

Equilibria in Schelling Games: Computational Hardness and Robustness

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ABSTRACT

In the simplest game-theoretic formulation of Schelling's model of segregation on graphs, agents of two different types each select their own vertex in a given graph so as to maximize the fraction of agents of their type in their occupied neighborhood. Two ways of modeling agent movement here are either to allow two agents to swap their vertices or to allow an agent to jump to a free vertex.

The contributions of this paper are twofold. First, we prove that deciding the existence of a swap-equilibrium and a jump-equilibrium in this simplest model of Schelling games is NP-hard, thereby answering questions left open by Agarwal et al. [AAAI '20] and Elkind et al. [IJCAI '19]. Second, we introduce two measures for the robustness of equilibria in Schelling games in terms of the minimum number of edges or the minimum number of vertices that need to be deleted to make an equilibrium unstable. We prove tight lower and upper bounds on the edge- and vertex-robustness of swap-equilibria in Schelling games on different graph classes.

KEYWORDS

Schelling's Model of Segregation; Graph Games; Modification Robustness; Equilibrium Analysis

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1 INTRODUCTION

Schelling's model of segregation [26, 27] is a simple random process that aims at explaining segregation patterns frequently observed in real life, e.g., in the context of residential segregation [24, 29]. In Schelling's model, one considers agents of two different types. Each agent is initially placed uniformly at random on an individual vertex of some given graph (also called *topology*), where an agent is called *happy* if at least a τ -fraction of its neighbors is of its type for some given tolerance parameter $\tau \in (0, 1]$. Happy agents do not change location, whereas, depending on the model, unhappy agents either randomly swap vertices with other unhappy agents

or randomly jump to empty vertices. Schelling [26, 27] observed that even for tolerant agents with $\tau \sim \frac{1}{3}$, segregation patterns (i.e., large connected areas where agents have only neighbors of their type) are likely to occur. Over the last 50 years, Schelling's model has been thoroughly studied both from an empirical (see, e.g., [9, 13]) and a theoretical (see, e.g., [3, 4, 6, 17]) perspective in various disciplines including computer science, economics, physics, and sociology. Most works focused on explaining under which circumstances and how quickly segregation patterns occur.

In Schelling's model it is assumed that unhappy agents move randomly and only care about whether their tolerance threshold is met. As this seems unrealistic, Schelling games, which are a game-theoretic formulation of Schelling's model where agents move strategically in order to maximize their individual utility, have recently attracted considerable attention [1, 5, 11, 15, 18]. However, there is no unified formalization of the agents' utilities in the different game-theoretic formulations. In this paper, we assume that all agents want to maximize the fraction of agents of their type in their occupied neighborhood. In contrast, for instance, Chauhan et al. [11] and Echzell et al. [15] assumed that the utility of an agent a depends on the minimum of its threshold parameter τ and the fraction of agents of a 's type in the occupied neighborhood of a (as done in [1, 5], we assume that $\tau = 1$). Along a different dimension, in the works of Chauhan et al. [11] and Agarwal et al. [1], the utility of (some) agents also depends on the particular vertex they occupy. For instance, Agarwal et al. [1] assume that there exist some agents which are *stubborn* and have a favorite vertex in the graph which they never leave (in our model, as also done before [5, 15, 18], we assume that the agents' behavior does not depend on their specific vertex, so there are no stubborn agents). The main focus in previous works and our work lies on the analysis of certain pure equilibria in Schelling games, where it is typically either assumed that agents can swap their vertices or jump to empty vertices. While we mostly focus on swap-equilibria, we also partly consider jump-equilibria. As the existence and other specifics of equilibria crucially depend on the underlying topology, a common approach is to consider different graph classes [5, 11, 18].

Our paper is divided into two parts. First, we prove that deciding the existence of a swap- or jump-equilibrium in Schelling games (as formally defined in Section 2.1) is NP-hard. Second, we introduce a new perspective for the analysis of equilibria in Schelling

games: Stability under changes. Considering the original motivation of modeling residential segregation, it might for example occur that agents move away and leave the city. In the context of swap-equilibria on which we focus in the second part, this corresponds to deleting the vertex occupied by the leaving agent (or all edges incident to it), as unoccupied vertices can never get occupied again and are also irrelevant for computing agents' utilities. An interesting task now is to find a more "robust" equilibrium that remains stable in such a changing environment, as for non-robust equilibria it could be that a small change can cause the reallocation of all agents. We formally define a worst-case measure for the *robustness* of an equilibrium with respect to vertex/edge deletion as the maximum integer r such that the deletion of any set of at most r vertices/edges leaves the equilibrium stable. Clearly, the robustness depends on the underlying topology. That is why we study different graph classes with respect to the robustness of equilibria.

Related Work. Most of the works on Schelling games focused on one of three aspects: existence and complexity of computing equilibria [1, 5], game dynamics [5, 11, 15], and price of anarchy and stability [1, 5, 18]. The first area is closest to our paper, so we review some results here. On the negative side, Agarwal et al. [1] showed that a jump- and swap-equilibrium may fail to exist even on tree topologies and that checking their existence is NP-hard on general graphs in the presence of stubborn agents. On a tree, the existence of a jump- and swap equilibrium can be checked in polynomial time [1]. On the positive side, Agarwal et al. [1] showed that a jump-equilibrium always exists on stars, paths, and cycles. Concerning swap-equilibria, Echzell et al. [15] showed that a swap-equilibrium always exists on regular graphs (and in particular cycles). Recently, Bilò et al. [5] proved that a swap-equilibrium is guaranteed to exist on paths and grids, and they obtained further results for the restricted case where only adjacent agents are allowed to swap.

Lastly, different from the three above mentioned directions, Bullinger et al. [8] and Deligkas et al. [14] studied finding Pareto-optimal assignments and assignments maximizing the summed utility of all agents in Schelling games and Chan et al. [10] proposed a generalization of Schelling games where, among others, multiple agents can occupy the same vertex.

Analyzing the robustness of outcomes of decision processes has become a popular topic in algorithmic game theory [2, 7, 19, 25, 28]. For instance, in the context of hedonic games, Igarashi et al. [16] studied stable outcomes that remain stable even after some agents have been deleted and, in the context of stable matching, Mai and Vazirani [22, 23] and Chen et al. [12] studied stable matchings that remain stable even if the agents' preferences partly change.

Our Contributions. The contributions of this paper are twofold. In the first, more technical part, we prove that deciding the existence of a jump- or swap-equilibrium in Schelling games where all agents want to maximize the fraction of agents of their type in their occupied neighborhood is NP-hard. Notably, our technically involved results strengthen results by Agarwal et al. [1], who proved the NP-hardness of these problems making decisive use of the existence of stubborn agents (which never leave their vertices).

Having analyzed the existence of equilibria, in the second, more conceptual part, we introduce a notion for the robustness of an equilibrium under vertex/edge deletion: We say that an equilibrium

Table 1: Overview of robustness values of swap-equilibria for various graph classes. For each considered class, a swap-equilibrium always exists (see [5, 15] and Section 4.2). Moreover, there exists a Schelling game on a graph from this class with two swap-equilibria whose robustness match the depicted lower and upper bound. For bounds marked with \dagger , on a graph from this class, an equilibrium with this robustness is guaranteed to exist in every Schelling game with at least four agents of the one and two agents of the other type.

	edge-robustness		vertex-robustness	
	lower bound	upper bound	lower bound	upper bound
Cliques (Ob. 2)	0^\dagger	0^\dagger	$ V(G) ^\dagger$	$ V(G) ^\dagger$
Cycles (Pr. 4.4)	0^\dagger	0^\dagger	0^\dagger	0^\dagger
Paths (Th. 4.5)	0^\dagger	$ E(G) ^\dagger$	0^\dagger	$ V(G) ^\dagger$
Grids (Th. 4.6)	0	1	0	1
α -Caterpillars (Pr. 4.10)	α	$ E(G) ^\dagger$	α	$ V(G) ^\dagger$

has vertex/edge-robustness r if it remains stable under the deletion of any set of at most r vertices/edges but not under the deletion of $r+1$ vertices/edges. We study the existence of swap-equilibria with a given robustness. We restrict our attention to swap-equilibria, as for jump-equilibria, the robustness heavily depends on both the underlying topology and the specific numbers of agents of each type. Providing meaningful bounds on the robustness of a jump-equilibrium is therefore rather cumbersome.

In our analysis of the robustness of swap-equilibria, we follow the approach from most previous works and investigate the influence of the structure of the topology [1, 5]. That is, we show tight upper and lower bounds on the edge- and vertex-robustness of swap-equilibria in Schelling games on topologies from various graph classes, summarized in Table 1. We prove that the edge- and vertex-robustness of swap-equilibria on a graph class can be arbitrarily far apart, as on cliques it is always sufficient to delete a single edge to make every swap-equilibrium unstable while every swap-equilibrium remains stable after the deletion of any subset of vertices. In contrast to this, all of our other lower and upper bounds are the same (and tight) for vertex- and edge-robustness and can be proven using similar arguments. We further show that on paths there exists a swap-equilibrium that can be made unstable by deleting a single edge or vertex and a swap-equilibrium that remains stable after the deletion of any subset of edges or vertices, implying that the difference between the edge/vertex-robustness of the most and least robust equilibrium can be arbitrarily large. This suggests that, in practice, one should be cautious when dealing with equilibria if robustness is important.

As an example of a nontrivial graph class where every swap-equilibrium has robustness larger than zero, we define *α -star-constellation graphs* (see Section 2.2 for a definition). We show that every swap-equilibrium on an α -star-constellation graph has edge- and vertex-robustness at least α . Lastly, we prove that a swap-equilibrium always exists on a subclass of *α -star-constellation graphs* and caterpillar graphs which we call α -caterpillars (see Figure 1).

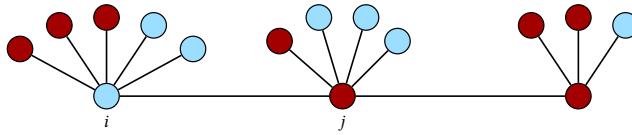


Figure 1: Schelling game with $|T_1| = 7$ and $|T_2| = 8$. Agents from T_1 are drawn in light blue and agents from T_2 in dark red. Let \mathbf{v} be the depicted assignment. It holds that $u_i(\mathbf{v}) = u_j(\mathbf{v}) = \frac{1}{3}$ and $u_i(\mathbf{v}^{i \leftrightarrow j}) = u_j(\mathbf{v}^{i \leftrightarrow j}) = \frac{1}{2}$. Thus, \mathbf{v} is not a swap-equilibrium, as i and j have a profitable swap. The displayed graph is a 2-star-constellation graph and a 2-caterpillar.

The proofs for results marked by (★) can be found in the full version of our paper [21].

2 PRELIMINARIES

Let \mathbb{N} be the set of positive integers and \mathbb{N}_0 the set of non-negative integers. For two integers $i < j \in \mathbb{N}_0$, we denote by $[i, j]$ the set $\{i, i+1, \dots, j-1, j\}$ and by $[j]$ the set $[1, j]$. Let $G = (V, E)$ be an undirected graph. Then, $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G . For a subset $S \subseteq E$ of edges, $G - S$ denotes the graph obtained from G by deleting all edges from S . For a subset $V' \subseteq V$ of vertices, $G[V']$ denotes the graph G induced by V' . Overloading notation, for a subset $V' \subseteq V$, we sometimes write $G - V'$ to denote the graph G induced by $V \setminus V'$, that is, $G - V' = G[V \setminus V']$. For a vertex $v \in V(G)$, we denote by $N_G(v)$ the set of vertices adjacent to v in G . The degree $\deg_G(v) := |N_G(v)|$ of v is the number of vertices adjacent to v in G . Lastly, $\Delta(G) := \max_{v \in V(G)} \deg_G(v)$ is the maximum degree of a vertex in G .

2.1 Schelling Games

A *Schelling game* is defined by a set $N = [n]$ of $n \in \mathbb{N}$ (strategic) agents partitioned into two *types* T_1 and T_2 and an undirected graph $G = (V, E)$ with $|V| \geq n$, called the *topology*. The strategy of agent $i \in N$ consists of picking some *position* $v_i \in V$ with $v_i \neq v_j$ for $i \neq j \in N$. The *assignment vector* $\mathbf{v} = (v_1, \dots, v_n)$ contains the positions of all agents. A vertex $v \in V$ is called *unoccupied* in \mathbf{v} if $v \neq v_i$ for all $i \in N$. In the following, we refer to an agent i and its position v_i interchangeably. For example, we say agent i has an edge to agent j if $\{v_i, v_j\} \in E$. For an agent $i \in T_l$ with $l \in \{1, 2\}$, we call all other agents $F_l = T_l \setminus \{i\}$ of the same type *friends* of i . The set of i 's *neighbors* on topology G is $N_i^G(\mathbf{v}) := \{j \neq i \mid \{v_i, v_j\} \in E(G)\}$, and $a_i^G(\mathbf{v}) := |N_i^G(\mathbf{v}) \cap F_l|$ is the number of friends in the neighborhood of i in \mathbf{v} .

Given an assignment \mathbf{v} , the utility of agent i on topology G is:

$$u_i^G(\mathbf{v}) := \begin{cases} 0 & \text{if } N_i^G(\mathbf{v}) = \emptyset, \\ a_i^G(\mathbf{v}) / |N_i^G(\mathbf{v})| & \text{otherwise.} \end{cases}$$

If the topology is clear from the context, we omit the superscript G .

Given some assignment \mathbf{v} , agent $i \in N$, and an unoccupied vertex v , we denote by $\mathbf{v}^{i \rightarrow v} = (v_1^{i \rightarrow v}, \dots, v_n^{i \rightarrow v})$ the assignment obtained from \mathbf{v} where i *jumps* to v , that is, $v_i^{i \rightarrow v} = v$ and $v_j^{i \rightarrow v} = v_j$ for all $j \in N \setminus \{i\}$. A jump of an agent i to vertex v is called *profitable* if it improves i 's utility, that is, $u_i(\mathbf{v}^{i \rightarrow v}) > u_i(\mathbf{v})$. Note that an agent can jump to any unoccupied vertex. For two agents $i \neq j \in N$

and some assignment \mathbf{v} , we define $\mathbf{v}^{i \leftrightarrow j} = (v_1^{i \leftrightarrow j}, \dots, v_n^{i \leftrightarrow j})$ as the assignment that is obtained by *swapping* the vertices of i and j , that is, $v_i^{i \leftrightarrow j} = v_j$, $v_j^{i \leftrightarrow j} = v_i$, and $v_k^{i \leftrightarrow j} = v_k$ for all $k \in N \setminus \{i, j\}$. Note that any two agents (independent of the vertices they occupy) can perform a swap. A swap of two agents $i, j \in N$ is called *profitable* if it improves i 's and j 's utility, that is, $u_i(\mathbf{v}^{i \leftrightarrow j}) > u_i(\mathbf{v})$ and $u_j(\mathbf{v}^{i \leftrightarrow j}) > u_j(\mathbf{v})$ (see Figure 1 for an example). Note that a swap between agents of the same type is never profitable.

An assignment \mathbf{v} is a *jump/swap-equilibrium* if no profitable jump/swap exists. Note that a jump-equilibrium is simply a Nash equilibrium for our Schelling game. Following literature conventions, in cases where we allow agents to swap, we assume that $n = |V(G)|$, while in cases where we allow agents to jump, we assume that $n < |V(G)|$.

2.2 Graph Classes

A *path* of length n is a graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} \mid i \in [n-1]\}$. A *cycle* of length n is a graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} \mid i \in [n-1]\} \cup \{\{v_n, v_1\}\}$. We call a graph $G = (V, E)$ where every pair of vertices is connected by an edge a *clique*. For $x, y \geq 2$, we define the $(x \times y)$ -grid as the graph $G = (V, E)$ with $V = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \leq x, b \leq y\}$ and $E = \{\{(a, b), (c, d)\} \mid |a-c| + |b-d| = 1\}$. An x -star with $x \in \mathbb{N}$ is a graph $G = (V, E)$ with $V = \{v_0, \dots, v_x\}$ and $E = \{\{v_0, v_i\} \mid i \in [x]\}$; the vertex v_0 is called the *central vertex* of the star. We say that a connected graph $G = (V, E)$ is an α -star-constellation graph for some $\alpha \in \mathbb{N}_0$ if it holds for all $v \in V$ with $\deg_G(v) > 1$ that $|\{w \in N_G(v) \mid \deg_G(w) = 1\}| \geq |\{w \in N_G(v) \mid \deg_G(w) > 1\}| + \alpha$. That is, the graph G consists of stars where the central vertices can be connected by edges such that every central vertex is adjacent to at least α more degree-one vertices than other central vertices. Thus, an α -star constellation graph consists of (connected) stars forming a *constellation* of stars, which gives this class its name. An α -caterpillar is an α -star-constellation graph where the graph restricted to non-degree-one vertices forms a path (see Figure 1 for an example).

3 NP-HARDNESS OF EQUILIBRIA EXISTENCE

In this section, we prove the NP-hardness of the following two problems:

SWAP/(JUMP)-EQUILIBRIUM [S/(J)-Eq]

Input: A connected topology G and a set $N = [n]$ of agents with $|V(G)| = n$ ($|V(G)| > n$) partitioned into types T_1 and T_2 .

Question: Is there an assignment \mathbf{v} of agents to vertices where no two agents have a profitable swap (no agent has a profitable jump)?

Agarwal et al. [1] proved that deciding the existence of a swap- or jump-equilibrium in a Schelling game with *stubborn* and *strategic* agents is NP-hard. In their model, a strategic agent wants to maximize the fraction of agents of its type in its occupied neighborhood (like the agents in our definition) and a stubborn agent has a favorite vertex which it never leaves (in our definition, such agents do not exist). Formally, Agarwal et al. [1] proved the NP-hardness of the following problems:

SWAP/(JUMP)-EQUILIBRIUM WITH STUBBORN AGENTS [S/(J)-EQ-STUB]

Input: A connected topology G and a set $N = [n] = R \cup S$ of agents with $|V(G)| = n$ ($|V(G)| > n$) partitioned into types T_1 and T_2 and a set $V_S = \{s_i \in V(G) \mid i \in S\}$ of vertices, where R is the set of strategic and S the set of stubborn agents.

Question: Is there an assignment v of agents to vertices with $v_i = s_i$ for $i \in S$ such that no two strategic agents have a profitable swap (no strategic agent has a profitable jump)?

Both known hardness reductions heavily rely on the existence of stubborn agents. We show that it is possible to polynomial-time reduce S/J-EQ-STUB to S/J-EQ. In the following two subsections, we first prove this for swap-equilibria and afterwards consider jump-equilibria.

3.1 Swap-Equilibria

This subsection is devoted to proving the following theorem:

THEOREM 3.1. *S-EQ is NP-complete.*

Note that membership in NP is trivial, as we can check whether an assignment is a swap-equilibrium by iterating over all pairs of agents and checking whether they have a profitable swap. For NP-hardness, in our polynomial-time many-one reduction we reduce from a restricted version of S-EQ-STUB:

LEMMA 3.2 (★). *S-EQ-STUB remains NP-hard even on instances satisfying the following two conditions:*

- (1) *For every vertex $v \notin V_S$ not occupied by a stubborn agent there exist two adjacent vertices $s_i, s_j \in V_S$ occupied by stubborn agents $i \in T_1$ and $j \in T_2$.*
- (2) *For both types there are at least five strategic and three stubborn agents.*

Idea Behind Our Reduction. Given an instance of S-EQ-STUB on a topology G' with vertices of stubborn agents $V_{S'}$, we construct a Schelling game without stubborn agents on a topology G that simulates the given game. To create G , we modify the graph G' by adding new vertices and connecting these new vertices to vertices from $V_{S'}$. Moreover, we replace each stubborn agent by a strategic agent and add further strategic agents. In the construction, we ensure that if there exists a swap-equilibrium v' in the given game, then v' can be extended to a swap-equilibrium v in the constructed game by replacing each stubborn agent with a strategic agent of the same type and filling empty vertices with further strategic agents. One particular challenge here is to ensure that the strategic agents that replace stubborn agents do not have a profitable swap in v . For this, recall that, by Lemma 3.2, we assume that in G' , every vertex not occupied by a stubborn agent in v' is adjacent to at least one stubborn agent of each type. Thus, in v' each strategic agent i is always adjacent to at least one friend and has utility $u_i^{G'}(v') \geq 1/\Delta(G')$. Conversely, by swapping with agent i , an agent j of the other type can get utility at most $u_j^{G'}(v'^{i \leftrightarrow j}) \leq \Delta(G') - 1/\Delta(G')$. Our idea is now to “boost” the utility of a strategic agent j that replaces a stubborn agent in v by adding enough degree-one neighbors only adjacent to v_j in G , which we fill with agents of j 's type when extending v' to v such that $u_j^G(v) \geq \Delta(G') - 1/\Delta(G') \geq u_j^{G'}(v'^{i \leftrightarrow j})$.

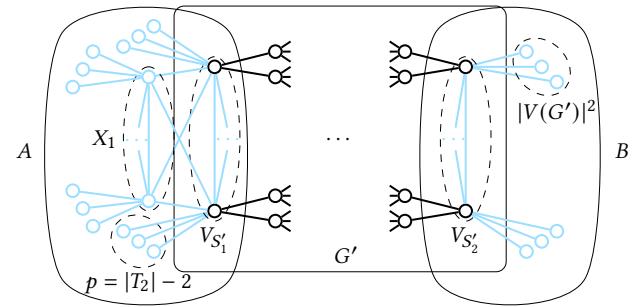


Figure 2: The constructed topology G . Modifications made to the given graph G' are colored light blue. Note that the vertices in $V_{S'_1} \cup X_1$ and $V_{S'_2}$ each form a clique.

Moreover, we ensure that if there exists a swap-equilibrium v in the constructed game, then v restricted to $V(G')$, where some (strategic) agents are replaced by the designated stubborn agents of the same type, is a swap-equilibrium in the given game. Note that the neighborhoods of all vertices in $V(G') \setminus V_{S'}$ are the same in G and G' and thus every swap that is profitable in the assignment in the given game would also be profitable in v . So the remaining challenge here is to design G in such a way that the vertices occupied by stubborn agents of some type in the input game have to be occupied by agents of the same type in every swap-equilibrium in the constructed game. We achieve this by introducing an asymmetry between the types in the construction.

Construction. We are given an instance \mathcal{I}' of S-EQ-STUB consisting of a connected topology G' , a set of agents $[[V(G')]] = N' = R' \cup S'$ partitioned into types T'_1 and T'_2 with at least five strategic and three stubborn agents of each type, and a set $V_{S'} = \{s_i \in V(G') \mid i \in S'\}$ of vertices occupied by stubborn agents with each vertex $v \notin V_{S'}$ being adjacent to two vertices $s_i, s_j \in V_{S'}$ with $i \in T'_1$ and $j \in T'_2$. We denote the sets of vertices occupied by stubborn agents from T'_1 and T'_2 as $V_{S'_1}$ and $V_{S'_2}$, respectively. We construct an instance \mathcal{I} of S-EQ consisting of a topology $G = (V, E)$ and types T_1 and T_2 as follows.

The graph G (sketched in Figure 2) is an extended copy of the given graph G' and contains all vertices and edges from G' . We add three sets of vertices M_1, X_1 , and M_2 as specified below. For every vertex $v \in V_{S'_2}$, we insert $|V(G')|^2$ degree-one vertices only adjacent to v and add them to M_2 . We connect the vertices in $V_{S'_2}$ to form a clique. Let $q := \Delta(G') + |V(G')|^2 + |V_{S'_2}|$ and note that q is an upper bound on the degree of a vertex from $V_{S'_2}$. Let X_1 be a set of $s - |V_{S'_1}|$ vertices, where $s := q \cdot (|T'_2| + |M_2| + \Delta(G)) + 1$ (note that $s > |V_{S'_1}|$). Thus, $|V_{S'_1} \cup X_1| = s$ (we use this property to introduce the mentioned asymmetry between the two types). We connect the vertices in $V_{S'_1} \cup X_1$ to form a clique. Let $p := |T'_2| + |M_2| - 2 > \Delta(G')^2$ (this choice of p is important to ensure that vertices in $V_{S'_1}$ are occupied by agents from T_1 in each swap-equilibrium of the constructed game). For every vertex $v \in V_{S'_1} \cup X_1$, we insert p degree-one vertices only adjacent to v and add them to M_1 . Notably, the neighborhood of all vertices in $V(G') \setminus V_{S'}$ is the same in G' and G .

The set $N = T_1 \cup T_2$ of agents is defined as follows. We have $|T_1| = |T'_1| + |M_1| + |X_1|$ agents in T_1 and $|T_2| = |T'_2| + |M_2|$ agents in T_2 . By the construction of X_1 above, we have that $|V_{S'_1} \cup X_1| = q \cdot (|T'_2| + |M_2| + \Delta(G')) + 1 = q \cdot (|T_2| + \Delta(G')) + 1$. It also holds that $p = |T'_2| + |M_2| - 2 = |T_2| - 2$.

Proof of Correctness. Let $A := M_1 \cup V_{S'_1} \cup X_1$ and $B := M_2 \cup V_{S'_2}$ (the vertices from A should be occupied by agents from T_1 , while the vertices from B should be occupied by agents from T_2). We start with sketching the (easier) direction of the correctness:

LEMMA 3.3 (★). *If the given instance \mathcal{I}' of S-EQ-STUB admits a swap-equilibrium, then the constructed instance \mathcal{I} of S-EQ admits a swap-equilibrium.*

PROOF SKETCH. Given a swap-equilibrium v' on G' for the given instance \mathcal{I}' , we construct a swap-equilibrium v on G for the constructed instance \mathcal{I} . In v , the vertices in $V(G')$ are occupied by agents of the same type as the agents in v' , while all vertices in $M_1 \cup X_1$ are occupied by agents from T_1 and all vertices in M_2 are occupied by agents from T_2 . This, in particular, implies that all vertices from A are occupied by agents from T_1 and all vertices from B by agents from T_2 . Note that in v all agents on vertices from $M_1 \cup X_1 \cup M_2$ have utility 1, while all agents on vertices $V(G') \setminus V_{S'}$ have the same utility as in v' . Thus, any profitable swap must involve at least one agent i with $v_i \in V_{S'}$. However, it can be shown that each such agent i has utility $u_i^G(v) \geq \Delta(G')/\Delta(G')+1$ and can thus never have a profitable swap with an agent j with $v_j \in V_{S'}$. Concerning the remaining case of swapping i and an agent j with $v_j \in V(G') \setminus V_{S'}$ (that is, outside of A and B), using Property 1 from Lemma 3.2 and the fact that the neighborhood of v_j is identical in G and G' , it can be shown that $u_i^G(v^{i \leftrightarrow j}) \leq \Delta(G')-1/\Delta(G')$. From this it follows that there is no profitable swap in v . \square

It remains to prove the more involved reverse direction:

LEMMA 3.4 (★). *If the constructed instance \mathcal{I} of S-EQ admits a swap-equilibrium, then the given instance \mathcal{I}' of S-EQ-STUB admits a swap-equilibrium.*

PROOF SKETCH. The main challenge here is to prove that in every swap-equilibrium v for \mathcal{I} every vertex in A is occupied by an agent from T_1 . Observe that the induced subgraph $G[A]$ contains $q \cdot (|T_2| + \Delta(G')) + 1$ stars (with centers from $V_{S'_1} \cup X_1$, which are possibly further connected) each consisting of $p+1 = (|T_2|-1)$ vertices. If there exists a star which contains agents of both types, then there always exists an agent i on a degree-one vertex with zero utility in this star. It is then possible to prove that i always has a profitable swap. If there exists a star in $G[A]$ which is fully occupied by agents from T_2 , then we can show that the agent i on the central vertex of this star has utility at most $1/q$ (as the central vertices of all other stars in A need to be occupied by agents from T_1). Further, we can show that it is profitable to swap i with an agent from T_1 adjacent to the remaining agent from T_2 (that needs to be placed outside of A). It follows that all vertices from A are occupied by agents from T_1 . From this we can conclude that all vertices from B are occupied by agents from T_2 in v . Using this, it is easy to show that v restricted to $V(G')$ with stubborn agents of the respective types on the vertices in $V_{S'}$ is a swap-equilibrium in \mathcal{I}' . \square

3.2 Jump-Equilibria

Inspired by the reduction for S-EQ described above, we can prove that deciding the existence of a jump-equilibrium is NP-hard as well. While the constructions behind both reductions use the same underlying general ideas, the proof for J-EQ is more involved. The main challenge here is that in every assignment some vertices remain unoccupied. For instance, we do not only need to prove that only agents from T_1 are placed on vertices from A (which is more challenging because we have to deal with possibly unoccupied vertices) but also that all vertices from A are occupied.

THEOREM 3.5 (★). *J-EQ is NP-complete.*

4 ROBUSTNESS OF EQUILIBRIA

Having established in Section 3 that deciding the existence of an equilibrium is NP-hard, we now introduce the concept of *robustness* of an equilibrium. We consider both the robustness of an equilibrium with respect to the deletion of edges and with respect to the deletion of vertices, where deleting a vertex also implies deleting the agent occupying the vertex from the Schelling game.

Definition 4.1. For a Schelling game I on a graph G , an equilibrium v in I is *r-edge-robust* (*r-vertex-robust*) for some $r \in \mathbb{N}_0$ if v is an equilibrium in I on the topology $G - S$ for all subsets of edges $S \subseteq E(G)$ (for all subsets of vertices $S \subseteq V(G)$) with $|S| \leq r$. The *edge-robustness* (*vertex-robustness*) of v is the largest $r \in [0, |E(G)|]$ ($r \in [0, |V(G)|]$) for which v is *r-edge-robust* (*r-vertex-robust*).

Note that given an equilibrium v that becomes unstable after deleting r edges/vertices, deleting further edges/vertices can make v stable again, as any assignment is stable if we delete all edges/vertices. That is why in Definition 4.1 we require that v is stable for all S with $|S| \leq r$ and not only for all S with $|S| = r$.

We focus on the robustness of swap-equilibria. While for jump-equilibria the introduced concepts are also meaningful, already obtaining lower and upper bounds on the robustness of a jump-equilibrium on a single fixed graph is problematic, as the robustness of a jump-equilibrium significantly depends on the number of unoccupied vertices. For instance, if the number of agents of both types is small, on most graphs the agents can be placed such that all agents are only adjacent to friends making the equilibrium quite robust.

Note that as we restrict our attention to swap-equilibria, deleting agents and vertices is equivalent, as in this context a once unoccupied vertex can never become occupied again and is also irrelevant for computing utilities. Thus, all our results on vertex-robustness also apply to the robustness with respect to the deletion of agents. However, when analyzing jump-equilibria one would have to distinguish the two and consider the robustness of a jump-equilibrium with respect to the deletion of agents (but not vertices) and the deletion of vertices (and the agents occupying them).

4.1 First Observations

We start by observing that if for one type there exists only a single agent, then there never is a profitable swap. Hence, in this case, every assignment trivially is a swap-equilibrium of edge-robustness $|E(G)|$ and vertex-robustness $|V(G)|$. That is why, in the following, we assume that $\min\{|T_1|, |T_2|\} \geq 2$. Further, if we consider vertex-robustness, then in case that $|T_1| = 2$ and $|T_2| = 2$,

after deleting an agent from one type the other agent from this type will always have zero utility and cannot be part of a profitable swap, implying that any swap-equilibrium has vertex-robustness $|V(G)|$. Thus, in the following, considering vertex-robustness, we assume that $|T_1| \geq 3$ and $|T_2| \geq 2$.

Focusing on edge-robustness for a moment, only the deletion of edges between agents of the same type has an influence on the stability of a swap-equilibrium. This is stated more precisely in the following proposition.

PROPOSITION 4.2 (★). *Let \mathbf{v} be a swap-equilibrium for a Schelling game on topology G . Let $S \subseteq E(G)$ be a set of edges such that \mathbf{v} is not a swap-equilibrium on $G - S$. Then,*

- (i) *S contains at least one edge between agents of the same type.*
- (ii) *\mathbf{v} is also not a swap-equilibrium on $G - S'$, where $S' \subseteq S$ is the subset of edges from S that connect agents of the same type.*
- (iii) *For every set $A \subseteq \{\{v_i, v_j\} \in E(G) \mid i, j \in T_1 \vee i, j \in T_2\}$ of edges between agents of the same type, \mathbf{v} is also not a swap-equilibrium on $G - (S \cup A)$.*

For vertex-robustness, one can similarly observe that only deleting a vertex occupied by an agent a adjacent to at least one vertex occupied by a friend of a can make a swap-equilibrium unstable.

Next, note that the utility of an agent only depends on its neighborhood. Thus, whether two agents i and j have a profitable swap in $G - S$ only depends on the edges/vertices incident/adjacent to v_i and v_j in S . Combining this with the observation that no profitable swap can involve an agent on an isolated vertex, it follows that if a swap-equilibrium cannot be made unstable by deleting $2 \cdot (\Delta(G) - 1)$ edges/vertices, then it cannot be made unstable by deleting an arbitrary number of edges/vertices:

OBSERVATION 1. *Let \mathbf{v} be a swap-equilibrium for a Schelling game on G . If \mathbf{v} is $2 \cdot (\Delta(G) - 1)$ -edge-robust, \mathbf{v} has edge-robustness $|E(G)|$ and if \mathbf{v} is $2 \cdot (\Delta(G) - 1)$ -vertex-robust, \mathbf{v} has vertex-robustness $|V(G)|$.*

The simple fact that the utility of an agent only depends on its neighborhood leads to a polynomial-time algorithm to determine whether a given swap-equilibrium \mathbf{v} has edge-robustness $r \in \mathbb{N}_0$: We simply iterate over all pairs of agents i and j and check whether we can delete at most r edges between v_i and adjacent vertices occupied by friends of i and between v_j and adjacent vertices occupied by friends of j such that the swap of i and j becomes profitable (note that the stability of \mathbf{v} only depends on the number of such deleted edges in the neighborhood of each agent, not the exact subset of edges). For vertex-robustness, an analogous approach works.

PROPOSITION 4.3 (★). *Given a Schelling game with n agents, a swap-equilibrium \mathbf{v} , and an integer $r \in \mathbb{N}_0$, one can decide in $O(n^2 \cdot r)$ time whether \mathbf{v} is r -edge/vertex-robust.*

Note, however, that finding a swap-equilibrium whose vertex- or edge-robustness is as high as possible is NP-hard, as we have proven in Theorem 3.1 that already deciding whether a Schelling game admits some swap-equilibrium is NP-hard.

4.2 Robustness of Equilibria on Different Graph Classes

In this subsection, we analyze the influence of the topology on the robustness of swap-equilibria. We first analyze cliques where each

swap-equilibrium has edge-robustness zero and vertex-robustness $|V(G)|$. Subsequently, we turn to cycles, paths, and grids and find that there exists a swap-equilibrium on all these graphs with edge-robustness and vertex-robustness zero. For paths, we observe that the difference between the edge/vertex-robustness of the most and least robust equilibrium can be arbitrarily large. Finally, with α -star-constellation graphs for $\alpha \in \mathbb{N}_0$, we present a class of graphs on which all swap-equilibria have at least edge/vertex-robustness α .

We start by observing that on a clique every assignment is a swap-equilibrium. From this it directly follows that every swap-equilibrium has vertex-robustness $|V(G)|$, as deleting a vertex from a clique results in another clique. In contrast, each swap-equilibrium can be made unstable by deleting one edge. Thereby, the following observation also proves that the difference between the edge- and vertex-robustness of a swap-equilibrium can be arbitrarily large:

OBSERVATION 2. *In a Schelling game on a clique G with $|T_1| \geq 2$ and $|T_2| \geq 2$, every swap-equilibrium \mathbf{v} has edge-robustness zero and vertex-robustness $|V(G)|$.*

PROOF. It remains to prove that the edge-robustness is always zero. Let $i \neq j \in T_1$, $e := \{v_i, v_j\} \in E(G)$, and $l \in T_2$. As G is a clique, it holds that $u_i^G(\mathbf{v}) = \frac{|T_1|-1}{|T_1|+|T_2|-1}$ and $u_l^G(\mathbf{v}) = \frac{|T_2|-1}{|T_1|+|T_2|-1}$. Swapping i and l is profitable in \mathbf{v} on $G - \{e\}$ for both i and l , as

$$u_i^{G-\{e\}}(\mathbf{v}^{i \leftrightarrow l}) = \frac{|T_1|-1}{|T_1|+|T_2|-1} > \frac{|T_1|-2}{|T_1|+|T_2|-2} = u_i^{G-\{e\}}(\mathbf{v}) \text{ and}$$

$$u_l^{G-\{e\}}(\mathbf{v}^{i \leftrightarrow l}) = \frac{|T_2|-1}{|T_1|+|T_2|-2} > \frac{|T_2|-1}{|T_1|+|T_2|-1} = u_l^{G-\{e\}}(\mathbf{v}).$$

□

For a cycle G , we can show that in a swap-equilibrium \mathbf{v} , every agent is adjacent to at least one friend. Then, picking an arbitrary agent $i \in T_1$ that has utility $1/2$ in \mathbf{v} and deleting i 's neighbor from T_1 or the edge between i and its neighbor from T_1 makes \mathbf{v} unstable.

PROPOSITION 4.4 (★). *In a Schelling game on a cycle G with $|T_1| \geq 2$ and $|T_2| \geq 2$, every swap-equilibrium \mathbf{v} has edge-robustness zero. For $|T_1| \geq 3$ and $|T_2| \geq 2$, every swap-equilibrium \mathbf{v} has vertex-robustness zero.*

Next, we turn to paths and prove that every Schelling game on a path with sufficiently many agents from both types has an equilibrium with edge-/vertex-robustness zero and one with edge-robustness $|E(G)|$ and vertex-robustness $|V(G)|$. This puts paths in a surprisingly sharp contrast to cycles. The reason for this is that on a path, we can always position the agents such that there exists only one edge between agents of different types, yielding a swap-equilibrium with edge-robustness $|E(G)|$ and vertex-robustness $|V(G)|$. This is not possible on a cycle.

THEOREM 4.5. *For a Schelling game on a path G with $|T_1| \geq 4$ and $|T_2| \geq 2$, there exists a swap-equilibrium \mathbf{v} that has edge-robustness and vertex-robustness zero and a swap-equilibrium \mathbf{v}' that has edge-robustness $|E(G)|$ and vertex-robustness $|V(G)|$.*

PROOF. Let $V(G) = \{w_1, \dots, w_n\}$ and $E(G) = \{\{w_i, w_{i+1}\} \mid i \in [n-1]\}$. In \mathbf{v} , vertices w_1 and w_2 are occupied by agents from T_1 , vertices w_3 to $w_{|T_2|+2}$ are occupied by agents from T_2 , and the remaining $|T_1| - 2 \geq 2$ vertices are occupied by agents from T_1 (see

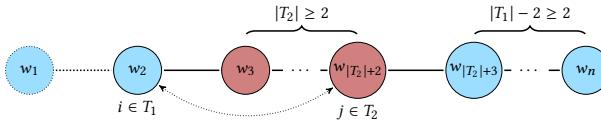


Figure 3: The swap-equilibrium with robustness zero from Theorem 4.5. After deleting $\{w_1, w_2\} \in E(G)$ or $w_1 \in V(G)$, swapping i and j is profitable.

Figure 3 for a visualization). As all agents have at most one neighbor of the other type and at least one neighbor of the same type, for each pair i, j of agents of different types it holds that $u_i(v^{i \leftrightarrow j}) \leq 1/2 \leq u_i(v)$. Thus, v is a swap-equilibrium. Further, after deleting the edge between w_1 and w_2 or deleting the vertex w_1 , swapping the agent on w_2 with the agent on $w_{|T_2|+2}$ is profitable. It follows that v has edge-robustness and vertex-robustness zero.

In v' , the agents from T_1 occupy the first $|T_1|$ vertices and agents from T_2 the remaining vertices. Let $S \subseteq E(G)$ or $S \subseteq V(G)$ and consider $G - S$. As for $j \in [|T_1| - 1] \cup [|T_1| + 2, n]$, in $G - S$, the agent on w_j got deleted, has no neighbor, or is only adjacent to friends, it can never be involved in a profitable swap. Further, swapping the agent on $w_{|T_1|}$ and the agent on $w_{|T_1|+1}$ can also never be profitable, since after the swap none of the two is adjacent to a friend. Thus, v' is a swap-equilibrium on $G - S$. \square

If $\max\{|T_1|, |T_2|\} \leq 3$, which is not covered by Theorem 4.5, then in every swap-equilibrium the path is split into two subpaths and agents from T_1 occupy one subpath and agents from T_2 occupy the other subpath. As argued in the proof of Theorem 4.5, such an assignment has edge-robustness $|E(G)|$ and vertex-robustness $|V(G)|$.

Turning to grids, which besides paths form the class which has been most often considered in the context of Schelling's segregation model, for both vertex- and edge-robustness, we show using some more involved arguments that every swap-equilibrium has either robustness one or zero and that there exists an infinite class of Schelling games on grids admitting a swap-equilibrium with robustness zero and one with robustness one.

- THEOREM 4.6 (★).** (1) In a Schelling game with $|T_1| \geq 2$ and $|T_2| \geq 2$ on an $(x \times y)$ -grid with $x \geq 2, y \geq 2$, the edge-robustness of a swap-equilibrium is at most one.
(2) In a Schelling game with $|T_1| \geq 4$ and $|T_2| \geq 4$ on an $(x \times y)$ -grid with $x \geq 3$ and $y \geq 3$, the vertex-robustness of a swap-equilibrium is at most one.
(3) In a Schelling game with $|T_1| = |T_2|$ on an $(x \times y)$ -grid with even $x \geq 4$ and $y \geq 2$, there exists a swap-equilibrium v with edge- and vertex-robustness zero and a swap-equilibrium v' with edge- and vertex-robustness one.

Lastly, motivated by the observation that on all previously considered graph classes there exist swap-equilibria with zero edge-robustness and on all considered graph classes except cliques there exist swap-equilibria with zero vertex-robustness, we investigate α -star-constellation graphs, a generalization of stars and α -caterpillars. We prove that every swap-equilibrium in a Schelling game on an α -star-constellation graph is α -vertex-robust and α -edge-robust. We also show that a swap-equilibrium on an α -star-constellation graph may fail to exist but that we can precisely

characterize swap-equilibria on such graphs. Using this characterization, we design a polynomial-time algorithm for S-Eq on α -star-constellation graphs and show that there always exists a swap-equilibrium on an α -caterpillar, that is, an α -star-constellation graph which restricted to non-degree-one vertices forms a path.

THEOREM 4.7. In a Schelling game on an α -star-constellation graph for $\alpha \in \mathbb{N}_0$, every swap-equilibrium v is α -edge and α -vertex-robust.

PROOF. Let v be a swap-equilibrium on an α -star-constellation graph G for some $\alpha \in \mathbb{N}_0$. We make a case distinction based on whether or not there exists an agent i on a degree-one vertex adjacent to an agent j of the other type in v . If this is the case, then assume without loss of generality that $i \in T_1$ and $j \in T_2$ and observe that it needs to hold that all agents $j' \in T_2 \setminus \{j\}$ are only adjacent to friends, as otherwise j' and i have a profitable swap. Now consider the topology $G - S$ for some subset $S \subseteq E(G)$ or some subset $S \subseteq V(G)$. Then, for all $j' \in T_2 \setminus \{j\}$, agent j' cannot be involved in a profitable swap in $G - S$, as j' got deleted, is only adjacent to friends, or placed on an isolated vertex. Moreover, there also cannot exist a profitable swap for j , as no agent from T_1 is adjacent to an agent from $T_2 \setminus \{j\}$. Hence, v is $|E(G)|$ -edge-robust and $|V(G)|$ -vertex-robust.

Now, assume that all agents on a degree-one vertex are only adjacent to friends in v and consider the topology $G - S$ for $S \subseteq E(G)$ or $S \subseteq V(G)$ with $|S| \leq \alpha$. Note that, in $G - S$, only agents $i \in T_1$ and $j \in T_2$ with $\deg_G(v_i) > 1$ and $\deg_G(v_j) > 1$ and $\deg_{G-S}(v_i) \geq 1$ and $\deg_{G-S}(v_j) \geq 1$ can be involved in a profitable swap, since all other agents either occupy an isolated vertex or are only adjacent to friends in $G - S$. For vertex-robustness, it additionally needs to hold that $v_i, v_j \notin S$. Since G is an α -star-constellation graph and we delete at most α edges or α vertices, it holds that both v_i and v_j are adjacent to at least as many degree-one vertices as non-degree-one vertices in $G - S$. By our assumption, the agents on degree-one vertices adjacent to v_i are friends of i and the agents on degree-one vertices adjacent to v_j are friends of j . Hence, swapping i and j cannot be profitable, as $u_k^{G-S}(v) \geq 1/2$ and $u_k^{G-S}(v^{i \leftrightarrow j}) \leq 1/2$ for $k \in \{i, j\}$. \square

Theorem 4.7 has no implications for the existence of swap-equilibria on α -star-constellation graphs. Indeed, we observe that there is no swap-equilibrium in a Schelling game with $|T_1| = 5$ and $|T_2| = 7$ on the 1-star-constellation graph depicted in Figure 4. Notably, to the best of our knowledge the graph from Figure 4 is the first known graph without a swap-equilibrium that is not a tree.

PROPOSITION 4.8. A Schelling game on an α -star-constellation graph G may fail to admit a swap-equilibrium, even if G is a split graph, that is, the vertices of the graph can be partitioned into a clique and an independent set.

PROOF. Consider the Schelling game with $|T_1| = 5$ many agents of type T_1 and $|T_2| = 7$ many agents of type T_2 on the graph G from Figure 4, which consists of three 3-stars whose central vertices form a clique. Observe that as all stars in G consist of four vertices and neither $|T_1| = 5$ nor $|T_2| = 7$ are divisible by four, in any assignment v , there exists a degree-one vertex occupied by an agent $i \in T_1$ such that the adjacent central vertex is occupied by an

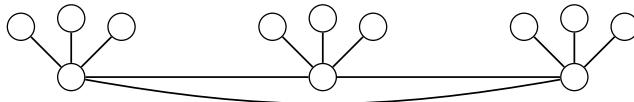


Figure 4: There is no swap-equilibrium in a Schelling game with $|T_1| = 5$ and $|T_2| = 7$ on this 1-star-constellation graph.

agent $j \in T_{l'}$ of the other type with $l \neq l'$. Let $v \neq v'$ be the other two central vertices. We make a case distinction based on whether the agents on the degree-one vertices adjacent to v and v' have the same type as their respective neighbor on the central vertex. If this is the case, then since we have $|T_1| < 8$ and $|T_2| < 8$, the vertices v and v' cannot be occupied by agents of the same type. Assume by symmetry, without loss of generality, that an agent $j' \in T_{l'}$ occupies vertex v and an agent $i' \in T_l$ occupies vertex v' . Then, we have $u_{j'}(v) < 1$ and swapping i and j' is profitable, as it holds that $u_i(v) = 0 < u_i(v^{i \leftrightarrow j'})$ and $u_{j'}(v) < 1 = u_{j'}(v^{i \leftrightarrow j'})$.

Otherwise, there is an agent on a degree-one vertex that has a different type than the agent on the adjacent central vertex v or v' . This implies that there is an agent $j' \neq j$ from $T_{l'}$ with $u_{j'}(v) < 1$. Then, similarly to the case above, swapping i and j' is profitable. \square

On the positive side, we can precisely characterize swap-equilibria in Schelling games on α -star-constellation graphs.

THEOREM 4.9 (★). *Let G be an α -star-constellation graph with $\alpha \in \mathbb{N}_0$ and let \mathbf{v} be an assignment in some Schelling game on G . The assignment \mathbf{v} is a swap-equilibrium if and only if at least one of the following two conditions holds.*

- (1) *Every vertex $v \in V(G)$ with $\deg_G(v) = 1$ is occupied by an agent from the same type as the only adjacent agent in \mathbf{v} .*
- (2) *There exists an agent $i \in T_l$ for some $l \in \{1, 2\}$ such that all other agents $i' \in T_l \setminus \{i\}$ are only adjacent to friends in \mathbf{v} .*

Using Theorem 4.9, we now argue that there is a subclass of α -star-constellation graphs, namely α -caterpillars, on which a swap-equilibrium always exists. Consider a Schelling game on an α -caterpillar G with w_1, \dots, w_ℓ being the non-degree-one vertices forming the central path ($\{\{w_i, w_{i+1}\} \mid i \in [\ell - 1]\} \subseteq E(G)$). It is easy to construct a swap-equilibrium \mathbf{v} on G by assigning for each $i \in \{1, \dots, \ell\}$ agents from T_1 to w_i and to adjacent degree-one vertices, until all agents from T_1 have been assigned; in which case the remaining vertices are filled with agents from T_2 . As \mathbf{v} fulfills Condition 2 from Theorem 4.9, \mathbf{v} is a swap-equilibrium and it is easy to see that \mathbf{v} has edge-robustness $|E(G)|$ and vertex-robustness $|V(G)|$. Notably, this assignment somewhat resembles the swap-equilibrium with edge-robustness $|E(G)|$ and vertex-robustness $|V(G)|$ on a path from Theorem 4.5. In contrast, extending the swap-equilibrium with robustness zero on a path from Theorem 4.5 such that all agents on degree-one vertices are of the same type as their only neighbor, in some Schelling games on α -caterpillars, it is possible to create a swap-equilibrium with edge- and vertex-robustness only α :

PROPOSITION 4.10 (★). *For a Schelling game on an α -caterpillar with $\alpha \in \mathbb{N}_0$, there is a swap-equilibrium with edge-robustness $|E(G)|$ and vertex-robustness $|V(G)|$. For every $\alpha \in \mathbb{N}_0$, there is a Schelling*

game on an α -caterpillar with a swap-equilibrium with edge-robustness and vertex-robustness α .

The characterization of swap-equilibria from Theorem 4.9 also yields a polynomial-time algorithm (using dynamic programming for SUBSET SUM) to decide for a Schelling game on an α -star-constellation graph whether it admits a swap-equilibrium.

COROLLARY 4.11 (★). *For a Schelling game on an α -star-constellation graph with $\alpha \in \mathbb{N}_0$, one can decide in polynomial time whether a swap-equilibrium exists.*

5 CONCLUSION

We proved that even in the simplest variant of Schelling games where all agents want to maximize the fraction of agents of their type in their occupied neighborhood, deciding the existence of a swap- or jump-equilibrium is NP-complete. Moreover, we introduced a notion for the robustness of an equilibrium under vertex or edge deletions and proved that the robustness of different swap-equilibria on the same topology can vary significantly. In addition, we found that the minimum and the maximum robustness of swap-equilibria vary depending on the underlying topology.

There are multiple possible directions for future research. First, independent of properties of the given graph, in our reduction showing the NP-hardness of deciding the existence of a swap- or jump-equilibrium, we construct a graph that is non-planar and which has a non-constant maximum degree. The same holds for graphs constructed in the reductions from Agarwal et al. [1] for showing NP-hardness in the presence of stubborn agents. Thus, the computational complexity of deciding the existence of equilibria on planar or constant-degree graphs (properties that typically occur in the real world) in Schelling games with or without stubborn agents is open. Second, Bilò et al. [5] recently introduced the notions of local swap (jump)-equilibria where only adjacent agents are allowed to swap places (agents are only allowed to jump to adjacent vertices). To the best of our knowledge, the computational complexity of deciding the existence of a local swap- or jump-equilibrium is unknown even if we allow for stubborn agents. Third, while we showed that on most considered graphs swap-equilibria can be very non-robust, it might be interesting to search for graphs guaranteeing a higher equilibrium robustness; here, graphs with a high minimum degree and/or high connectivity seem to be promising candidates. Fourth, besides looking at the robustness of equilibria with respect to the deletion of edges or vertices, one may also study adding or contracting edges or vertices. Fifth, instead of analyzing the robustness of a specific equilibrium, one could also investigate the robustness of a topology regarding the existence of an equilibrium. Lastly, for an equilibrium, it would also be interesting to analyze empirically or theoretically how many reallocations of agents take place on average after a certain change has been performed until an equilibrium is reached again.

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