For the "only if" part, let $\bar{x} = (x_v)_{v \in \mathcal{V}}$ be a PSNE of the BNPG game. We claim that we have $x_{d_1} = 1, x_{u_i} = x_{u'_i} = 1, \forall i \in [n], x_{u'_{n+1}} = 0, x_{d_2} = 0$. To prove this, we consider all cases for $(x_{u_i})_{i \in [n]}$.

(1) Case – $\forall i \in [n]\ x_{u_i} = 1$: We have $x_{d_2} = 0$ as $n_{d_2} > 0$ otherwise d_2 would deviate. This implies that $x_{u'_{n+1}} = 0$ since $n_{u'_{n+1}} \leq 1$(as $x_{d_2} = 0$). Now we consider the following sub-cases (according to the values of x_{d_1} and $x_{u'_i}, i \in [n]$):
- ($x_{d_1} = 1, \exists k \in [n]$ such that $x_{u'_k} = 0$). Here $x_{u'_k}$ will then deviate to 1 as $n_{u'_k} = 2$. Hence, it is not a PSNE.
- ($x_{d_1} = 1, \forall i \in [n]\ x_{u'_i} = 1$). This is exactly what we claim thus we have nothing to prove in this case.
- ($x_{d_1} = 0, \exists k \in [n]$ such that $x_{u'_k} = 1$). Here $x_{u'_k}$ will then deviate to 0 as $n_{u'_k} = 1$. Hence, it is not a PSNE.
- ($x_{d_1} = 0, \forall i \in [n]\ x_{u'_i} = 0$). The player d_1 will deviate to 1 as $n_{d_1} = 0$. Hence, it is not a PSNE.

(2) Case – $\exists k_1, k_2 \in [n]$ such that $x_{u_{k_1}} = 1$ and $x_{u_{k_2}} = 0$: We have $x_{d_2} = 0$ as $n_{d_2} > 0$ otherwise d_2 would deviate. This implies that $x_{u'_{n+1}} = 0$ since $n_{u'_{n+1}} \leq 1$ (as $x_{d_2} = 0$). Now we consider the following sub-cases (according to the values of x_{d_1} and $x_{u'_i}, i \in [n]$):
- ($x_{d_1} = 1, \forall i \in [n]\ x_{u'_i} = 0$). Here u'_{k_1} will deviate to 1 as $n_{u'_{k_1}} = 2$. So, it isn't a PSNE.
- ($x_{d_1} = 1, \forall i \in [n]\ x_{u'_i} = 1$). Here u'_{k_2} will deviate to 0 as $n_{u'_{k_2}} = 1$. So, it isn't a PSNE.
- ($x_{d_1} = 1, \exists i, j \in [n]$ such that $x_{u'_i} = 1$ and $x_{u'_j} = 0$). Here d_1 will deviate to 0 as $0 < n_{d_1} < n$ (there are at least 2 neighbours of d_1 which play 0 and at least 1 neighbour of d_1 which plays 1). Hence, it is not a PSNE.
- ($x_{d_1} = 0, \forall i \in [n]\ x_{u'_i} = 0$). Here d_1 will deviate to 1 as $n_{d_1} = 0$. So, it isn't a PSNE.
- ($x_{d_1} = 0, \exists i \in [n]$ such that $x_{u'_i} = 1$). Here u'_i will deviate to 0 as $n_{u'_i} \leq 1$ and hence, it is not a PSNE.

(3) Case – $\forall i \in [n]\ x_{u_i} = 0$: For every $i \in [n]$, we must have $x_{u'_i} = 0$ so that u'_i doesn't deviate. We have the following sub-cases (according to the values of x_{d_1}, x_{d_2} and $x_{u'_{n+1}}$):
- ($x_{d_1} = 0, x_{u'_{n+1}} = 0$). Here d_1 deviates to 1 as $n_{d_1} = 0$ and hence, it is not a PSNE.
- ($x_{d_1} = 0, x_{u'_{n+1}} = 1$). Here u'_{n+1} deviates to 0 as $n_{u'_{n+1}} \leq 1$. So, it isn't a PSNE.
- ($x_{d_1} = 1, x_{u'_{n+1}} = 0, x_{d_2} = 0$). Here d_2 deviates to 1 as $n_{d_2} = 0$. So, it isn't a PSNE.
- ($x_{d_1} = 1, x_{u'_{n+1}} = 0, x_{d_2} = 1$). Here u'_{n+1} deviates to 1 as $n_{u'_{n+1}} = 2$ and hence, it is not a PSNE.
- ($x_{d_1} = 1, x_{u'_{n+1}} = 1, x_{d_2} = 0$).Here u'_{n+1} deviates to 0 as $n_{u'_{n+1}} = 1$ and hence, it is not a PSNE.
- ($x_{d_1} = 1, x_{u'_{n+1}} = 1, x_{d_2} = 1$).Here d_2 deviates to 0 as $n_{d_2} > 0$. So, it isn't a PSNE.

So if $\bar{x} = (x_v)_{v \in \mathcal{V}}$ is a PSNE of the BNPG game, then we have $x_{d_1} = 1, \forall i \in [n], x_{u'_i} = 1, x_{u'_{n+1}} = 0, \forall i \in [n]\ x_{u_i} = 1, x_{d_2} = 0$. Now consider the set $\mathcal{F} = \{\{v_i, v_j\} : x_{a_{\{i,j\}}} = 1, \{v_i, v_j\} \in \mathcal{E}'\}$. Note that $\forall i \in [n]$, the number of neighbors of u_i playing 1 excluding u'_i and d_2 (which in this case is $n_{u_i} - 1$ as $x_{d_2} = 0, x_{u'_i} = 1$) is the same as the number of edges in $\mathcal{F}$ which are incident on v_i in $\mathcal{G}$. Since $\forall i, n_{u_i} - 1 \in K(v_i)$, the number of edges in $\mathcal{F}$ incident on v_i in General Factor instance is an element of $K(v_i)$. Hence, the General Factor instance is a yes instance. □

Corollary 4.4. *Exists-PSNE for BNPG games is* W[1]*-hard parameterized by treewidth and pathwidth.*

We next consider the diameter (d) of the graph as our parameter and prove para-NP-hardness in Observation 1. It follows immediately from the fact that the reduced instance in the NP-completeness proof of Exists-PSNE for BNPG games in [29] has diameter 2.

Observation 1. *Exists-PSNE for BNPG games is* NP*-complete even for graphs of diameter at most* 2. *In particular, the* Exists-PSNE *problem for BNPG games is para-*NP*-hard parameterized by diameter.*

We next consider a variant of Exists-PSNE where at most k_0 (respectively k_1) players are playing 0 (respectively 1) in the PSNE. We denote this variant as k_0-Exists-PSNE (resp. k_1-Exists-PSNE). Obviously there is a brute force XP algorithm which runs in time $O^*\left(n^{k_0}\right)$ (respectively $O^*\left(n^{k_1}\right)$). We show that k_0-Exists-PSNE (resp. k_1-Exists-PSNE) is W[2]-hard parameterized by k_0 (respectively k_1). For this, we reduce from the Dominating Set problem parameterized by the size of dominating set which is known to be W[2]-hard [6].

Theorem 4.5 (★). *k_0-Exists-PSNE for BNPG games is* W[2]*-hard parameterized by k_0.*

Theorem 4.6 (★). *k_1-Exists-PSNE for BNPG games is* W[2]*-hard parameterized by k_1 even for fully homogeneous BNPG games.*

Till now we have mostly focused on heterogeneous BNPG games. We next consider fully homogeneous BNPG games and show the following by reducing from the Exists-PSNE problem on heterogeneous BNPG games.

Theorem 4.7. *The following results hold even for fully homogeneous games.*

(1) *Exists-PSNE is* NP*-complete even if the diameter of the graph is at most* 4.
(2) *Exists-PSNE is* W[1]*-hard with respect to the parameter treedepth of the graph.*
(3) *k_0-Exists-PSNE is* W[2]*-hard parameterized by k_0.*

Proof. We first present a reduction from the Exists-PSNE problem on heterogeneous BNPG games to the Exists-PSNE problem on fully homogeneous BNPG games. Let $(\mathcal{G} = (\mathcal{V} = \{v_i : i \in [n]\}, \mathcal{E}), (g_v)_{v \in \mathcal{V}}, (c_v)_{v \in \mathcal{V}})$ be any heterogeneous BNPG game. We now construct the graph $\mathcal{H} = (\mathcal{V}', \mathcal{E}')$ for the instance of the fully homogeneous BNPG game.

$$\mathcal{V}' = \{u_i : i \in [n]\} \cup \bigcup_{i \in [n]} \mathcal{V}_i,$$
$$\text{where } \mathcal{V}_i = \{a^i_j : j \in [2 + (i-1)n]\}, \forall i \in [n]$$
$$\mathcal{E}' = \{\{u_i, u_j\} : \{v_i, v_j\} \in \mathcal{E}\} \cup \bigcup_{i \in [n]} \mathcal{E}_i,$$
$$\text{where } \mathcal{E}_i = \{\{a^i_j, u_i\} : j \in [2 + (i-1)n]\}, \forall i \in [n]$$

Let us define $f(x) = \lfloor \frac{x-2}{n} \rfloor + 1, h(x) = x - 2 - (f(x) - 1)n$. We now define best-response strategies β for the fully homogeneous BNPG game on $\mathcal{H}$.

$$\beta(k) = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = 1 \\ \{0,1\} & \text{if } \Delta g_{v_{f(k)}}(h(k)) = c_{v_{f(k)}}, k > 1 \\ 1 & \text{if } \Delta g_{v_{f(k)}}(h(k)) > c_{v_{f(k)}}, k > 1 \\ 0 & \text{if } \Delta g_{v_{f(k)}}(h(k)) < c_{v_{f(k)}}, k > 1 \end{cases}$$

This finishes description of our fully homogeneous BNPG game on $\mathcal{H}$. We now claim that there exists a PSNE in the heterogeneous BNPG game on $\mathcal{G}$ if and only if there exists a PSNE in the fully homogeneous BNPG game on $\mathcal{H}$.

For the "only if" part, let $x^* = (x^*_v)_{v\in\mathcal{V}}$ be a PSNE in the heterogeneous BNPG game on $\mathcal{G}$. We now consider the following strategy profile $\bar{y} = (y_v)_{v\in\mathcal{V}'}$ for players in $\mathcal{H}$.

$$\forall i \in [n] y_{u_i} = x^*_{v_i}; y_w = 1 \text{ for other vertices } w$$

Clearly the players in $\cup_{i\in[n]}\mathcal{V}_i$ do not deviate as their degree is 1 and $\beta(0) = \beta(1) = 1$. In $\bar{y}$, we have $n_{u_i} = n_{v_i} + 2 + (i-1)n \geq 2$ and $n_{v_i} \leq n - 1$ for every $i \in [n]$. If $x^*_{v_i} = 1$, then we have $\Delta g_{v_i}(n_{v_i}) \geq c_{v_i}$. We have $f(n_{u_i}) = i$ and $h(n_{u_i}) = n_{v_i}$. This implies that $\Delta g_{v_{f(n_{u_i})}}(h(n_{u_i})) \geq c_{v_{f(n_{u_i})}}$. So u_i does not deviate as 1 is the best-response. If $x^*_{v_i} = 0$, then we have $\Delta g_{v_i}(n_{v_i}) \leq c_{v_i}$. This implies that $\Delta g_{v_{f(n_{u_i})}}(h(n_{u_i})) \leq c_{v_{f(n_{u_i})}}$. So u_i does not deviate as 0 is the best-response. Hence, $\bar{y}$ is a PSNE.

For the "if" part, suppose there exists a PSNE $(x^*_v)_{v\in\mathcal{V}'}$ in the fully homogeneous BNPG game on $\mathcal{H}$. Clearly $x^*_v = 1$ for all $v \in \cup_{i\in[n]}\mathcal{V}_i$ as $n_v \leq 1$. Now we claim that the strategy profile $\bar{x} = (x_{v_i} = x^*_{u_i})_{i\in[n]}$ forms a PSNE for the heterogeneous BNPG game on $\mathcal{G}$. We observe that if $x^*_{u_i} = 1$, then $\Delta g_{v_{f(n_{u_i})}}(h(n_{u_i})) \geq c_{v_{f(n_{u_i})}}$ for $i \in [n]$. This implies that $\Delta g_{v_i}(n_{v_i}) \geq c_{v_i}$. So $x_{v_i} = 1$ is the best-response for v_i in $\mathcal{G}$ and hence, she does not deviate. Similarly, If $x^*_{u_i} = 0$, then $\Delta g_{v_{f(n_{u_i})}}(h(n_{u_i})) \leq c_{v_{f(n_{u_i})}}$. This implies that $\Delta g_{v_i}(n_{v_i}) \leq c_{v_i}$. So $x_{v_i} = 0$ is the best-response for $v_i \in \mathcal{V}$ and hence, it won't deviate. Hence, $\bar{x}$ is a PSNE in the heterogeneous BNPG game on $\mathcal{G}$.

We now prove the three statements in the theorem as follows.

(1) We observe that, if the diameter of $\mathcal{G}$ is at most 2, then the diameter of $\mathcal{H}$ is at most 4. Hence, the result follows from Observation 1.
(2) We observe that the treedepth of $\mathcal{H}$ is at most 1 more than the treedepth of $\mathcal{G}$. Hence, the result follows from Theorem 4.3.
(3) We observe that there exists a PSNE where at most k players play 0 in the heterogeneous BNPG game on $\mathcal{G}$ if and only if there exists a PSNE where at most k players play 0 in the fully homogeneous BNPG game on $\mathcal{H}$. Hence, the result follows from Theorem 4.5.

□

We next show that Exists-PSNE for fully homogeneous BNPG games is para-NP-hard parameterized by the maximum degree of the graph again by reducing from heterogeneous BNPG games.

Theorem 4.8 (★). Exists-PSNE *for fully homogeneous BNPG games is* NP*-complete even if the maximum degree* Δ *of the graph is at most* 9.

4.2 XP Algorithm for the parameter treewidth

Our next result is an XP algorithm for the Exists-PSNE problem when parameterized by treewidth. Towards that, we introduce the notion of "feasible function" in Definition 4.9 and prove a related algorithmic result in Lemma 4.10.

Definition 4.9. Let $\mathcal{G}$ =($\mathcal{V}$,$\mathcal{E}$) be a graph with maximum degree Δ. Let $f : V \to [\Delta] \cup \{0\}$ be a function where $V \subseteq \mathcal{V}$. We call a function f feasible if there exists a strategy profile S of all the players in $\mathcal{G}$ such that for each $u \in V$, number of neighbours of u playing 1 in the strategy profile S is $f(u)$.

Lemma 4.10 (★). *Let* $\mathcal{G}$ =($\mathcal{V}$,$\mathcal{E}$) *be a graph with maximum degree* Δ. *Let* $V \subseteq \mathcal{V}$. *Then the set of all feasible functions* $f : V \to [\Delta] \cup \{0\}$ *can be computed in time* $O^*(\Delta^{|V|})$.

We now present a $O^*(\Delta^{O(k)})$ time XP algorithm for Exists-PSNE where k is the treewidth of the input graph. Note that the running time of $O^*(\Delta^{O(k)})$ implies that Exists-PSNE is fixed-parameter tractable for the combined parameter "treewidth+maximum degree".

Theorem 4.11. *Let* $\mathcal{G}$ *be an n-vertex graph given together with its tree decomposition of treewidth at most k. Then there is an algorithm running in time* $O^*(\Delta^{O(k)})$ *for* Exists-PSNE *in BNPG game on* $\mathcal{G}$ *where* Δ *is the maximum degree of graph* $\mathcal{G}$.

Proof Sketch. Let $(\mathcal{G} = (\mathcal{V},\mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$ be any instance of Exists-PSNE for BNPG games. Let $(\beta_v(.))_{v\in\mathcal{V}}$ be the set of the best response functions. Let $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ be a nice tree decomposition of the input n-vertex graph $\mathcal{G}$ that has width at most k. Let $\mathcal{T}$ be rooted at some node r. For a node t of $\mathcal{T}$, let V_t be the union of all the bags present in the subtree of $\mathcal{T}$ rooted at t, including X_t. We solve the Exists-PSNE problem using dynamic programing. Let $N_1(X_t)$ denote set of vertices in $\mathcal{V} \setminus V_t$ which is adjacent to at least one vertex in X_t. Let $N_2(X_t)$ denote set of vertices in $V_t \setminus X_t$ which is adjacent to at least one vertex in X_t.Let $c[t, (x_v)_{v\in X_t}, (d^1_v)_{v\in X_t}, (d^2_v)_{v\in X_t}] = 1$ (resp. 0) denote that there exists (resp. doesn't exist) a strategy profile S of all the players in $\mathcal{G}$ such that for each $u \in X_t$, u plays x_u, number of neighbours of u in $N_1(X_t)$ (resp. $N_2(X_t)$) playing 1 is d^1_u (resp. d^2_u) and none of the vertices in V_t deviate in the strategy profile S. Before we proceed, we would like to introduce some notations. Let V be a set of vertices and $S_1 = (x_v)_{v\in V}$, $S_2 = (x_v)_{v\in V\setminus\{w\}}$ be two tuples. Then $S_1 \setminus \{x_w\} := S_2$ and $S_2 \cup \{x_w\} := S_1$. Also, we denote an empty tuple by ϕ. Clearly $c[r, \phi, \phi, \phi]$ indicates whether there is a PSNE in $\mathcal{G}$ or not. We now present the recursive equation to compute $c[t, (x_v)_{v\in X_t}, (d^1_v)_{v\in X_t}, (d^2_v)_{v\in X_t}]$ for various types of node in $\mathcal{T}$.

Leaf Node: For a leaf node t we have that $X_t = \phi$. Hence, $c[t, \phi, \phi, \phi] = 1$.

Join Node: For a join node t, let t_1, t_2 be its two children. Note that $X_t = X_{t_1} = X_{t_2}$.

Now we proceed to compute $c[t, (x_v)_{v\in X_t}, (d^1_v)_{v\in X_t}, (d^2_v)_{v\in X_t}]$. Let $\mathcal{F}$ be a set of tuples $(d'_v)_{v\in X_t}$ such that there is a strategy profile S such that for each $v \in X_t$, its response is x_v, the number of neighbours in $N_1(x)$, $V_{t_1} \setminus X_{t_1}$ and $V_{t_2} \setminus X_{t_2}$ playing 1 is $d^1_v, d'_v, d^2_v - d'_v$ respectively. Using Lemma 4.10 we can find the set $\mathcal{F}$ in time $O^*(\Delta^k)$. Then $c[t, (x_v)_{v\in X_t}, (d^1_v)_{v\in X_t}, (d^2_v)_{v\in X_t}]$ is equal to the following formula:

$$0 \vee \bigvee_{(d'_v)_{v\in X_t} \in \mathcal{F}} \big(c[t_1, (x_v)_{v\in X_t}, (d^1_v + d^2_v - d'_v)_{v\in X_t}, (d'_v)_{v\in X_t}] \wedge c[t_2, (x_v)_{v\in X_t}, (d^1_v + d'_v)_{v\in X_t}, (d^2_v - d'_v)_{v\in X_t}]\big)$$

Introduce Node: Let t be an introduce node with a child t' such that $X_t = X_{t'} \cup \{u\}$ for some $u \notin X_{t'}$. Let $S' = (x_v)_{v\in X_t}$ be a strategy profile of vertices in X_t. Let n'_v denote the number of neighbours of v playing 1 in S'. Let $g : \mathcal{V} \times \mathcal{V} \to \{0,1\}$ be a function such that $g(\{u,v\}) = 1$ if and only if $\{u,v\} \in \mathcal{E}$. We now proceed to compute $c[t, S', (d^1_v)_{v\in X_t}, (d^2_v)_{v\in X_t}]$. If there is no strategy profile S where $\forall v \in X_t$,the number of neighbours of v in $N_1(X_t)$ (resp. $N_2(X_t)$) playing 1 is d^1_v (resp. d^2_v) , then clearly $c[t, S', (d^1_v)_{v\in X_t}, (d^2_v)_{v\in X_t}] = 0$. Due to Lemma 4.10, we can check the previous statement in $O^*(\Delta^k)$ by considering a bipartite subgraph of $\mathcal{G}$ between X_t and $N_1(X_t)$ (or $N_2(X_t)$). Otherwise, we have the following:

$$c[t, S', (d^1_v)_{v\in X_t}, (d^2_v)_{v\in X_t}] = \begin{cases} 0 \text{ if } \exists v \in X_t, x_v \notin \beta_v(n'_v + d^1_v + d^2_v) \\ c[t', S' \setminus \{x_u\}, (d^1_v + g(\{v,u\}))_{v\in X_{t'}}, (d^2_v)_{v\in X_{t'}}] \text{ if } x_u = 1 \\ c[t', S' \setminus \{x_u\}, (d^1_v)_{v\in X_{t'}}, (d^2_v)_{v\in X_{t'}}] \text{ otherwise} \end{cases}$$

Forget Node: Let t be a forget node with a child t' such that $X_t = X'_t \setminus \{w\}$ for some $w \in X_{t'}$. Let $S_0 = (x_v)_{v\in X_t} \cup \{x_w = 0\}$, $S_1 = (x_v)_{v\in X_t} \cup \{x_w = 1\}$ be two strategy profiles of vertices in X'_t. Let $g : \mathcal{V} \times \mathcal{V} \to \{0,1\}$ be a function such that $g(\{u,v\}) = 1$ if and only if $\{u,v\} \in \mathcal{E}$. We now have the following:

$$c[t, (x_v)_{v\in X_t}, (d^1_v)_{v\in X_t}, (d^2_v)_{v\in X_t}] = \bigvee_{d^1_w, d^2_w : 0 \le d^1_w, d^2_w \le \Delta} \big(c[t', S_0, (d^1_v)_{v\in X_{t'}}, (d^2_v)_{v\in X_{t'}}] \vee c[t', S_1, (d^1_v)_{v\in X_{t'}}, (d^2_v - g(\{v,w\}))_{v\in X_{t'}}\big)$$

Due to space constraints, we refer the reader to the full version of our paper for the proof of correctness of the above recursive equations. Now we consider the time complexity of our algorithm. Total number of cells in the dynamic programming table which we created is $O^*(\Delta^{O(k)})$. For each cell, we spend at most $O^*(\Delta^{O(k)})$ time if we are computing the table in a bottom up fashion. Hence, the running time is $O^*(\Delta^{O(k)})$. □

4.3 Tractable Results

To conclude our fine-grained analysis of the Exists-PSNE problem, we bridge the gap between the tractablility and intractibility by showing some tractable results. Our first result is an FPT algorithm for Exists-PSNE for strict games when parameterized by the vertex cover number.

Theorem 4.12 (★). *There is a $O^*(2^{vc(\mathcal{G})})$ time algorithm for* Exists-PSNE *for strict BNPG games where vc($\mathcal{G}$)is the vertex cover number.*

Our next result shows that we can always find a PSNE for additive BNPG games in $O(n)$ time. This complements the intractable result for subadditive BNPG games.

Observation 2 (★). *There exists an $O(n)$ time algorithm to find a PSNE in an additive BNPG game.*

We next consider circuit rank and distance from complete graph as parameter. These parameters can be thought of distance from tractable instances (namely tree and complete graph). They are defined as follows.

Definition 4.13. Let the number of edges and number of vertices in a graph $\mathcal{G}$ be m and n respectively. Then d_1 (circuit rank) is defined to be $m - n + c$ (c is the number of connected components in the graph) and d_2 (distance from complete graph) is defined to be $\frac{n(n-1)}{2} - m$. Note that circuit rank is not the same as feedback arc set.

Yu et al. presented an algorithm for Exists-PSNE on trees in [29]. It turns out that their algorithm can be appropriately modified to get the following observation.

Observation 3. *[29] Given a BNPG game on a tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, a subset of vertices $\mathcal{U} \subseteq \mathcal{V}$ and a strategy profile $(x_u)_{u\in\mathcal{U}} \in \{0,1\}^{\mathcal{U}}$, there is a polynomial time algorithm for deciding if there exists a PSNE $(y_v)_{v\in\mathcal{V}} \in \{0,1\}^{\mathcal{V}}$ for the BNPG game such that $x_u = y_u$ for every $u \in \mathcal{U}$.*

Now by using the observation 3 as a subroutine, we exhibit an FPT algorithm for the parameter circuit rank.

Theorem 4.14. *There is an algorithm running in time $O^*(4^{d_1})$ for* Exists-PSNE *in BNPG games where d_1 is the circuit rank of the input graph.*

Proof. Let $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$ be any instance of Exists-PSNE for BNPG games. Let the graph $\mathcal{G}$ have c connected components namely, $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1), \ldots, \mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c)$. For every $i \in [c]$, we decide if there exists a PSNE in $\mathcal{G}_i$; clearly there is a PSNE in $\mathcal{G}$ if and only if there is a PSNE in $\mathcal{G}_i$ for every $i \in [n]$. Hence, in the rest of the proof, we focus on the algorithm to decide the existence of a PSNE in $\mathcal{G}_i$. We compute a minimum spanning tree $\mathcal{T}_i$ in the connected component $\mathcal{G}_i$. Let $\mathcal{E}'_i \subset \mathcal{E}_i$ be the set of edges which are not part of $\mathcal{T}_i$; let $|\mathcal{E}'_i| = d^i_1$ and $\mathcal{V}'_i = \{v^i_1, v^i_2, \ldots, v^i_l\} \subseteq \mathcal{V}_i$ be the set of vertices which are endpoints of at least one edge in $\mathcal{E}'_i$. Of course, we have $|\mathcal{V}'_i| = \ell \le 2d^i_1$. For every tuple $t = (x'_v)_{v\in\mathcal{V}'_i} \in \{0,1\}^l$, we do the following.

(1) For each $v \in \mathcal{V}'_i$, let n^t_v be the number of neighbours of v in $\mathcal{G}_i[\mathcal{E}'_i]$ (subgraph of $\mathcal{G}_i$ containing the set of nodes $\mathcal{V}_i$ and the set of edges $\mathcal{E}'_i$) who play 1 in t. We now define g^t_v for every player $v \in \mathcal{V}$ as follows.

$$g^t_v(k) = \begin{cases} g_v(k + n^t_v) & \text{if } v \in \mathcal{V}'_i \\ g_v(k) & \text{otherwise} \end{cases}$$

(2) We now decide if there exists a PSNE $(y_v)_{v\in\mathcal{V}_i} \in \{0,1\}^{\mathcal{V}_i}$ in the BNPG game $(\mathcal{T}_i, (g^t_v)_{v\in\mathcal{V}_i}, (c_v)_{v\in\mathcal{V}_i})$ such that $y_v = x'_v$ for every $v \in \mathcal{V}'_i$; this can be done in polynomial time due to Observation 3. If such a PSNE exists, then we output yes.

If we fail to find a PSNE for every choice of tuple t, then we output no. The running time of the above algorithm (for $\mathcal{G}_i$) is $O^*\left(2^{|\mathcal{V}'_i|}\right)$. Hence the overall running time of our algorithm is $O^*\left(\sum_{i=1}^{c} 2^{|\mathcal{V}'_i|}\right) \le O^*\left(2^{2d_1}\right) = O^*\left(4^{d_1}\right)$. We now argue correctness of our algorithm. We observe that it is enough to argue correctness for one component.

In one direction, let $x^* = (x_v^*)_{v\in\mathcal{V}_i}$ be a PSNE in the BNPG game $(\mathcal{G}_i, (g_v)_{v\in\mathcal{V}_i}, (c_v)_{v\in\mathcal{V}_i})$. We now claim that $(x_v^*)_{v\in\mathcal{V}_i}$ is also a PSNE in the BNPG game on $(\mathcal{T}_i, (g_v^t)_{v\in\mathcal{V}_i}, (c_v)_{v\in\mathcal{V}_i})$ where $t = (x_v^*)_{v\in\mathcal{V}_i'}$. Let $n_v^{\mathcal{G}_i}$ and $n_v^{\mathcal{T}_i}$ be the number of neighbors of $v \in \mathcal{V}_i$ in $\mathcal{G}_i$ and $\mathcal{T}_i$ respectively who play 1 in x^*. With n_v^t defined as above, we have $n_v^{\mathcal{G}_i} = n_v^{\mathcal{T}_i} + n_v^t$ for $v \in \mathcal{V}_i'$ and $n_v^{\mathcal{G}_i} = n_v^{\mathcal{T}_i}$ for $v \in \mathcal{V}_i \setminus \mathcal{V}_i'$. Hence, we have $\Delta g_v^t(n_v^{\mathcal{T}_i}) = \Delta g_v(n_v^{\mathcal{T}_i}+n_v^t) = \Delta g_v(n_v^{\mathcal{G}_i})$ for $v \in \mathcal{V}_i'$ and $\Delta g_v^t(n_v^{\mathcal{T}_i}) = \Delta g_v(n_v^{\mathcal{G}_i})$ for $v \in \mathcal{V}_i \setminus \mathcal{V}_i'$. If $x_v^* = 1$ where $v \in \mathcal{V}_i$, then $\Delta g_v(n_v^{\mathcal{G}_i}) \geq c_v$ and thus we have $\Delta g_v^t(n_v^{\mathcal{T}_i}) \geq c_v$. Hence, v does not deviate in $\mathcal{T}_i$. Similarly, if $x_v^* = 0$ where $v \in \mathcal{V}_i$, then $\Delta g_v(n_v^{\mathcal{G}_i}) \leq c_v$ and thus we have $\Delta g_v^t(n_v^{\mathcal{T}_i}) \leq c_v$. Hence, v does not deviate in $\mathcal{T}_i$. Hence $(x_v^*)_{v\in\mathcal{V}}$ is also a PSNE in BNPG game $(\mathcal{T}_i, (g_v^t)_{v\in\mathcal{V}_i}, (c_v)_{v\in\mathcal{V}_i})$ where $t = (x_v^*)_{v\in\mathcal{V}_i'}$ (which means our Algorithm returns YES).

In the other direction, let $(x_v^*)_{v\in\mathcal{V}_i}$ be the PSNE in BNPG game on $(\mathcal{T}_i, (g_v^t)_{v\in\mathcal{V}_i}, (c_v)_{v\in\mathcal{V}_i})$ where $t = (x_v^*)_{v\in\mathcal{V}_i'}$ (which means our Algorithm returns YES). We claim that $(x_v^*)_{v\in\mathcal{V}_i}$ is also a PSNE in BNPG game $(\mathcal{G}_i, (g_v)_{v\in\mathcal{V}_i}, (c_v)_{v\in\mathcal{V}_i})$. If $x_v^* = 1$ for $v \in \mathcal{V}_i$, then $\Delta g_v^t(n_v^{\mathcal{T}_i}) \geq c_v$. This implies that $\Delta g_v(n_v^{\mathcal{G}_i}) \geq c_v$ and thus v does not deviate in $\mathcal{G}_i$. Similarly, if $x_v^* = 0$ for $v \in \mathcal{V}_i$, then $\Delta g_v^t(n_v^{\mathcal{T}_i}) \leq c_v$. This implies that $\Delta g_v(n_v^{\mathcal{G}_i}) \leq c_v$ and thus v does not deviate in $\mathcal{G}_i$. Hence $(x_v^*)_{v\in\mathcal{V}}$ is also a PSNE in BNPG game on $(\mathcal{G}_i, (g_v)_{v\in\mathcal{V}_i}, (c_v)_{v\in\mathcal{V}_i})$. □

Yu et al. presented an algorithm for Exists-PSNE on complete graphs in [29]. It turns out that their algorithm can be appropriately modified to get the following observation.

Observation 4. *[29] Given a BNPG game on a complete graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and an integer k, there is a polynomial time algorithm for deciding if there exists a PSNE where exactly k players play 1 and returns such a PSNE if it exists.*

Now by using the observation 4 as a subroutine, we exhibit an FPT algorithm for the parameter distance from complete graph.

Theorem 4.15. *There is an algorithm running in time $O^*(4^{d_2})$ for* Exists-PSNE *in BNPG games where d_2 is the distance from complete graph.*

Proof. Let $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$ be any instance of Exists-PSNE for BNPG games. If $d_2 \geq \frac{n}{2}$, then iterating over all possible strategy profiles takes time $O^*(2^n) \leq O^*(4^{d_2})$. So allow us to assume for the rest of the proof that $d_2 < \frac{n}{2}$. Let us define $\mathcal{V}' = \{u \in \mathcal{V} : \exists v \in \mathcal{V}, v \neq u, \{u, v\} \notin \mathcal{E}\}$; we have $|\mathcal{V}'| \leq 2d_2$.

For every strategy profile $y = (y_u)_{u\in\mathcal{V}'}$, we do the following. For each $v \in \mathcal{V} \setminus \mathcal{V}'$, let n_v' be the number of neighbors of v in $\mathcal{V}'$ who play 1 in y. We now define $g_v'(\ell) = g_v(\ell + n_v')$ for every $\ell \in \mathbb{N} \cup \{0\}$ and every player $v \in \mathcal{V} \setminus \mathcal{V}'$. For every $k \in \{0, \ldots, |\mathcal{V} \setminus \mathcal{V}'|\}$, we decide (using the algorithm in Observation 4) if there exists a PSNE $x^k = (x_v^k)_{v\in\mathcal{V}\setminus\mathcal{V}'}$ in the BNPG game $(\mathcal{G}[\mathcal{V} \setminus \mathcal{V}'], (g_v')_{v\in\mathcal{V}\setminus\mathcal{V}'}, (c_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ where exactly k players play 1. If x^k exists, then we output yes if $((y_u)_{u\in\mathcal{V}'}, (x_v^k)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ forms a PSNE in the BNPG game $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$.

If the above procedure fails to find a PSNE, then we output no. The running time of the above algorithm is $O^*\left(2^{|\mathcal{V}'|}\right) \leq O^*\left(4^{d_2}\right)$. We now argue correctness.

Clearly, if the algorithm outputs yes, then there exists a PSNE for the input game. On the other hand, if there exists a PSNE $((y_u)_{u\in\mathcal{V}'}, (x_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}) \in \{0,1\}^{\mathcal{V}}$ in the input game, then let us consider the iteration of our algorithm with the guess $(y_u)_{u\in\mathcal{V}'}$. Let the number of players playing 1 in $(x_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}$ be k. If $x_v = 1$ where $v \in \mathcal{V} \setminus \mathcal{V}'$, then $\Delta g_v(n_v' + k - 1) \geq c_v$ and thus we have $\Delta g_v'(k-1) \geq c_v$. Similarly, if $x_v = 0$ where $v \in \mathcal{V} \setminus \mathcal{V}'$, then $\Delta g_v(n_v' + k) \leq c_v$ and thus we have $\Delta g_v'(k) \leq c_v$. Hence, we observe that $(x_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}$ forms a PSNE in the BNPG game $(\mathcal{G}[\mathcal{V} \setminus \mathcal{V}'], (g_v')_{v\in\mathcal{V}\setminus\mathcal{V}'}, (c_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$. Let $(x_v')_{v\in\mathcal{V}\setminus\mathcal{V}'}$ be the PSNE of the BNPG game $(\mathcal{G}[\mathcal{V} \setminus \mathcal{V}'], (g_v')_{v\in\mathcal{V}\setminus\mathcal{V}'}, (c_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ where exactly k players play 1 returned by the algorithm in Observation 4. We observe that every player in $\mathcal{V}'$ has the same number of neighbors playing 1 in both the strategy profiles $((y_u)_{u\in\mathcal{V}'}, (x_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ and $((y_u)_{u\in\mathcal{V}'}, (x_v')_{v\in\mathcal{V}\setminus\mathcal{V}'})$. So no player in $\mathcal{V}'$ will deviate in the strategy profile $((y_u)_{u\in\mathcal{V}'}, (x_v')_{v\in\mathcal{V}\setminus\mathcal{V}'})$. If $x_v' = 1$ where $v \in \mathcal{V} \setminus \mathcal{V}'$, then $\Delta g_v'(k-1) \geq c_v$ and thus we have $\Delta g_v(n_v' + k - 1) \geq c_v$. Hence, v does not deviate in the strategy profile $((y_u)_{u\in\mathcal{V}'}, (x_v')_{v\in\mathcal{V}\setminus\mathcal{V}'})$. Similarly, if $x_v' = 0$ where $v \in \mathcal{V} \setminus \mathcal{V}'$, then $\Delta g_v'(k) \leq c_v$ and thus we have $\Delta g_v(n_v' + k) \leq c_v$. Hence, v does not deviate in the strategy profile $((y_u)_{u\in\mathcal{V}'}, (x_v')_{v\in\mathcal{V}\setminus\mathcal{V}'})$. Hence, $((y_u)_{u\in\mathcal{V}'}, (x_v')_{v\in\mathcal{V}\setminus\mathcal{V}'})$ also forms a PSNE in the input BNPG game and thus the algorithm outputs yes. This concludes the correctness of our algorithm. □

We finally show that a PSNE always exists for fully homogeneous BNPG games for some important graph classes and such a PSNE can be found in $O(n)$ time.

Theorem 4.16 (★). *There is always a PSNE in a fully homogeneous BNPG game for paths, complete graphs, cycles, and bi-cliques. Moreover, we can find a PSNE in $O(n)$ time.*

5 CONCLUSION AND FUTURE WORK

We have studied parameterized complexity of the Exists-PSNE problem for the BNPG games with respect to various important graph parameters. We exhibited intractibility w.r.t. the parameters like maximum degree, diameter, treedepth, number of players playing 1 and 0. We complemented this by showing FPT algorithms parameterized by circuit rank, treewidth+maximum degree, and the distance from complete graph. We also showed that PSNE always exists in a fully homogeneous BNPG game for paths, complete graphs, cycles and bi-cliques.

Our work leaves some important questions open. For example, can we show PPAD-Hardness for finding Nash Equilibrium in BNPG games. Another immediate research direction is to study if our algorithmic results could be extended to other types of more general public goods games. Another research direction could be to look at social welfare functions in the context of BNPG game. We can also consider BNPG games with altruism introduced in [28] and try to resolve its parameterized complexity.

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