Start with the empty graph $g = \emptyset_n$.
while (not done):
let $e = ij \notin E(g)$ maximize quantity $C_i[g + ij] - C_i[g]$.
among those satisfying $C_i[g] < \theta_i$ and $C_j[g] < \theta_j$.
$g = g \cup \{e\}$
return g.

Figure 3: Greedy algorithm for finding APSN.

*a). For **every network** g and centralities $(C_i)_{i \in V(g)}$ satisfying axiom 1 there exist thresholds (θ_i) s.t. g is an APSN for the truncated centrality game with thresholds θ_i.*

*b). If agents' original centrality measures satisfy axiom 1, then for all families (θ_i) of thresholds, the APSN for the truncated centrality games with thresholds θ_i, **if they exist**, can be characterized as the graphs with "Pareto optimal centralities", i.e. graphs h satisfying:*
- *for every $ij \notin E(h)$, $C_i[h] \geq \theta_i$ or $C_j[h] \geq \theta_j$, and*
- *for every edge $ij \in E(h)$, removing ij from h would yield a network l with $C_i[l] < \theta_i$ and $C_j[l] < \theta_j$.*

*c). For all $\theta_i \geq 0$, **APSN exist** in all truncated centrality games with centralities (C_i) satisfying Axioms 1, 4 and 5.*

Proof. a. Let $\theta_i = C_i[g]$. We claim that g is an APSN for the truncated centrality game with centralities C_i and thresholds θ_i. Indeed, consider an edge ij of g. Nodes i, j don't want to drop edge ij, since their current centrality values are θ_i, θ_j while, by Axiom 1, their centralities would decrease below these values if they dropped ij. Let now i, j be vertices such that $ij \notin g$. Since their current centrality values are θ_i, θ_j (at the threshold), adding edge ij would not increase their truncated centralities, while incurring the extra (positive) cost of edge ij. So adding edge ij is not an improving move.

b. First, by essentially repeating the proof at point a., it is easy to see that graphs with Pareto optimal centralities are APSN. The opposite direction is equally easy: consider an APSN h and two vertices i, j. If $ij \notin E(h)$ then adding edge ij must not be an improving move for at least one of i, j. Since C_i, C_j are increasing, the only possibility is that $C_i[h] \geq \theta_i$ (so that adding edge ij does not increase the truncated centrality of i and, in fact, decrease its utility, because of the extra cost of edge ij) or, similarly, $C_j[h] \geq \theta_j$. Consider now the case when $ij \in E(h)$. Because centralities are increasing and removing edge ij is not an improving move, removing ij must strictly decrease truncated centralities for both nodes i, j. This is only possible if the centralities of i, j in the resulting network l satisfy $C_i[l] < \theta_i$ and $C_j[l] < \theta_j$.

c. First of all, a comment about the result at point b.: it does **not** establish the existence of APSN, since it is not clear that the conditions in the characterization are actually feasible. This is what we show next, under the hypothesis that all centralities satisfy axioms 1,4,5.

We will prove the existence of APSN as follows. Consider the algorithm in Figure 3. We claim that its outcome g is an APSN. Indeed, none of the missing edges could be added to g: if $ab \notin E(g)$ then $C_a[g] \geq \theta_a$ or $C_b[g] \geq \theta_b$ at the moment when edge ab was considered for inclusion. Since centralities only increase during the algorithm, the condition is valid at the end of the algorithm as well.

On the other hand, consider an edge $ab \in E(g)$ with $C_a[g] \geq \theta_a$. In order to apply point (b), we aim to prove that for every edge $ad \in E(g)$, $C_a[g - ad] < \theta_a$. Let af be the last edge adjacent to a added to g by the algorithm in Figure 3. By the algorithm, adding af is the first moment the centrality of node a increases beyond value $\geq \theta_a$. So $C_a[g - af] < \theta_a$. We will prove that in fact $C_a[g - ad] \leq C_a[g - af] < \theta_a$. If $d = f$ then our claim is true. Otherwise, edge ad must have been added to g before edge af. Consider the moment when adding edge ad. Let g_0 be the network before the addition. By Axiom 4, adding edge ad maximized the centrality increases of both nodes a, d. Since af was a candidate for edge addition, $C_a[g_0 + af] \leq C_a[g_0 + ad]$. By the fact that C_a satisfies Axiom 5, $C_a[g_1 + af] \leq C_a[g_1 + ad]$, where $g_1 \supseteq g_0$ is the graph $g - \{af, ad\}$. But $g_1 + af = g - ad$ and $g_1 + ad = g - af$. □

5 DEGREE HOMOPHILY YIELDS RICH-CLUB APSN

Next we study centrality games for degree-homophilic centrality measures. The following result shows that APSN in this case have a "rich club" hierarchical structure:

Theorem 5. *Let h be an APSN for the centrality game with upward degree homophilic centralities with function $f(\cdot)$ satisfying $f(0) = -1$ and $f(x) \geq x$ for every $x \geq 1$.*

Let m be the maximum degree of a node in h. Let $n_1^* = min\{k : f(k) \geq m\}$ and, for $i \geq 2$, $n_i^* = min\{r : f(r) \geq n_{i-1}^*\}$. Clearly $n_1^* \geq n_2^* \geq \ldots$ (and one can assume w.l.o.g., by removing multiple copies of the same value, that $n_1^* > \ldots > n_r^* = 1$ for some $r \geq 1$)

a). If $deg(i), deg(j) \geq n_1^*$ then $ij \in E(h)$.

b). If $deg(i), deg(j) \in [n_k^*, n_{k-1}^*]$ for some $k \geq 2$ then $ij \in E(h)$ ("alike nodes connect to each other")

c). If $k \geq 2$, $deg(i) \leq n_k^*$, $deg(j) > n_{k-1}^*$ then $ij \notin E(h)$.

Proof. We use the definition of degree homophily:

a). Since $deg(i) \geq n_1^*$ and f is monotonic, $f(deg(i)) \geq f(n_1^*) \geq m \geq deg(j)$, and similarly $f(deg(j)) \geq deg(i)$. If i, j were not connected, then adding ij would be an improving move for both of them.

b). Similar to (a): as $deg(i) \geq n_k^*$ and f is monotonic, $f(deg(i)) \geq f(n_k^*) = n_{k-1}^* \geq deg(j)$, so $f(deg(i)) \geq deg(j)$, and similarly $f(deg(i)) \geq deg(j)$. If i, j were not connected, then adding ij would be an improving move.

c). We have $deg(i) \leq n_k^*$ so $f(deg(i) - 1) \leq f(n_k^* - 1) < n_{k-1}^* \leq deg(j) - 1$. Hence $deg(j) - 1 > f(deg(i) - 1)$, so removing edge ij is an improving move for j, since in the graph $h = g - ij$ adding edge ij is not an improving move for j.

□

Corollary 2. *Let $a_1 > a_2 > \ldots > a_p > a_{p+1} = 1$ be a sequence of integers such that $a_i - 1 > f(a_{i+1} - 1)$ for all $i = 1, \ldots, p$. Then all graphs of type $K_{a_1} + K_{a_2} + \ldots + K_{a_p} + rK_0$ ("stratified clique graphs") are APSN for upward degree homophilic centrality games* *with function f and conversely, all APSN are unions of cliques with this structure.*

Proof. We need the following simple

Lemma 1. *Let g be a network and $ij \in E(g)$. Then removing edge ij from g is an improving move iff $deg(j) - 1 > f(deg(i) - 1)$ or $deg(i) - 1 > f(deg(j) - 1)$.*

Proof. Removing edge ij is an improving move **iff** for at least one of the two nodes i, j its betweenness centrality stays the same when removing the edge. In this case adding edge ij to $h = g - ij$ is **not** an improving move and vice-versa: if adding edge ij to h is not an improving then one of i, j has the same betweenness centrality in g as in h, hence removing edge ij is an improving move in g.

By definition, adding edge ij to h is not improving iff $deg_h(i) > f(deg_h(j))$ or $deg_h(j) > f(deg_h(i))$. □

Consider now a sequence $a_1 > a_2 > \ldots > a_p > a_{p+1} = 1$ be a sequence of integers such that $a_i - 1 > f(a_{i+1} - 1)$ for all $i = 1, \ldots, p$. We first need to prove that all graphs of type $K_{a_1} + K_{a_2} + \ldots + K_{a_p} + lK_0$, $l \geq 0$, are APSN.

This is easy, by applying points a),b),c) of the theorem: let, indeed, y, z be nodes in the same clique K_{a_r}, $1 \leq r \leq p$. We need to show that removing edge yz is not an improving move. Since they are in the same clique, the degrees of y, z are both equal to $a_r - 1$. Since $a_r - 1 \leq f(a_r - 1)$ (because $a_r \geq 2$ and $f(x) \geq x$ for $x \geq 1$), the desired conclusion follows by Lemma 1.

Let now y, z be nodes in different cliques, $y \in K_{a_r}$, $z \in K_{a_s}$, $a_r > a_s$. We have $deg(y) = a_r - 1 > f(a_s - 1) = f(deg(z))$. By the definition, adding edge yz is **not** an improving move.

Since $0 > f(0) = -1$ connecting any isolated node to any other node is **not** an improving move. So g is an APSN.

Conversely, let g be an APSN. By applying points a) and b) of the Theorem, we get that g has edges between every two vertices whose degrees are in the same interval $[n_i^*, n_{i-1}^*]$, where by convention $n_0^* = m$.

To infer the fact that g has the structure claimed in the corollary we need to prove that no other edges are present. Point c) of the theorem excludes edges between node whose degrees are **not** in the same interval.

The only potential trouble is that there might be a node x of degree n_i^* who is connected with nodes whose degrees are in both intervals $[n_{i+1}^*, n_i^*]$ and $[n_i^*, n_{i-1}^*]$, thus "joining two cliques". We will show that something like this doesn't happen by induction on i.

Case $i = 1$: Let z be a node of maximum degree m. Let A be the set of nodes with degree in the range $[n_1^*, n_0^*]$. Then all the nodes in A are connected to each other. z is not connected to any node outside A. If there were some other node w in A that is connected to a node outside A then w would have degree higher than m, a contradiction. Hence nodes in A form a connected component that is a clique.

The induction step: Assume we have obtained $l - 1$ connected components that are cliques of size $a_1 > a_2 > \ldots > a_{l-1}$ satisfying the condition $a_i - 1 > f(a_{i+1} - 1)$ for $i = 1, \ldots, l - 2$. Applying the reasoning in the induction case $i = 1$ to the remaining graph we obtain a connected component of size a_l that is a clique. Furthermore $a_{l-1} - 1 > f(a_l - 1)$, since nodes in the l'th clique component are not connected to those in the $l - 1$'st component.

It is possible that the tail of the resulting sequence $a_1, \ldots, a_s$ is composed of components of size 1, that is isolated nodes. The required condition is satisfied, since $1 - 1 > f(1 - 1) = f(0) = -1$. □

In the previous theorem the condition $f(x) \geq x$ is necessary: as the next result shows, without it the structure of APSN is much simpler:

Theorem 6. *Consider a centrality game with upward degree homophilic centralities with common function $f(x) = x - 1$. Then no edge addition can be an improving move.*

Assume that, additionally, for every agents i, j such that $ij \notin g$ and $deg(i) \leq deg(j)$ we have $C_i[g + ij] \leq C_i[g]$. Then the unique APSN for the centrality game is the empty network $\emptyset_n$.

Proof. Adding a missing edge ij can never be an improving move: to be so, one would need, simultaneously that $deg(i) \leq deg(j) - 1$ and $deg(j) \leq deg(i) - 1$, which is impossible.

For similar reasons, removing an existing edge ij is always improving for one of the endpoints. Indeed, assume that h is a network containing edge ij and, w.l.o.g. $deg(i) \leq deg(j)$. Let $g = h - ij$. Then $u_i(g) - u_i(h) = C_i[g] - C_i[h] + c > 0$. So removing edge ij is an improving move for i. □

Observation 1. *The Banzhaf-Michalak centrality satisfies the conditions of Theorem 6. Indeed, assume $ij \notin g$ and $deg(i) \leq deg(j)$. Then $C_i[g + ij] - C_i[g] = \frac{1}{2^{deg(i)+1}} + \frac{1}{2^{deg(j)+1}} - \frac{1}{2^{deg(i)}} = \frac{1}{2^{deg(j)+1}} - \frac{1}{2^{deg(i)+1}} \leq 0$.*

6 DOMINATION AND APSN IN BETWENNESS CENTRALITY GAMES

In this section we completely characterize APSN for betweenness centrality games. First, simple computations provide examples of APSN with components that are not complete graphs: networks $C_4 + nK_1$, $n \geq 0$. What about the general structure of APSN ? We will show that the domination relation plays a decisive role in their characterization. To accomplish this, we first prove:

Lemma 2. *The following statements are true:*

- *Adding any bridge edge ij weakly increases i's betweenness centrality, strictly unless i was isolated. Consequently adding a bridge edge is improving for i, unless i was isolated. Conversely, a disconnecting edge removal is improving for i iff i was a pendant node.*
- *Adding any non-bridge edge ij weakly increases i's betweenness centrality.*

We now prove the following result, which gives an unexpected (and fairly elegant) algorithmic characterization of APSN for betweenness games using the domination relation:

Theorem 7. *Graphs g that are APSN for betweenness centrality games consist of isolated vertices plus at most one connected component C with at least two vertices which satisfies the following condition: $deg(l) \geq 2$ for every $l \in C$, $diam(C) = 2$ and for every $i \neq j \in C$, $ij \in E(g)$ **if and only if** sets $N(i) \setminus \{j\}$ and $N(j) \setminus \{i\}$ are incomparable, i.e. if **none of** i, j **dominates the other.***

Proof. First, it is easy to see that the networks that satisfy the condition of Theorem 7 are APSN: indeed, by Lemma 2 isolated vertices have no incentive to connect to anyone else, as their utility would decrease. Consider, on the other hand two vertices i, j in a large component C.

If $ij \in E(g)$ then, by the condition of the theorem, there exist vertices $k \in N(i)\setminus\{j\}$ and $l \in N(j)\setminus\{i\}$. Since $d(k, j) \leq 2$ it follows that $k - i - j$ is a shortest path between i and j that would disappear if we dropped edge ij, decreasing the betweenness centrality of i and ultimately its utility. Similarly, if we dropped edge ij the utility of j would also decrease, hence it is **not** an improving move.

On the other hand if $ij \notin E(g)$ then $N(i) \setminus \{j\}$ and $N(j) \setminus \{i\}$ are comparable. Assume w.l.o.g. that $N(i) \setminus \{j\} \subseteq N(j) \setminus \{i\}$. Then every shortest path between two vertices $s, t \neq i$ that goes through i stays a shortest path when we add edge ij: This is clear when $s, t \neq j$, so assume w.l.o.g. $t = j$. Then $d(s, j) = 1$. Adding edge ij creates no new shortest paths, hence it is not an improving move for i.

Let us now prove the converse direction, that APSN satisfy the conditions in the theorem. A first statement to prove is that any APSN has at most one component with at least two vertices. Indeed, if there were more than two such connected components then, by Lemma 2, joining them by an edge ij would be an improving move for both i, j. Second, we claim that this nontrivial component has diameter 2: indeed, it cannot have diameter 1, as complete graphs are not APSN. Assume there was a (shortest) path of length 3 $p-q-r-s$ between two vertices p, s. Then p, s would increase their utility by connecting since, for instance, now there is a shortest path from q to s going through p. Third, this component has no pendant vertices: if a node had degree 1, it would have, by Lemma 2, an incentive to disconnect.

Consider a connected APSN g and a pair $ij \notin g$, and assume w.l.o.g. that adding edge ij decreases the utility of i for small $\epsilon > 0$, so that the move is not improving. Hence adding edge ij does not increase the betweenness of i. Paths contributing positively to the betweenness of i **before adding** ij are between nodes $k_1, k_2 \in N(i)$ that are not connected, so that a shortest path between k_1, k_2 goes through i. Then adding edge ij does not change the fraction corresponding to k_1, k_2 in the betweenness of i. Consider now the shortest paths between $k_1 \in N(i)\setminus\{j\}$ and j. Since the betweenness of i does not increase as a result of adding edge ij, k_1 must be connected to j. Hence j dominates i.

Consider now an edge $ij \in g$. Since the removal of edge ij is not improving for either i or j, it means that there exists a shortest path between some vertices $s_1 \neq i \neq t_1$ that employs edge ij. As the diameter of g is two, one of s_1, t_1 (say t_1) must be j, hence $s = s_1$ is a neighbor of i that is not a neighbor of j. Similarly, there must be a vertex $t = t_2$ that is a neighbor of j that is not a neighbor of i. Hence none of i, j dominates the other. □

Observation 2. *Complete bipartite graphs $K_{a,b}$, $a, b \geq 2$ satisfy the conditions of the theorem, hence they are APSN. One could believe that these are* ***all*** *connected APSN with at least 2 vertices, but this is* ***not*** *true: a counterexample, found using computer simulations, is the graph g in Figure 4. g is not bipartite as it has, e.g. triangle 459.*

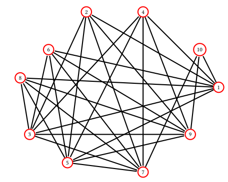

Figure 4: Non-bipartite APSN for betweenness games.

7 OTHER RESULTS/CONJECTURES

In this section we study centrality games for some measures that appear to satisfy none of Axioms 1,2,3: eccentricity centrality, random walk betweenness and eigenvector centrality. First, we show that eccentricity centrality is very close to obeying Axiom 2:

Lemma 3. *Let g be a network, i a node in g and j another node such that $ij \notin g$. The following are true:*

- *If $j \notin Conn(i)$ then $EC(i, g + ij) \leq EC(i, g)$.*
- *If $j \in Conn(i)$ then $EC(i, g + ij) \geq EC(i, g)$. The inequality is strict iff j is on all shortest paths to all vertices k farthest in g from i.*

In spite of this result, the structure of APSN for eccentricity centrality games is quite different from the one for centrality games with measures satisfying Axiom 2:

Theorem 8. *All vertices in connected components of size at least three of an APSN have degree at least two. On the other hand all connected, eccentricity-one graphs with min. degree 2 and at least two nodes with degree at most $n - 2$ are APSN. There exist (Fig. 5) APSN with eccentricity two.*

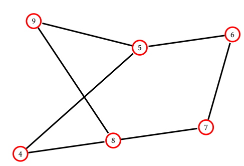

Figure 5: APSN for eccentricity centrality games.

We weren't able to obtain a full characterization of APSN in this case, or analytical results for random walk betweenness and eigenvector centrality. However, computer simulations suggest that the following statements are true. The first one is interesting due to apparent difference with the case of betweenness:

Conjecture 1. *For random walk betweenness centrality the only APSN are the empty graph $\emptyset_n$ and the complete K_n.*

As for eigenvector centrality, although it seems not to have any monotonicity properties, experimental evidence is consistent with the following conjecture, that seems to situate this measure together with the monotonic ones:

Conjecture 2. *The complete graphs K_n are the only asymptotically pairwise stable networks for eigenvector centrality.*

8 LEARNING AGENT THRESHOLDS

In spite of the previous result, we can still talk about learning agent utility functions. However, the problem that we will deal with is **not** that of learning agent centralities (which we will, in fact, assume known), but **agent thresholds.** In other words, we want to answer the following variant of Q3: *Can we learn (something about) agents' thresholds from the structure of stable networks ?*

The learning model we will assume is a type of *oracle learning* [1]. Specifically[4], oracle queries are pairs (g, i) consisting of a network g and an agent index i. Given query (g, i) the oracle will either reply with an APSN h such that $C_i[h] > C_i[g]$, or with "NONE", in the case such an APSN h does not exist.

It is important to realize that thresholds may fail to be fully identifiable simply due to *the coarse resolution of centralities*: for instance, any values between two consecutive integers (e.g. 2.3 and 2.7) are completely equivalent as thresholds for degree centrality, since degrees in graphs are integral, and jumps in centrality (as a result of an edge flip) have a magnitude at least one. The best we can hope for in such a scenario is to identify the interval [2,3] as an interval that contains the threshold. The interval corresponds to a single edge flip in a network that decreased the centrality of the given node below the threshold value.

A potential issue with the identification of thresholds is the fact that (consistent with the model in our Corollary 1) we only get APSN as oracle answers. If for all APSN g, $C_i[g] < \theta_i$ then **all** estimates provided by the oracle on the value of the threshold are too low. If this doesn't happen, we can prove:

THEOREM 9. *Given an agent i, assume that there exists an APSN h with $\theta_i \leq C_i[h]$. Then there exists an algorithm that uses oracle queries and outputs an APSN g and edge ij s.t. $C_i[g-ij] \leq \theta_i \leq C_i[g]$. For linear centralities the algorithm runs in polynomial time.*

9 RELATED LITERATURE

The area of network games is quite large, and a comprehensive survey is impossible. We list here two such overviews: the first one, most relevant to our interest is [38]. Another one with an algorithmic bent is [51]. The model that we are concerned with is a variant of the symmetric connection model [37] (see also [36]). Some notable subsequent work includes [21, 28, 39]. Many alternative models have been investigated. A more preeminent one is [4].

Our work owes much to the axiomatic approach to network centralities. For significant work in this area see [5, 8, 9, 47–49, 54].

More related work exists in the theoretical computer science literature: for example, Hopcroft and Sheldon [32] discuss an oriented model in which nodes have control over outgoing edges. There is no cost for changing their links, and their purpose is to increase their Pagerank. They show that the Nash equilibria in this game have a fairly sophisticated structure (see also Chen et al. [15]). Undirected versions of this game have been studied [3]. Recently Kouroupas et al. [41] have studied a model in which the utility of a node is a product of two factors: content quality multiplied by the traffic level. In this model pure Nash equilibria always exist. On the other hand Avin et al. [2] prove that preferential attachment models can be seen as Nash equilibria of some network games.

Other related work comes from the sociology literature [13, 33, 44]. For instance, in the Buskens and Van De Rijt model every node strives to fill "structural holes" (including lack of connnectedness) between nodes. This is somewhat analogous to maximizing betweenness, but the precise model (and the results) are different.

Our model allows heterogeneity in agents' utilities, corresponding to distinct measures of centrality. Heterogeneous network formation models have been studied before, e.g. Galeotti et al. [26].

Finally, several papers (e.g. [6, 18]) have treated the problem of improving the centrality of a node by adding or removing links. Our work is different in several respects: first of all, in our setting *all* agents aim to improve their respective centralities. Second, in our model maintaining a link has a (small) cost.

10 CONCLUSIONS, POSSIBLE EXTENSIONS

We have shown that our models can accommodate a wide range of agent centrality objectives. Still, we do not see our results as adequate enough yet for the analysis of real-life networks. They have, instead, more of a proof-of-concept nature, and could conceivably be made more realistic in many ways. Some variations (we believe) worth investigating are listed below:

Probabilistic edge addition/removal: In real life an edge may only form with some probability even though both agents would benefit from it. Studying such a variation could produce networks with core structures that are dense but not quite complete.

Strong and weak links, forced links, affiliation models: In the model we have discussed all the links are weak links. A natural extension allows for both strong and weak links. This would entail using two types of costs: fixed, constant costs for the strong links, small ("ϵ") costs for weak links. A second, orthogonal, distinction that could be useful is that of forced versus free links. We assumed implicitly that link formation is completely under the control of the agent. Often this is not so: there are social ties in real life that could be regarded as fixed, since severing them entails a significant cost. Forced links may be a consequence of *affiliation*: people meet as the result of joining the same clubs. A possibly relevant model is the *social effort model* of [11]. Another one is the *social clubs* model of [24]. For centrality in affiliation networks see [23].

Manipulating link strength: agents could manipulate *link strength*, rather than completely severing them.

Tagged networks: agents have a *tag* and care about the tags of their neighbors, like in Schelling's segregation model.

Spatial agents: Agents interaction may result from placement in space. A standard reference for spatial connection models is [28].

Multilayer networks: Sometimes (e.g. [20]) link formation may encompass multiple, correlated, link types. E.g. two coworkers may end up being friends as well. It would be interesting to formulate multilayer extensions of the Jackson-Wolinsky model.

Overlapping communities: For centrality in such models see [27, 30, 50, 52].

Dynamic models: Finally, our concepts of network stability are steady-state concepts. It would be interesting to study the emerging networks in *dynamic* models of network formation with a similar philosophy.

[4] perhaps a more natural model would be one that centrality of node i in APSN g, $C_i[g]$) can be bigger/smaller than some arbitrary target value. The point is that our restricted query model is good enough.

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