subsidized-faithful.

First, the subsidized-faithfulness requires that, if agent i changes her report from v_i to $v_i|_X$, either (a) i receives X and her subsidy

[3] For a given set of vectors, an *increasingly maximal element* is an element such that the smallest entry is as large as possible; within this, the next smallest entry is as large as possible; and so on.

[4] For matroidal valuations, a clean Lorenz dominating allocation is equivalent to a clean utilitarian optimal allocation A such that the smallest entry of $\mathrm{sv}^\uparrow(A)$ is as large as possible; within this, the next smallest entry is as large as possible; and so on [5, 11]. This certifies the equivalence between an increasingly maximal element of S_2 and a clean Lorenz dominating allocation.

does not decrease or (b) i receives a proper subset of X and her subsidy strictly increases. We say that a mechanism is *subsidized-faithful* if

$$v_i(X) + p_i \leq v_i(A'_i) + p'_i \tag{1}$$

for any valuation function $(v_1, \ldots, v_n)$, agent $i \in N$, and subset $X \subseteq A_i$, where (A, p) and (A', p') are the allocations with subsidies returned by the mechanism when agents report $P = (v_1, \ldots, v_i, \ldots, v_n)$ and $P' = (v_1, \ldots, v_i|_X, \ldots, v_n)$, respectively. Note that the strong faithfulness (i.e., $A'_i = X$ instead of (1)) does not hold for the SE mechanism because $A \in$ **cLD** is chosen arbitrarily.

Next, the subsidized-monotonicity means that the (true) utility of an agent is monotone with respect to restriction of her report. Formally, we say that a mechanism is *subsidized-monotone* if the utility of an agent is monotone with respect to the restriction, i.e.,

$$v_i(A_i) + p_i \leq v_i(A'_i) + p'_i$$

for any valuation function $(v_1, \ldots, v_n)$, agent $i \in N$, and subsets $X \subseteq Y \subseteq M$, where (A, p) and (A', p') are the allocations with subsidies returned by the mechanism when agents report $P = (v_1, \ldots, v_i|_X, \ldots, v_n)$ and $P' = (v_1, \ldots, v_i|_Y, \ldots, v_n)$, respectively. The two properties of subsidized-faithfulness and subsidized-monotonicity ensure the truthfulness of a mechanism.

Lemma 3.10. *A mechanism is truthful if it is subsidized-faithful and subsidized-monotone.*

We show that the SE mechanism is subsidized-monotone and subsidized-faithful by the matroidal M-convex structure of **cLD** and the way to distribute subsidies.

Lemma 3.11. *The SE mechanism is subsidized-faithful and subsidized-monotone.*

By combining Lemmas 3.10 and 3.11, we obtain the truthfulness of the SE mechanism.

3.4 Without the free-disposal assumption

In Theorem 3.1, we presented the so-called SE mechanism, which attains truthfulness, utilitarian optimality, and envy-freeness with each agent receiving a subsidy of 0 or 1. In the mechanism's output, however, the allocation may not be complete (i.e., some items may be left unallocated). In some situations, this disposal of items is not ideal. For example, consider a shift scheduling at a call center or a production factory; all shifts must be allocated to employees in order not to stop the operation, even if no one may find that time slot valuable. Another example is the allocation of research papers to reviewers; every paper must be reviewed by a certain number of reviewers irrespective of whether the paper is attractive or not. Unfortunately, the following theorem shows that no mechanism outputs a complete allocation while attaining all the properties of the SE mechanism (i.e., truthfulness, Lorenz domination, and envy-freeness with each agent receiving a subsidy of at most 1).

Theorem 3.12. *If a truthful mechanism is envy-free, and returns a complete Lorenz dominating allocation, it requires a subsidy of $\Omega(m)$, even when there are two agents with binary additive valuations.*

For matroidal valuations, we provide an algorithm that returns a Lorenz dominating allocation and simultaneously attains completeness and envy-freeness with each agent receiving a subsidy of at most 1 while tolerating a violation of truthfulness.

Theorem 3.13. *For matroidal valuations, there is a polynomial-time algorithm for computing an allocation with a subsidy that is complete, utilitarian optimal, and envy-free, with each agent receiving a subsidy of* 0 *or* 1 *and the total subsidy being at most $n - 1$.*

We construct the allocation required in the theorem by extending an arbitrary clean Lorenz dominating allocation $A = (A_1, A_2, \ldots, A_n)$; that is, we initialize A to be the one computed in Step 1 of the SE mechanism. By Theorem 3.1, A then maximizes the utilitarian social welfare $\sum_{i \in N} v_i(A_i)$ and is envy-freeable with a subsidy of at most 1 for each agent. Therefore, we can obtain a desired allocation if we can allocate items in $M \setminus \bigcup_{i \in N} A_i$ while preserving the utilitarian optimality and the bound 1 of the subsidy for each agent. Note that, for binary additive valuations, this task is trivial because an item unallocated in A has a value of 0 for all agents by the utilitarian optimality; hence allocating it to any agent does not cause envy. However, a similar argument does not apply to matroidal valuations, as shown by the following example.

Example 3.14. Let $N = \{1, 2, 3\}$ and $M = \{e_1, e_2, e_3, e_4, e_5\}$ and define the matroidal valuations v_1, v_2, v_3 by $v_1(X) = |X \cap \{e_1, e_2\}|$, $v_2(X) = |X \cap \{e_1, e_2, e_3\}|$, and $v_3(X) = |X \cap \{e_1, e_2, e_3\}| + \min\{1, |X \cap \{e_4, e_5\}|\}$. Then $A = (A_1, A_2, A_3) = \{\{e_1, e_2\}, \{e_3\}, \{e_4\}\}$ is a clean Lorenz dominating allocation. It is not difficult to see that we cannot increase the utility of any agent by allocating e_5, which is currently unallocated. However, if we allocate e_5 to agent 2, the amount $v_3(A_2) - v_3(A_3)$ of envy agent 3 has towards 2 changes from 0 to 1. To eliminate envy for the resultant allocation $A' = (A'_1, A'_2, A'_3) = \{\{e_1, e_2\}, \{e_3, e_5\}, \{e_4\}\}$, we need to pay at least one dollar to agent 2 because her envy towards agent 1 is $v_2(A'_1) - v_2(A'_2) = 1$. Then $v_3(A'_2) + p_2 \geq 3$ while $v_3(A'_3) = 1$, and to eliminate the envy of agent 3 towards agent 2, we must pay at least 2 dollars to agent 3.

We present the *subsidized egalitarian with completion* (SEC) algorithm, which extends any clean Lorenz dominating allocation to a complete allocation while preserving the property that each agent requires at most 1 subsidy.

To describe our algorithm, we introduce the notion of envy graphs. For an allocation A, its envy graph G_A is the complete weighted directed graph whose node set is the agent set N; for each $i, j \in N$, the arc (i, j) has weight $w(i, j) = v_i(A_j) - v_i(A_i)$, which represents the amount of envy of i towards j. This value can be negative if i prefers her bundle to j's bundle. A *walk* Q in G_A is a sequence of nodes $(i_1, i_2, \ldots, i_k)$, and its weight is defined as $w(Q) = \sum_{t=1}^{k-1} w(i_t, i_{t+1})$. A walk is a *path* if all nodes are distinct, and a *cycle* if $i_1, i_2, \ldots, i_{k-1}$ are all distinct and $i_1 = i_k$. The following theorem is a combination of Theorems 1 and 2 in [28].

Theorem 3.15 (Halpern and Shah [28]). *For any allocation $A = (A_1, \ldots, A_n)$ and any nonnegative real $q \in \mathbb{R}_+$, the following two conditions are equivalent:*

- *A is envy-freeable with a subsidy of at most q for each agent.*
- *G_A has neither a positive-weight cycle nor a path with a weight larger than q.*

When these conditions hold, if we set p_i as the maximum weight of any path starting at i in G_A for each $i \in N$, then (A, p) is envy-free.

Note that Theorem 3.15 is shown for general valuations. In the case of matroidal valuations, which are integer-valued, each arc in G_A has an integer weight.

Subsidized Egalitarian with Completion

1. Allocate items according to an arbitrarily chosen $A \in$ **cLD**.
2. For each unallocated item $e \in M \setminus \bigcup_{i \in N} A_i$, do as follows:
   (a) Take an agent i arbitrarily.
   (b) Let $A^{i,e} := (A_1, \dots, A_i \cup \{e\}, \dots, A_n)$. If $G_{A^{i,e}}$ has a positive-weight path ending at i, then take such a path P_i arbitrarily, update i by the initial agent of P_i, and repeat (b). Else, go to (c).
   (c) Update $A \leftarrow A^{i,e}$ (i.e., $A_i \leftarrow A_i \cup \{e\}$).
3. Give 1 subsidy to each agent $i \in N$ such that the envy graph G_A has a path of weight 1 starting at i.

Lemma 3.16. *The following conditions hold throughout the SEC algorithm: (i) $A = (A_1, \dots, A_n)$ is utilitarian optimal, and (ii) G_A has neither a path of weight more than 1 nor a positive-weight cycle.*

By condition (ii) in Lemma 3.16 and Theorem 3.15, the allocation with a subsidy returned by the SEC algorithm is envy-free, with each agent receiving a subsidy of 0 or 1. Furthermore, there is at least one agent $i \in N$ such that G_A has no path of weight 1 starting at i (since otherwise there exists a positive-weight cycle in G_A, which contradicts (ii)). Thus, the total subsidy is at most $n - 1$. By the algorithm and condition (i) in Lemma 3.16, the allocation is complete and utilitarian optimal. To complete the proof of Theorem 3.13, we now estimate the time complexity. The following claim guarantees that the algorithm does not fall into an infinite loop at Step 2 (b).

Lemma 3.17. *In Step 2, for each item e, any agent is chosen as i in (b) at most once, and hence (b) is repeated at most n times.*

Note that Step 1 is the same as that of the SE mechanism. Steps 2 and 3 can be computed by the method used by Halpern and Shah [27], i.e., by applying the Floyd–Warshall algorithm [21, 43]. Thus, the algorithm runs in polynomial time.

4 SUPERADDITIVE VALUATIONS

In this section, we consider a class of valuations that do not possess the substitution property, namely, a class of superadditive valuations. We provide a truthful mechanism that achieves envy-freeness and utilitarian optimality, with each agent receiving a subsidy of at most m. Although the upper bound of the subsidy seems too large, we show that this amount of subsidy is essentially required.

Holmström [29] proved that when the set V of valuations satisfies the *convexity* condition, the Groves mechanisms are the only utilitarian optimal and truthful mechanisms. This result is carried to superadditive valuations, which satisfy convexity. Moreover, for superadditive valuations, some rules of the Groves mechanisms, including the VCG mechanism, satisfy envy-freeness [36].

We require that the subsidy for each agent must be non-negative; to fulfill this goal, we can use the following mechanism:

VCG with an upfront subsidy m

1. Allocate items according to an arbitrarily chosen A^* in $\arg\max_A \sum_{j \in N} v_j(A_j)$.
2. Give $m - \left(\max_A \sum_{j \neq i} v_j(A_j) - \sum_{j \neq i} v_j(A^*_j)\right)$ subsidy to each $i \in N$.

Note that the second term of the subsidy (i.e., $\max_A \sum_{j \neq i} v_j(A_j) - \sum_{j \neq i} v_j(A^*_j)$) is equal to the standard VCG payment. Thus, this mechanism is equivalent to the following mechanism; first, each agent obtains an upfront subsidy m (where $m \geq v_i(M)$ holds $\forall i \in N$). Then, items are allocated using the standard VCG, where each agent pays the VCG payment from the upfront subsidy.

Theorem 4.1. *For superadditive valuations, the VCG with an upfront subsidy m is truthful, utilitarian optimal, and envy-free, and each subsidy is in $[0, m]$.*

For additive valuations, a utilitarian optimal allocation can be computed in polynomial time by allocating each item to the agent who likes the most. Hence, the above mechanism is polynomial-time implementable for a class of additive valuations. However, generally, the problem is NP-hard for superadditive valuations (see, e.g., [34, Proposition 11.5]).

We discuss in Section 4 of the full version [25] that unlike the SE mechanism, Groves mechanisms cannot achieve a limited amount of subsidy even when valuations are binary additive. Here, we define a *Groves mechanism* to be a generalization of a VCG with an upfront subsidy m, where we replace the right term of the subsidy rule in Step 2 with an arbitrary function h that only depends on valuations of the other agents $j \neq i$.

Now, is there any other mechanism that can reduce the amount of subsidy while achieving envy-freeness and utilitarian optimality? The next theorem shows that we need the amount of m for each agent to achieve such objectives, even when valuations are additive.

Theorem 4.2. *For any $\epsilon > 0$, if a mechanism is envy-free and utilitarian optimal, it requires a subsidy of $m(n-1) - \epsilon$ in total, even when n agents have additive valuations such that the value of each item is at most 1.*

5 CONCLUDING REMARKS

We studied the mechanism design for allocating an indivisible resource with limited subsidy. Although our work is concerned with utilitarian optimality, studying the compatibility of truthfulness and fairness with other efficiency requirements, such as completeness and non-wastefulness, is a natural direction. Specifically, the mechanism in Section 4 can allocate all items and achieves the bound of m for additive valuations; it would be interesting to see whether the amount of m is necessary to achieve a truthful, envy-free, and complete mechanism for additive valuations.

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