Bounded Approval Ballots: Balancing Expressiveness and Simplicity for Multiwinner Elections

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ABSTRACT

Approval ballots have been celebrated for many voting scenarios [16], in particular because of the low cognitive burden they put on the voters. This however, comes at the cost of expressiveness that can be problematic when voters have sophisticated preferences. We consider voters who, in addition to usual approval, may wish to express incompatibilities, dependencies, and/or substitution effects between the alternatives. We introduce, and evaluate a new type of ballot—*bounded approval ballots*—which captures these effects while being almost as easy as regular approval ballots to cast.

KEYWORDS

Multiwinner Voting; Approval Voting; Ballot Design

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1 INTRODUCTION

Let us focus on the case study of *Goodman's Pipes and Tubes Ltd.*, a company that is about to elect its expert committee. The committee consists of four people who advise the board on business strategy questions. The following six candidates are up for election: *Anna* and *Chris* are two of the leading engineers in the company; *Ben* from the human resources; *Diana* from the legal department; *Elena* from the advertisement department; the external craftsman *Frank*; and *Gustavo* who is responsible for material purchases. The three board members *Rob*, *Su*, and *Tim* have the following opinions.

Rob: "I'm happy with Anna's, Chris', Elena's, Frank's, and Gustavo's work. They are reliable, work hard, and have been around long enough, so each of them will improve the committee with their own expertise. "

Su: "We are an engineering company. Expertise on materials, production, and craftsmanship should be our focus, so there should be one, or better two of Anna, Chris, Frank, and Gustavo. Having more of them is also fine, although I don't see a big advantage for that. I think there should also be expertise from Diana in the committee. However, Diana and Chris shouldn't be together because they argue a lot. "

Tim: "I am pretty sure that Chris, Gustavo, Ben, and Diana will do a good job. However, every time I discuss something with either Ben or Diana, the other one is angry because they say that the legal department and the human resources must work closely together. So if we give one of them a seat in the committee, the other one must also get a seat, or else we can just as well give no seat to either of the two. "

The committee selection example above is a canonical instance of *multiwinner elections* [11]. In this context, the most common way of asking one's opinion is to use approval ballots: the voters indicate which of the alternatives they approve of [15]. Rob would for instance express his preference by approving of Anna, Chris, Elena, Frank, and Gustavo. However, Su's and Tim's statements cannot be expressed as approval ballots. It is true that Su in principle approves of Anna, Chris, Diana, Frank, and Gustavo. However, by using an approval ballot she cannot state that two of them are just as good as three, nor that Diana and Chris are incompatible. Similarly, Tim cannot express in an approval ballot that either Ben and Diana must be in the committee, or neither of them. Our goal is to find a ballot format to account for the type of preferences illustrated above, *i.e.*, approvals, incompatibilities, substitutions, and dependencies. Obviously, very expressive ballot formats (*e.g.* rankings over subsets of alternatives) could be used to express those, and even more complicated, preferences. This approach is however not satisfactory. Indeed, we believe that more expressive ballots should still be practical, *i.e.*, not imposing a high cognitive burden on the voters, and scaling reasonably well as the number of alternatives increases. Moreover, even though some voters can be interested in submitting complex ballots, only proposing complex ballots can prevent some others to participate. We thus want to develop ballot formats that still allow for simple ballots to be submitted.

Contribution. To achieve the goals described above, we introduce *bounded approval ballots.* A bounded approval ballot is a collection of bounded approval sets: sets of approved alternatives that are enriched with three bounds: a *lower bound* (minimum number of alternatives that have to be selected), a *saturation point* (number of selected alternatives after which no additional satisfaction is derived), and an *upper bound* (maximum number of alternatives

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that should be selected). Approval ballots are still valid (with simple reformatting) and treated exactly as in usual multiwinner approval voting to be convenient for the voters who don't want to take the effort to submit a more complicated ballot. Moreover, for voters who want to submit more sophisticated ballots (incompatibilities, substitutions and dependencies) the cognitive burden is no higher than setting some bounds for the ballot.

Related Work. Multiwinner voting (as a special case of voting in combinatorial domains [8]) has become a widely studied research area over the past years. We refer to Faliszewski et al. [11] for an overview of multiwinner voting rules and typical applications. Interestingly, the two most often used ballot types are approval ballots and ordinal ballots (rankings). Most of the research in the field focuses on the development of voting rules for such ballots (for example to guarantee fairness [1, 7]) rather than on the design and the study of these ballot types. Closest to what we are trying to achieve here are conditional preferences, where the preferences of a voter are conditioned on the status of a given variable. Several proposals have been discussed to express those opinions, the most famous probably being *conditional approval ballots* [3], *conditional preference networks* [5], and *lexicographic preference trees* [4]. A stream of research for combinatorial auctions focuses on modeling complex utility functions using expressive languages [9]. Sandholm [18] studies bidding languages, where atomic bids are joined with logical connectives (allowing for substitution and incompatibility effects), while Hoos and Boutilier [6, 13] interconnect bids and logical formulas over the alternatives. However, all these bidding languages are rather complicated to express, at least compared to the very simple approval ballot format. Finally, several proposals have been made to extend approval ballots, mainly in the context of participatory budgeting, a generalization of multiwinner elections [2]. Jain et al. [14] partition the alternatives into categories to model interactions among them. Fairstein et al. [10] incorporate individual partitions of the alternatives to study substitution effects (in a slightly different way than we do).

2 PRELIMINARIES

A multiwinner election consists of a set of m alternatives (also called candidates) $\mathcal{A} = \{a_1, \ldots, a_m\}$, a profile $\mathfrak{B} = (\boldsymbol{B}_1, \ldots, \boldsymbol{B}_n)$ which is a list of ballots $\boldsymbol{B}_i$ of n voters $\mathcal{N} = \{1, \ldots, n\}$, and an integer $k \in \{1, \ldots, m\}$. We denote by $C_k = \{\pi \subseteq \mathcal{A} \mid |\pi| = k\}$ the set of all k-sized committees. The outcome of an irresolute multiwinner election is a set of winning committees $\{\pi_1, \pi_2, \ldots\} \subseteq C_k$. The ballot format is described in the next section. We will use $\oplus$ to denote the concatenation operator between two lists. The subtraction of list B from list A will be denoted through $A \ominus B$ (where for each element in B the first occurrence of the element in A is removed). We sometimes omit the brackets around a list of length one.

2.1 Bounded Approval Ballots

We now introduce bounded approval ballots, our generalization of approval ballots to allow for submitting more complex preferences.

Definition 1 (Bounded Sets and Ballots)**.** *Given a set of alternatives $\mathcal{A}$, a* bounded (approval) set *is a tuple $B^j = \langle A^j, \ell^j, s^j, u^j \rangle$ such that $A^j \subseteq \mathcal{A}$ and ℓ^j, s^j, u^j, respectively the lower bound, the saturation point, and the upper bound, are all integers such that $1 \leq \ell^j \leq s^j \leq u^j \leq |A^j|$. A* bounded (approval) ballot *$\boldsymbol{B}_i$, for voter $i \in \mathcal{N}$, is a list $\boldsymbol{B}_i = (B_i^1, \ldots, B_i^p)$ of bounded sets.*

A bounded set indicates that from all the alternatives in A^j, at least ℓ^j but no more than u^j have to be selected; while after s^j alternatives have been selected from A^j, the voter will not enjoy any additional satisfaction from selecting more alternatives.[1] This way, we achieve all of our initial modeling goals:

- *Standard approval ballots* can be expressed by setting $\ell^j = 1$, $s^j = u^j = |A^j|$: the more alternatives from A^j the better;
- *Incompatibilities* can be expressed by bounded sets with an upper bound $u^j = 1$: Selecting multiple alternatives from A^j is not desired by the voter because these alternatives are incompatible, but selecting one is desirable;
- *Substitution* can be expressed by bounded sets with $\ell^j = s^j = 1$ and $u^j = |A^j|$: Selecting one alternative from A^j is desired but additional alternatives are substitutes;
- *Dependencies* can be expressed by bounded sets where $\ell^j = |A^j|$: All alternatives from A^j rely on each other, and are only useful for the voter if all of them are present.

We illustrate these ballots on the example from the introduction.

Example 1. Rob only wants to provide an approval ballot that can be expressed by a simple bounded approval ballot consisting of only one bounded set: $\langle\{\text{Anna, Chris, Elena, Frank, Gustavo}\}, 1, 5, 5\rangle$.

Su's preference is more involved. The incompatibility between Diana and Chris can be expressed by $\langle\{\text{Diana, Chris}\}, 1, 1, 1\rangle$. Further, with $\langle\{\text{Anna, Chris, Frank, Gustavo}\}, 1, 2, 4\rangle$, we can express that one—or better two—of Anna, Chris, Frank, and Gustavo should be included but there is no further benefit for three or four. Su's ballot would thus consist of these two bounded sets.

Tim states that from Ben and Diana either both or none should be included, which can be expressed as $\langle\{\text{Ben, Diana}\}, 2, 2, 2\rangle$. Furthermore, both Chris and Gustavo are approved, which can be expressed as $\langle\{\text{Chris, Gustavo}\}, 1, 2, 2\rangle$. △

Finally, we introduce one useful notation: for a ballot $\boldsymbol{B}$ and an alternative $a \in \mathcal{A}$, we denote by $\boldsymbol{B}_{|a} = \{B^j \in \boldsymbol{B} \mid a \in A^j\}$ the bounded sets in $\boldsymbol{B}$ involving a.

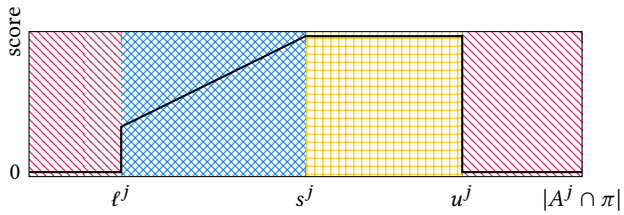
2.2 Scoring with Bounded Approval Ballots

We eventually want to aggregate the ballots that the voters submitted in order to determine a winning committee. In the following we provide different *scoring functions* which map profiles and committees to real numbers. These can then be used to define rules by simply selecting the committee with the highest score. We will investigate properties of the scoring functions later.

Definition 2 (Scoring Function)**.** *A scoring function score is a function taking as input a bounded approval ballot $\boldsymbol{B}$ and a committee π, and returning a real value score$(\boldsymbol{B}, \pi)$. We extend scoring functions to profiles s.t. for every profile $\mathfrak{B}$, score$(\mathfrak{B}, \pi) = \sum_{\boldsymbol{B} \in \mathfrak{B}}$ score$(\boldsymbol{B}, \pi)$.*

To capture the semantics of bounded sets described above, for a committee π and a bounded set $B^j = \langle A^j, \ell^j, s^j, u^j \rangle$, we want scoring functions to behave as depicted below.

[1] We assume that all alternatives in A^j are *approved* in the sense that for each $a \in A^j$ there is a committee in which the voter would like a to be part of.



The area represents that π violates a dependency or incompatibility stated in B^j. Thus, π should score 0. The area represents that each element in π is independently approved according to B^j—each contributes to the total score. Finally, the # area represents that the elements in π show substitution effects according to B^j—additional items contribute no additional score.

Assuming neutrality, *i.e.,* that all alternatives are treated the same, we introduce a function that should be interpreted as the average score of the alternatives appearing in B^j. When within the lower bound and the saturation point, each alternative from $A^j \cap \pi$ fully contributes to B^j's score, *i.e.*, they score 1 each. If the saturation point is exceeded, the s^j points are equally split among the alternatives. If the lower or the upper bound is violated, all the alternatives score 0. The formal definition of this function φ is:

$$\varphi(B^j, \pi) = \begin{cases} 1 & \text{if } \ell^j \leq |A^j \cap \pi| \leq s^j \\ \frac{s^j}{|A^j \cap \pi|} & \text{if } s^j < |A^j \cap \pi| \leq u^j \\ 0 & \text{otherwise.} \end{cases}$$

Example 2. Consider the following bounded set from Su's ballot: $B^j = \langle \{\text{Anna, Chris, Frank, Gustavo}\}, 1, 2, 4 \rangle$. For $\pi = \{\text{Anna, Ben, Chris, Diana}\}$ we have $\varphi(B^j, \pi) = 1$. That is, according to B^j each alternative in $\pi \cap A^j$ is fully approved, so in total π has two approvals—one for Anna and one for Chris. For $\pi' = \{\text{Anna, Ben, Chris, Frank}\}$ the saturation bound is exceeded, so we have $\varphi(B^j, \pi') = \frac{2}{3}$. △

Scoring functions are defined for bounded approval ballots (and profiles of them), and not for bounded sets as φ is defined. If each ballot consists of only one bounded set, this would be straightforward. However, as soon as there are several bounded sets, we need to aggregate the score of the individual bounded sets. Several usual operators can be considered here: averaging, taking the minimal or the maximal value, or simply summing up the scores. This results in the following scoring functions, all based on φ.

$$score_{min}(\boldsymbol{B}, \pi) = \sum_{a \in \pi} \min\left(\varphi(B^j, \pi) \mid B^j \in \boldsymbol{B}_{|a}\right)$$

$$score_{max}(\boldsymbol{B}, \pi) = \sum_{a \in \pi} \max\left(\varphi(B^j, \pi) \mid B^j \in \boldsymbol{B}_{|a}\right)$$

$$score_{avg}(\boldsymbol{B}, \pi) = \sum_{a \in \pi} \frac{1}{|\boldsymbol{B}_{|a}|} \sum_{B^j \in \boldsymbol{B}_{|a}} \varphi(B^j, \pi)$$

$$score_{tot}(\boldsymbol{B}, \pi) = \sum_{a \in \pi} \sum_{B^j \in \boldsymbol{B}_{|a}} \varphi(B^j, \pi)$$

Note that the semantics we developed is respected when each ballot consists of a single bounded set. Note further, that the functions coincide when each alternative is part of at most one bounded set per ballot, *i.e.*, when $|\boldsymbol{B}_{|a}| \leq 1$ for all $a \in \mathcal{A}$ and every ballot $\boldsymbol{B}$.

In addition to these four scoring functions, we also study another one that is not based on φ: $score_{app}$. It is a natural generalization of the approval score, as it counts the number of alternatives in π for which there exists at least one bounded set B^j for which the lower bound is exceeded in π, but not the saturation point:

$$score_{app}(\boldsymbol{B}, \pi) = |\{a \in \pi \mid \exists B^j \in \boldsymbol{B}_{|a} \text{ s.t. } \ell^j \leq |A^j \cap \pi| \leq s^j\}|.$$

Since, $score_{app}$ completely disregards substitution effects, it will not exactly fit the framework we detailed above. It will be shown to provide interesting axiomatic results still.

Before moving on to the axiomatic analysis, let us briefly discuss the computational complexity of the scoring functions. For all of them, finding a committee with maximal score cannot be done in polynomial-time, unless P = NP. For the φ-based rules it is easy to see that we can simulate the (approval version of the) Chamberlin–Courant rule with them by submitting a single bounded set per voter, where each bounded set has a saturation point of one and an upper bound involving all approved alternatives. The observation then follows from the fact that Chamberlin–Courant winner determination is NP-hard [19]. In the case of $score_{app}$, we can use the NP-hard problem Exact Cover by 3-Sets (see Garey and Johnson [12]) to show the claim. These downsides are, unfortunately, unavoidable when working with more expressive ballot formats.

3 ADEQUACY OF THE MODELIZATION

Following the classical method of social choice [20], we develop an axiomatic theory to investigate the behavior of scoring functions.

3.1 Axiomatic Theory

We encode, by the means of axioms, the idea that bounded approval ballots allow voters to express the different statements we are interested in; and that the scoring functions comply with the semantics we are aiming for.

We first define two axioms enforcing that a violated incompatibility or dependency should not increase the score.

Definition 3 (Incompatibility Adequacy)**.** *A scoring function score satisfies* incompatibility adequacy *if for every $A \subseteq \mathcal{A}$, and all ballots $\boldsymbol{B}$ and $\boldsymbol{B}' = \boldsymbol{B} \oplus \langle A, 1, 1, 1 \rangle$, the following holds:*

- *$score(\boldsymbol{B}, \pi) \leq score(\boldsymbol{B}', \pi)$ for every π with $|\pi \cap A| = 1$;*
- *$score(\boldsymbol{B}, \pi) \geq score(\boldsymbol{B}', \pi)$ for every π with $|\pi \cap A| \neq 1$.*

Definition 4 (Dependency Adequacy)**.** *A scoring function score satisfies* dependency adequacy *if for every $A \subseteq \mathcal{A}$, and all ballots $\boldsymbol{B}$ and $\boldsymbol{B}' = \boldsymbol{B} \oplus \langle A, |A|, |A|, |A| \rangle$, the following holds:*

- *$score(\boldsymbol{B}, \pi) \leq score(\boldsymbol{B}', \pi)$ for every π with $A \subseteq \pi$;*
- *$score(\boldsymbol{B}, \pi) \geq score(\boldsymbol{B}', \pi)$ for every π with $A \nsubseteq \pi$.*

Our next axiom concerns properly modeling substitution. Informally, if according to all bounded sets an item $a^\star$ is considered a substitute w.r.t. π, then adding $a^\star$ to π should not change the score.

Definition 5 (Substitution Adequacy)**.** *A scoring function score satisfies* substitution adequacy *if for every ballot $\boldsymbol{B}$ and committee π for which there exists an alternative $a^\star \in \mathcal{A} \setminus \pi$ such that for all bounded sets $B^j \in \boldsymbol{B}_{|a^\star}$, it is the case that $s^j \leq |A^j \cap \pi| \leq u^j - 1$, we have $score(\boldsymbol{B}, \pi) = score(\boldsymbol{B}, \pi \cup \{a^\star\})$.*

Next, we ensure that a scoring function treats approval ballots correctly, *i.e.*, that it behaves as the usual approval score for standard approval ballots.

Definition 6 (Approval Adequacy)**.** *A scoring function score satisfies* approval adequacy *if for every ballot $\boldsymbol{B}$ and committee π the following two conditions hold:*

(1) $score(\boldsymbol{B}, \pi) \leq |\left(\bigcup_{B^j \in \boldsymbol{B}} A^j\right) \cap \pi|$;

(2) $score(\boldsymbol{B}, \pi) = |\left(\bigcup_{B^j \in \boldsymbol{B}} A^j\right) \cap \pi|$ *whenever* $\ell^j \leq |A^j \cap \pi| \leq s^j$ *for all* $B^j \in \boldsymbol{B}$.

The final adequacy axiom requires the scoring function to return 0 if and only if there is a good reason to do so.

Definition 7 (Zero Adequacy)**.** *A scoring function score satisfies* zero-adequacy *if for every ballot $\boldsymbol{B}$ and committee π we have:*

$$score(\boldsymbol{B}, \pi) = 0 \quad \textit{iff} \quad \forall B^j \in \boldsymbol{B}, |A^j \cap \pi| > u^j \textit{ or } |A^j \cap \pi| < \ell^j.$$

We further introduce monotonicity axioms enforcing the scoring rules to be well-behaved in a dynamic environment.

The first one says that adding a bounded set which does not conflict with a committee π should not decrease the score of π.

Definition 8 (Ballot-Size Monotonicity)**.** *Let $\boldsymbol{B}$ be a ballot and π a committee. A scoring function score satisfies* ballot-size monotonicity *if for every bounded set $B = \langle A, \ell, s, u\rangle$ such that $\ell \leq |A \cap \pi| \leq u$, we have $score(\boldsymbol{B}, \pi) \leq score(\boldsymbol{B} \oplus B, \pi)$.*

Ballot-splitting monotonicity says that expressing an equivalent statement with one large ballot, or several smaller ones, should result in the same score.

Definition 9 (Ballot-Splitting Monotonicity)**.** *A scoring function score satisfies* ballot-splitting monotonicity *if for every committee π and every ballot $\boldsymbol{B}$ for which there exists a bounded set $B^{j^\star} \in \boldsymbol{B}$ such that $\ell^{j^\star} \leq |A^{j^\star} \cap \pi| \leq s^{j^\star}$, then, for $\boldsymbol{B}' = (\boldsymbol{B} \ominus B^{j^\star}) \oplus (\langle\{a\}, 1, 1, 1\rangle \mid a \in A^{j^\star} \cap \pi)$, we must have $score(\boldsymbol{B}, \pi) = score(\boldsymbol{B}', \pi)$.*

Finally, score monotonicity requires the score not to decrease when adding a suitable alternative to the committee.

Definition 10 (Score Monotonicity)**.** *A scoring function score satisfies* score monotonicity *if for every ballot $\boldsymbol{B}$ and committee π for which there exists an alternative $a^\star \in \mathcal{A} \setminus \pi$ such that for all bounded sets $B^j \in \boldsymbol{B}_{|a^\star}$ it is the case that $\ell^j \leq |A^j \cap \pi| \leq u^j - 1$, we have that $score(\boldsymbol{B}, \pi) \leq score(\boldsymbol{B}, \pi \cup \{a^\star\})$.*

3.2 Axiomatic Behavior of Scoring Functions

Now that we have introduced a complete axiomatic theory, we investigate the performance of the scoring functions we introduced regarding those axioms. We start with $score_{avg}$.

Theorem 3. *The scoring function $score_{avg}$ satisfies approval, incompatibility, dependency, and zero adequacy, as well as ballot-splitting monotonicity. It fails ballot-size monotonicity, score monotonicity, and substitution adequacy.*

Proof. Let $\boldsymbol{B}$ be a ballot and π a committee.

Approval Adequacy (✓) For every alternative $a \in \pi$, $score_{avg}$ scores the average fulfillment of the relevant bounded sets. The fulfillment being a number between 0 and 1, the average also is between 0 and 1. We thus have $score_{avg}(\boldsymbol{B}, \pi) \leq |\left(\bigcup_{B^j \in \boldsymbol{B}} A^j\right) \cap \pi|$.

Now assume that $\ell^j \leq |A^j \cap \pi| \leq s^j$ for all $B^j \in \boldsymbol{B}$. Then, for all $B \in \boldsymbol{B}$, we have $\varphi(B, \pi) = 1$. Each alternative in π then scores 1, meaning that $score_{avg}(\boldsymbol{B}, \pi) = |\left(\bigcup_{B^j \in \boldsymbol{B}} A^j\right) \cap \pi|$. ∘

Substitution Adequacy (✗) Let $\mathcal{A} = \{a_1, a_2, a_3\}$, $\pi = \{a_1, a_2\}$, and $\boldsymbol{B} = (\langle\{a_1, a_2\}, 1, 2, 2\rangle, \langle\{a_1, a_3\}, 1, 1, 2\rangle)$. Note that a_3 fulfills the conditions required for $a^\star$ in the definition of substitution adequacy (Definition 5). On the one hand, we have $score_{avg}(\boldsymbol{B}, \pi) = 2/2 + 1/1 = 2$. On the other hand, for $\pi' = \{a_1, a_2, a_3\}$, we have $score_{avg}(\boldsymbol{B}, \pi') = \frac{1+1/2}{2} + 1/1 + \frac{1/2}{1} = 9/4 > 2$. ∘

Incompatibility Adequacy (✓) Let $\boldsymbol{B}' = \boldsymbol{B} \oplus \langle A, 1, 1, 1\rangle$ be the ballot with added incompatibility.

First assume $|\pi \cap A| \neq 1$. It is clear that for each $a \in A$, $|\boldsymbol{B}'_{|a}| = 1 + |\boldsymbol{B}_{|a}| > |\boldsymbol{B}_{|a}|$ holds, *i.e.*, the normalization factor decreases. This decrement together with $\varphi(\langle A, 1, 1, 1\rangle, \pi) = 0$ results in a decreased score contribution of a and thus $score_{avg}(\boldsymbol{B}, \pi) > score(\boldsymbol{B}', \pi)$.

Now assume that $\pi \cap A = \{a\}$ for some $a \in \mathcal{A}$. We distinguish three cases. (1) If $\boldsymbol{B}_{|a} = \emptyset$, then $score(\boldsymbol{B}', \pi) = score_{avg}(\boldsymbol{B}, \pi) + 1 > score_{avg}(\boldsymbol{B}, \pi)$. (2) If $\boldsymbol{B}_{|a} \neq \emptyset$ and for all $B^j \in \boldsymbol{B}_{|a}$, $\ell^j \leq |A^j \cap \pi| \leq s^j$ holds. Then clearly $score_{avg}(\boldsymbol{B}, \pi) = score(\boldsymbol{B}', \pi)$. (3) Finally, assume $\boldsymbol{B}_{|a} \neq \emptyset$ but for some $B^j \in \boldsymbol{B}_{|a}$ either $|A^j \cap \pi| < \ell^j$ or $s^j < |A^j \cap \pi|$. Let $\alpha = \sum_{B^j \in \boldsymbol{B}_{|a}} \varphi(B^j, \pi)$ and $\beta = |\boldsymbol{B}_{|a}|$. By assumption, $\alpha < \beta$ holds. Note that α/β is the contribution of a to the score of ballot $\boldsymbol{B}$. Moreover, $\alpha+1/\beta+1$ is the contribution of a to the score of $\boldsymbol{B}'$. Since for any $0 \leq \alpha < \beta$ we have $\alpha/\beta < \alpha+1/\beta+1$, we immediately obtain $score_{avg}(\boldsymbol{B}, \pi) < score(\boldsymbol{B}', \pi)$. ∘

Dependency Adequacy (✓) Let $\boldsymbol{B}' = \boldsymbol{B} \oplus \langle A, |A|, |A|, |A|\rangle$ be the ballot with added dependency.

First, assume $A \not\subseteq \pi$. Trivially, if A is disjoint with the other bounded sets, then $score_{avg}(\boldsymbol{B}, \pi) = score(\boldsymbol{B}', \pi)$. Otherwise, note that for each $a \in A$ that also occurs in another bounded set, we have $|\boldsymbol{B}'_{|a}| = 1 + |\boldsymbol{B}_{|a}| > |\boldsymbol{B}_{|a}|$. The normalization factor thus decreases, and since $\varphi(\langle A, |A|, |A|, |A|\rangle, \pi) = 0$, the contribution of a to the score also decreases. Overall, $score_{avg}(\boldsymbol{B}, \pi) > score(\boldsymbol{B}', \pi)$ holds.

Now assume $A \subseteq \pi$. For each element $a \in A$ we distinguish three cases. (1) If $\boldsymbol{B}_{|a} = \emptyset$, then clearly a increases the total score by 1. (2) If $\boldsymbol{B}_{|a} \neq \emptyset$ and for all $B^j \in \boldsymbol{B}_{|a}$, $\ell^j \leq |A^j \cap \pi| \leq s^j$ holds, then a contributed 1 to the total score in $\boldsymbol{B}$ and also in $\boldsymbol{B}'$ so nothing changes. (3) Finally, assume $\boldsymbol{B}_{|a} \neq \emptyset$ but for some $B^j \in \boldsymbol{B}_{|a}$ either $|A^j \cap \pi| < \ell^j$ or $s^j < |A^j \cap \pi|$. Let $\alpha = \sum_{B^j \in \boldsymbol{B}_{|a}} \varphi(B^j, \pi)$ and $\beta = |\boldsymbol{B}_{|a}|$. By assumption, $\alpha < \beta$ holds. Note that α/β is the contribution of a to the score of ballot $\boldsymbol{B}$. Moreover, $\alpha+1/\beta+1$ is the contribution of a to the score of $\boldsymbol{B}'$. Since for any $0 \leq \alpha < \beta$ we have $\alpha/\beta < \alpha+1/\beta+1$, we have $score_{avg}(\boldsymbol{B}, \pi) < score(\boldsymbol{B}', \pi)$. ∘

Zero Adequacy (✓) Note that we always have $score_{avg}(\boldsymbol{B}, \pi) \geq 0$. Moreover, $score_{avg}(\boldsymbol{B}, \pi) = 0$ iff $\varphi(B^j, \pi) = 0$ for all $B^j \in \boldsymbol{B}$, which holds iff for all $B^j \in \boldsymbol{B}$, either $|A^j \cap \pi| > u^j$ or $|A^j \cap \pi| < \ell^j$. ∘

Ballot-Size Monotonicity (✗) Let $\mathcal{A} = \{a_1, a_2, a_3\}$, $\pi = \{a_1, a_2\}$, and $\boldsymbol{B} = (\langle\{a_1, a_2\}, 1, 2, 2\rangle)$. Observe that we have $score_{avg}(\boldsymbol{B}, \pi) = 1/1 + 1/1 = 2$. Now, let $\boldsymbol{B}' = \boldsymbol{B} \oplus \langle\{a_1, a_2, a_3\}, 1, 1, 3\rangle$. We would then get $score_{avg}(\boldsymbol{B}', \pi) = \frac{1+1/2}{2} + \frac{1+1/2}{2} = 3/2 < 2$. This shows that ballot-size monotonicity is not satisfied. ∘

Ballot-Splitting Monotonicity (✓) Assume $B^{j^\star} \in \boldsymbol{B}$ is a set with $\ell^{j^\star} \leq |A^{j^\star} \cap \pi| \leq s^{j^\star}$. Let $\boldsymbol{B}' = (\boldsymbol{B} \ominus B^{j^\star}) \oplus (\langle\{a\}, 1, 1, 1\rangle \mid a \in A^{j^\star} \cap \pi)$. Note that $|\boldsymbol{B}_{|a}| = |\boldsymbol{B}'_{|a}|$ for all $a \in \pi$. So overall splitting the

ballot has no effect on the normalization factor. Now, it is clear from the assumption that $\varphi(B^{j^\star}, \pi) = 1$. In $\boldsymbol{B}'$, for each of the newly created bounded set $B'^{j_i^\star}$, we have $\varphi(B'^{j_i^\star}, \pi) = 1$, too. Overall, nothing changes and $score_{avg}$ satisfies splitting monotonicity. ∘

Score Monotonicity (✗) Let $\mathcal{A} = \{a_1, a_2, a_3\}$, $\pi = \{a_2, a_3\}$, and $\boldsymbol{B} = (\langle\{a_1, a_2\}, 1, 1, 2\rangle, \langle\{a_1, a_3\}, 1, 1, 2\rangle)$. Note that a_1 fulfills the conditions required for $a^\star$ in the definition of score monotonicity (Definition 10). We have $score_{avg}(\boldsymbol{B}, \pi) = 1/1 + 1/1 = 2$. However, $score_{avg}(\boldsymbol{B}, \pi \cup \{a_1\}) = \frac{1/2+1/2}{2} + 1/2 + 1/2 < 2$. This shows that score monotonicity is not satisfied. □

Interestingly, the $score_{avg}$ scoring function fails substitution adequacy. As we shall see, this is due to its normalization factor. Indeed, $score_{tot}$ will not suffer this drawback, but suffers some others.

Theorem 4. *The scoring function $score_{tot}$ satisfies substitution, incompatibility, dependency, and zero adequacy, as well as ballot-size monotonicity, score monotonicity, and ballot-splitting monotonicity, but it fails approval adequacy.*

Proof. Let $\boldsymbol{B}$ be a ballot and π a committee.

Approval Adequacy (✗) Let $\mathcal{A} = \{a_1, a_2\}$, $\pi = \{a_1\}$, and $\boldsymbol{B} = (\langle\{a_1\}, 1, 1, 1\rangle, \langle\{a_1, a_2\}, 1, 2, 2\rangle)$. We have $score_{tot}(\boldsymbol{B}, \pi) = 2$ which is a clear violation of approval adequacy. ∘

Substitution Adequacy (✓) Consider a bounded approval ballot $\boldsymbol{B}$, a committee π and an alternative $a^\star \in \mathcal{A} \setminus \pi$ as in the definition of substitution adequacy (Definition 5). Let $B = \langle A, \ell, s, u\rangle$ be an arbitrary bounded set from $\boldsymbol{B}$ such that $a^\star \in A$. By the definition of $a^\star$, we know that $s \leq |A \cap \pi| \leq u - 1$. Hence, the contribution of B to $score_{tot}(\boldsymbol{B}, \pi)$ is s^j. Now, for $\pi' = \pi \cup \{a^\star\}$ we have $s + 1 \leq |A \cap \pi'| \leq u$. Hence, the contribution of B to $score_{tot}(\boldsymbol{B}, \pi')$ is also s^j. This applies to any bounded set including $a^\star$. Since the contributions of sets which don't include $a^\star$ are also unchanged, we have $score_{tot}(\boldsymbol{B}, \pi) = score_{tot}(\boldsymbol{B}, \pi')$. ∘

Incompatibility Adequacy (✓) Note that by adding a bounded set to a ballot the score cannot decrease. Further, if for the added ballot B^j, it holds that $u^j < |\pi \cap A^j|$, the score does not increase. ∘

Dependency Adequacy (✓) Note that by adding a bounded set to a ballot the score cannot decrease. Further, if for the added ballot B^j holds $\ell^j > |\pi \cap A^j|$, the score also does not increase. ∘

Zero Adequacy (✓) Note that we always have $score_{tot}(\boldsymbol{B}, \pi) \geq 0$. Moreover, $score_{tot}(\boldsymbol{B}, \pi) = 0$ iff $\varphi(B^j, \pi) = 0$ for all $B^j \in \boldsymbol{B}$, which holds iff for all $B^j \in \boldsymbol{B}$, either $|A^j \cap \pi| > u^j$ or $|A^j \cap \pi| < \ell^j$. ∘

Ballot-Size Monotonicity (✓) Note that by adding a bounded set B^j to a ballot the score cannot decrease. If for a committee π holds $\ell^j \leq |A^j \cap \pi| \leq u^j$, the score will even strictly increase. Thus, $score_{tot}$ satisfies ballot-size monotonicity. ∘

Ballot-Splitting Monotonicity (✓) Note that replacing B^j with $\ell^j \leq |A^j \cap \pi| \leq s^j$ by the bounded sets $(\langle\{a\}, 1, 1, 1\rangle \mid a \in A^j \cap \pi)$ means replacing a bounded set which contributes $|A^j \cap \pi|$ to the total score by $|A^j \cap \pi|$ many bounded sets which contribute 1 to the total score (w.r.t. π). This means no change for the score, *i.e.*, ballot-splitting monotonicity is satisfied. ∘

Score Monotonicity (✓) Let $a^\star$ be the alternative described in the definition. For bounded sets B^j with $\ell^j \leq |A^j \cap \pi| \leq s^j - 1$ it is immediate that the score contributions of alternatives in $A^j \cap \pi$ are the same in $score_{tot}(\boldsymbol{B}, \pi)$ and $score_{tot}(\boldsymbol{B}, \pi \cup \{a^\star\})$, and $a^\star$'s contribution counts on top. In bounded sets B^j with $s^j \leq |A^j \cap \pi| \leq u^j - 1$ (*i.e.*, where $a^\star$ is a substitute) we know that the contribution of B^j to $score_{tot}(\boldsymbol{B}, \pi)$ is s^j. This contribution is unchanged in $score_{tot}(\boldsymbol{B}, \pi \cup \{a^\star\})$. Thus, we can conclude that $score_{tot}(\boldsymbol{B}, \pi) \leq score_{tot}(\boldsymbol{B}, \pi \cup \{a^\star\})$. □

It turns out that also the other φ-based scoring functions $score_{min}$ and $score_{max}$ are not perfect from an axiomatic point as detailed in Table 1. The formal proofs are omitted due to space constraints.

Theorem 5. *The scoring function $score_{min}$ satisfies approval, incompatibility, dependency, and zero adequacy, as well as ballot-splitting monotonicity, but it fails substitution adequacy, ballot-size monotonicity, and score monotonicity.*

Theorem 6. *The scoring function $score_{max}$ satisfies substitution, incompatibility, dependency, and zero adequacy, as well as ballot-size monotonicity, score monotonicity, and ballot-splitting monotonicity, but it fails approval adequacy.*

Let us finally consider $score_{app}$. It fails several axioms, including substitution adequacy, as it completely ignores substitution effects.

Theorem 7. *The scoring function $score_{app}$ satisfies approval, incompatibility, and dependency adequacy, as well as ballot-size and ballot-splitting monotonicity. It fails substitution and zero adequacy, and score monotonicity.*

Proof. Approval adequacy follows directly from the definition of $score_{app}$. For ballot-size monotonicity, note that adding a bounded set never decreases the score. To prove the satisfaction of the other axioms, consider arbitrary ballot $\boldsymbol{B}$ and committee π.

Incompatibility Adequacy (✓) When we add bounded set $B^j = \langle A, 1, 1, 1\rangle$ to $\boldsymbol{B}$, the score cannot decrease as already stated above. However, it can increase only if there is some $a \in \pi$ with $a \in A^j$ and $1 = \ell^j \leq |A^j \cap \pi| \leq u^j = 1$, *i.e.*, if $|A \cap \pi| = 1$. If $|A \cap \pi| \neq 1$ it cannot increase. ∘

Dependency Adequacy (✓) When we add a bounded set indicating dependency $B^j = \langle A, |A|, |A|, |A|\rangle$ to $\boldsymbol{B}$, the score again cannot decrease. However, it can increase only if there is some $a \in \pi$ with $a \in A^j$ and $|A| = \ell^j \leq |A^j \cap \pi| \leq u^j = |A|$, *i.e.*, if $A \subseteq \pi$. If $A \nsubseteq \pi$ it cannot increase. ∘

Ballot-Splitting Monotonicity (✓) Let $B^j \in \boldsymbol{B}$ be a suitable set according to the definition of ballot-splitting monotonicity. Note that since $\ell^j \leq |A^j \cap \pi| \leq s^j$, each $a \in A^j \cap \pi$ contributes 1 to the score. By adding $\langle\{a\}, 1, 1, 1\rangle$ to the ballot, each a still contributes 1 to the score. However, for each $b \in \pi \setminus A$ there is no new bounded set concerning b, so they score just as before, too. All other alternatives in $\mathcal{A}$ score 0 just as before. Thus, the score is unchanged. ∘

Now we turn to the counterexample showing that substitution adequacy, zero adequacy, and score monotonicity are failed. Consider $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\boldsymbol{B} = (\langle\{a_1, a_2\}, 1, 1, 2\rangle)$. Let $\pi = \{a_1\}$. Note that a_2 is a suitable substitute. Let $\pi' = \pi \cup \{a_2\}$. Since $score_{app}(\boldsymbol{B}, \pi) = 1 > score_{app}(\boldsymbol{B}, \pi') = 0$, substitution is failed. We also see that a score of 0 is possible even though all upper and lower bound are respected for the committee π'. Thus, zero adequacy is failed. Finally, since adding a_2 to π decreased the score though the bounds are respected, score monotonicity is also failed. □

3.3 Impossibility and Possibility Results

The fact that $score_{avg}$, $score_{min}$, $score_{max}$, and $score_{app}$ fail substitution but satisfy approval adequacy, and that $score_{tot}$ satisfies substitution but fails approval adequacy, is actually a hint at a bigger result: it is impossible to satisfy both approval adequacy and substitution adequacy at the same time.

Theorem 8. *No scoring function satisfies approval adequacy and substitution adequacy simultaneously.*

Proof. Let *score* be a scoring function that satisfies approval adequacy and substitution. Throughout the proof, we will consider an instance with three alternatives: $\mathcal{A} = \{a_1, a_2, a_3\}$. Let us first look at the following profile $\mathfrak{B}$ composed of the single voter's ballot:

$$\boldsymbol{B} = (\langle\{a_1, a_3\}, 1, 1, 2\rangle, \langle\{a_2, a_3\}, 1, 1, 2\rangle).$$

For $\pi_1 = \{a_3\}$ approval adequacy implies $score(\boldsymbol{B}, \pi_1) = 1$. Note that a_1 is a suitable substitute for π_1 as defined in Definition 5. Thus, for $\pi_1' = \{a_1, a_3\}$ must hold $score(\mathfrak{B}, \pi_1') = score(\mathfrak{B}, \pi_1) = 1$ in order for *score* to satisfy substitution. Interestingly, alternative a_2 is a suitable substitute for π_1'. Thus, for $\pi_1'' = \{a_1, a_2, a_3\}$ substitution entails that $score(\mathfrak{B}, \pi_1'') = score(\mathfrak{B}, \pi_1') = 1$. Consider now the committee $\pi_2 = \{a_1, a_2\}$. Approval adequacy on $\boldsymbol{B}$ and π_2 implies $score(\boldsymbol{B}, \pi_2) = 2$. Alternative a_3 is a suitable substitute here, thus, for $\pi_2' = \{a_1, a_2, a_3\}$ we have $score(\mathfrak{B}, \pi_2') = score(\mathfrak{B}, \pi_2) = 2$. Since $\pi_2' = \pi_1''$, the contradiction between is immediate. □

This impossibility is quite stringent as it prevents us from modeling what we had in mind in the first place. One way to circumvent it is to restrict the ballots. For instance, whenever bounded sets are not overlapping, *i.e.*, no alternative appears in more than one bounded set per ballot, then all φ-based scoring functions coincide and thus satisfy both substitution and approval adequacy (and also any axiom that is satisfied by at least one of them).

Theorem 9. *For every ballot $\boldsymbol{B}$ such that for any two bounded sets B^j and $B^{j'}$ in $\boldsymbol{B}$, we have $A^j \cap A^{j'} = \emptyset$, $score_{min}$, $score_{max}$, $score_{avg}$, and $score_{tot}$ coincide and thus all satisfy approval, substitution, incompatibility, dependency, and zero adequacy, as well as ballot-size, ballot-splitting, and score monotonicity.*

Another approach to "escape" the impossibility would be to weaken the axioms. We first investigate weak-substitution adequacy that requires the score not to improve when adding a substitute to the committee—instead of simply scoring the same.

Definition 11 (Weak-Substitution Adequacy)**.** *A scoring function score satisfies* weak-substitution adequacy *if for every ballot $\boldsymbol{B}$ and committee π for which there exists an alternative $a^\star \in \mathcal{A} \setminus \pi$ such that for all bounded sets $B^j \in \boldsymbol{B}_{|a^\star}$, it is the case that $s^j \leq |A^j \cap \pi| \leq u^j - 1$, we have $score(\boldsymbol{B}, \pi) \geq score(\boldsymbol{B}, \pi \cup \{a^\star\})$.*

It is clear that substitution adequacy implies weak substitution adequacy. Furthermore, it is easy to see that weak-substitution adequacy together with score monotonicity implies substitution adequacy. We can conclude that no scoring function can satisfy score monotonicity, weak-substitution, and approval adequacy at the same time. Finally, note that the counterexamples used to show that $score_{avg}$, $score_{min}$, and $score_{max}$ fail substitution adequacy also show that these scoring functions fail weak-substitution adequacy. We can however prove that $score_{app}$ satisfies it.

Table 1: Summary of our axiomatic analysis. Suits show the impossibility that all axioms with same suit are combined.

$score_{\mathbf{x}}$	*min*	*max*	*avg*	*tot*	*app*
App. Adeq. ♣ ♠	✓	✓	✓	✗	✓
Subst. Adeq. ♣	✗	✗	✗	✓	✗
Incomp. Adeq.	✓	✓	✓	✓	✓
Dep. Adeq.	✓	✓	✓	✓	✓
Zero Adeq.	✗	✓	✓	✓	✗
Weak-Subst. Adeq. ♠	✗	✗	✗	✓	✓
Weak-App. Adeq.	✓	✓	✓	✗	✓
Ballot-Size Mon.	✗	✓	✗	✓	✓
Ballot-Split. Mon.	✓	✓	✓	✓	✓
Score Mon. ♠	✗	✗	✗	✓	✗

Proposition 10. *The scoring function $score_{app}$ satisfies weak-substitution adequacy.*

Proof. Consider a substitute $a^\star \in \mathcal{A} \setminus \pi$ with $s_j \leq |A^j \cap \pi| < u^j$ for all $B^j \in \boldsymbol{B}_{|a^\star}$. If $a^\star$ is added to π, it holds that $|A^j \cap (\pi \cup \{a^\star\})| = |A^j \cap \pi| + 1$ for all $B^j \in \boldsymbol{B}_{|a^\star}$ and $|A^j \cap (\pi \cup \{a^\star\})| = |A^j \cap \pi|$ for all other B^j. Together with the fact that $a^\star$ is a substitute follow the following two facts. (1) If for an alternative $a \in \pi$ exists no set $B^j \in \boldsymbol{B}_{|a}$ s.t. $\ell^j \leq |A^j \cap \pi| \leq s^j$ there is also no set $B^j \in \boldsymbol{B}_{|a}$ s.t. $\ell^j \leq |A^j \cap (\pi \cup \{a^\star\})| \leq s^j$. (2) There is no set $B^j \in \boldsymbol{B}_{|a^\star}$ s.t. $\ell^j \leq |A^j \cap (\pi \cup \{a^\star\})| \leq s^j$. Thus, the score does not increase. □

We now consider weakening approval adequacy. The impossibility result is largely based on the fact that bounded sets are overlapping. One could thus weaken approval adequacy to require scoring functions to coincide with the usual approval score only when bounded sets are not overlapping.

Definition 12 (Weak-Approval Adequacy)**.** *A scoring function score satisfies* weak-approval adequacy *if for every ballot $\boldsymbol{B}$ and committee π the following two conditions hold:*

(1) $score(\boldsymbol{B}, \pi) \leq \left|\left(\bigcup_{B^j \in \boldsymbol{B}} A^j\right) \cap \pi\right|$;

(2) $score(\boldsymbol{B}, \pi) = \left|\left(\bigcup_{B^j \in \boldsymbol{B}} A^j\right) \cap \pi\right|$ *whenever it holds that both* $\ell^j \leq |A^j \cap \pi| \leq s^j$ *and* $A^j \cap A^{j'} = \emptyset$ *for all* $B^j, B^{j'} \in \boldsymbol{B}$.

Note that approval adequacy implies weak-approval adequacy.

Can we find a scoring function satisfying both weak-approval and substitution adequacy at the same time? Yes, we could use the following scoring function, that essentially forbids overlapping ballots:

$$score(\boldsymbol{B}, \pi) = \begin{cases} score_{tot}(\boldsymbol{B}, \pi) & \text{if } A^j \cap A^{j'} = \emptyset \ \ \forall B^j, B^{j'} \in \boldsymbol{B} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that this function satisfies substitution adequacy since $score_{tot}$ does (for the first case), and any constant scoring function does as well (for the second case). Weak-approval adequacy is also trivially satisfied. Indeed, the above scoring function scores non-0 only for profiles satisfying the second condition of weak-approval

adequacy. This scoring function is quite artificial but proves possibility. We did not find an intuitively appealing scoring function that would satisfy both weak-approval and substitution adequacy.

4 EXPRESSIVENESS OF BOUNDED APPROVAL BALLOTS

We now want to study how expressive bounded approval ballots are. Because the way they are defined, we cannot discuss expressiveness of bounded approval ballots on their own, but need to consider them together with some scoring function.

In the following, we first study which kind of weak ordinal rankings over subsets of alternatives a bounded approval ballot—paired with a scoring function—can induce. Then, we show that the additional expressiveness can be crucial for the voters' satisfaction, when compared to standard approval ballots.

4.1 Limits to Expressiveness

For a given scoring function *score*, we study the limits of what a bounded approval ballot $\boldsymbol{B}$ can express. We work under the assumption that voters have ordinal preferences over committees and we measure expressiveness through the type of rankings over committees that can be induced by the scoring function, when ranking all committees based on their score for the ballot $\boldsymbol{B}$. Formally, every voter $i \in \mathcal{N}$ is equipped with a weak ranking over committees in C_k denoted by $\succeq_i$. We represent a weak ranking $\succeq$ over C_k as an ordered partition $\succeq = (C^1_\succeq, C^2_\succeq, \dots)$ of C_k where $C^1_\succeq$ contains the most preferred committees, and so on. The rank of a committee π in $\succeq$—denoted by $rank_\succeq(\pi)$—is the value $j \in \mathbb{N}$ such that $\pi \in C^j_\succeq$. For a given scoring function *score* and bounded approval ballot $\boldsymbol{B}$, let $\succeq^{score}_{\boldsymbol{B}}$ be the weak order over C_k such that for all $\pi, \pi' \in C_k$, $\pi \succeq^{score}_{\boldsymbol{B}} \pi'$ if and only if $score(\boldsymbol{B}, \pi) \geq score(\boldsymbol{B}, \pi')$. A scoring function can represent such a ranking $\succeq$ over C_k if there exists a bounded approval ballot $\boldsymbol{B}$ such that $\succeq^{score}_{\boldsymbol{B}}$ and $\succeq$ coincide. Everything is now set for us to delve into expressiveness. Our findings are summarized in Table 2. We start with arbitrary orders over C_k.

Proposition 11. *The scoring functions $score_{avg}$ and $score_{tot}$ can represent any arbitrary weak order $\succeq$ over C_k for any $k \geq 2$, while $score_{min}$, $score_{max}$, and $score_{app}$ cannot.*

Proof. Let $\succeq$ be an arbitrary ranking over C_k. To show that $score_{tot}$ can represent $\succeq$ we construct a ballot $\boldsymbol{B}$ as follows. For every $\pi \in C_k$, we add to $\boldsymbol{B}$ as many copies of $\langle \pi, k, k, k \rangle$ as $\binom{m}{k} - rank_\succeq(\pi)$. Since for $\pi, \pi' \in C_k$, it holds that $score_{tot}(\langle \pi, k, k, k \rangle, \pi')$ is k if $\pi = \pi'$ and zero if $\pi \neq \pi'$, the result then follows.

For $score_{avg}$, we extend the ballot described above, to bypass normalization, by enforcing that each alternative appears in an equal number of bounded sets. In particular, for all $\pi \in C_k$, we add sufficiently many copies of $\langle \pi, k-1, k-1, k-1 \rangle$ to $\boldsymbol{B}$, such that every $\pi \in C_k$ appears in exactly $\binom{m}{k}$ bounded sets in $\boldsymbol{B}$. This ensures that each alternative appears in exactly $\gamma = \binom{m}{k-1} \cdot \binom{m}{k}$ bounded sets. We thus have $\gamma \cdot score_{avg}(\boldsymbol{B}, \pi) = score_{tot}(\boldsymbol{B}, \pi)$ for all $\pi \in C_k$. The claim is thus derived from the above.

The claim for $score_{max}$ and $score_{app}$ follows from Proposition 12 (see below). For $score_{min}$, we can use a counting argument. Assume $\succeq$ is a strict ranking over C_k, *i.e.*, we have $|\succeq| = \binom{m}{k}$. We claim that for any fixed ballot $\boldsymbol{B}$, $score_{min}(\boldsymbol{B}, \pi)$ can take at most $\binom{k^2+k+1}{k}$ different values for different $\pi \in C_k$. This is because for a given bounded set $B = \langle A, \ell, s, u \rangle$, the size of the image of $\varphi(B, \pi)$, where π is the input, is at most $k^2 + 2$. Indeed, $\varphi(B, \pi)$ can be either 0, 1 or $s^j / |A^j \cap \pi|$ and that there are k^2 possible values for the latter (as if $s^j > k$, then $\varphi(B, \pi) \in \{0, 1\}$). For all k alternatives in π, $score_{min}$ takes the minimum of the relevant $\varphi(B, \pi)$ and then sum them up. The final score is then the sum of k (not necessarily distinct) values from a set of $k^2 + 2$ ones. Hence, the number of possible values for $score_{min}$ (and any $\pi \in C_k$) is bounded upwards by the number of multisets with cardinality k, taken from a set of size k^2+2. The latter is well known to be $\binom{k^2+k+1}{k}$, which is smaller than $|\succeq| = \binom{m}{k}$ as soon as $m > k^2 + k + 1$. This concludes the counting argument. □

Table 2: Expressiveness of the scoring functions.

$score_{\mathbf{x}}$	*min*	*max*	*avg*	*tot*	*app*
Arbitrary	✗	✗	✓	✓	✗
Trichotomous	✓	✗	✓	✓	✗
Dichotomous	✓	✓	✓	✓	✓

To understand where the limit in expressiveness lies for $score_{min}$, $score_{max}$, and $score_{app}$ we focus on specific classes of orders over C_k. A weak order $\succeq$ is said to be *dichotomous* if $|\succeq| = 2$, and *trichotomous* if $|\succeq| = 3$. We show that $score_{min}$ can capture the former, while $score_{max}$ and $score_{app}$ can only capture the latter.

Proposition 12. *The scoring function $score_{min}$ can represent any trichotomous weak order $\succeq$ over C_k for any $k \geq 2$, while $score_{max}$ and $score_{app}$ cannot.*

Proof. To represent a trichotomous order $\succeq = (C^1_\succeq, C^2_\succeq, C^3_\succeq)$ over C_k with $score_{min}$, we construct a ballot $\boldsymbol{B}$ as follows. First, we add $\langle \{a\}, 1, 1, 1 \rangle$ for each $a \in \mathcal{A}$ to $\boldsymbol{B}$. For now, any committee $\pi \in C_k$ would have a score of k. We diminish the score of all committees $\pi \in C^2_\succeq$ by adding $\langle \pi, 1, k-1, k \rangle$ to $\boldsymbol{B}$. Note that this does not impact the score of any committee $\pi \in C^1_\succeq$. For any $\pi \in C_k$, it now holds that each $a \in \pi$ receives a score of one if and only if there is no $\langle \pi, 1, k-1, k \rangle \in \boldsymbol{B}$, and $k-1/k$ otherwise. Finally, for any $\pi \in C^3_\succeq$, we add $\langle \pi, 1, k-1, k-1 \rangle$ to $\boldsymbol{B}$, so that all these committees score 0. We thus have three levels of score: k for committees in $C^1_\succeq$, $k-1/k$ for committees in $C^2_\succeq$ and 0 for committees in $C^3_\succeq$.[2]

For $score_{max}$, consider $\mathcal{A} = \{a, b, c, d\}$, $k = 2$ and the order $\succeq$ such that $C^1_\succeq = \{\{a, b\}, \{c, d\}\}$, $C^2_\succeq = \{\{a, d\}, \{b, c\}\}$ and $C^3_\succeq = \{\{a, c\}, \{b, d\}\}$. Consider an arbitrary ballot $\boldsymbol{B}$. It is important to note that in this case, the score of a committee would always be a multiple of $1/2$ (because $\varphi(B, \pi) \in \{0, 1/2, 1\}$ for all B and π). If $score_{max}(\boldsymbol{B}, \{a, c\}) = score_{max}(\boldsymbol{B}, \{b, d\}) = 0$, then $\boldsymbol{B}$ may not contain a bounded set with a lower bound of 1. Hence, the remaining committees can all either receive a score of 0 or 2, which cannot lead to a trichotomous order. Next, it is easy to see, that no committee $\{x, y\} \in C_k$ can achieve a score of $1/2$. The only way x can receive a score of $1/2$, is with $\langle \{x, y\}, 1, 1, 2 \rangle$ and then y necessarily yields a score of at least $1/2$, too. Hence, in order to achieve said ordering, it must hold that $score_{max}(\boldsymbol{B}, \{a, c\}) = score_{max}(\boldsymbol{B}, \{b, d\}) = 1$ and $score_{max}(\boldsymbol{B}, \{a, d\}) = score_{max}(\boldsymbol{B}, \{b, c\}) = 3/2$. For the latter to

[2]Note that this would not work for $k = 1$ as in this case, $score_{min}(\boldsymbol{B}, \pi)$ can only take two values—0 or 1—for any $\boldsymbol{B}$ and π.

hold (*i.e.*, an alternative to yield a score of 1/2), both $\langle\{a,d\},1,1,2\rangle$ and $\langle\{b,c\},1,1,2\rangle$ must be added to the ballot. This is a contradiction, because then $\{a,c\}$ yields a score of two.

For $score_{app}$ the situation is similar to $score_{max}$. To see that we are not able to represent trichotomous preferences, we can use the same counterexample, where only the first case applies. In particular, for trichotomous rankings and $k = 2$ we can always assume that the score for the least preferred committees must be zero, as $score_{app}(\boldsymbol{B},\{x,y\}) \in \{0,1,2\}$ holds by design. □

Proposition 13. *The scoring functions $score_{max}$ and $score_{app}$ can represent any dichotomous weak order $\succeq$ over C_k.*

Proof. To represent any dichotomous order $\succeq$ with $score_{max}$ or $score_{app}$, we may add the bounded set $\langle\pi,k,k,k\rangle \in \boldsymbol{B}$ to the ballot, for each committee $\pi \in C^1_{\succeq}$. For both scoring functions, the score for a committee π would then be k if $\pi \in C^1_{\succeq}$ or 0 otherwise. □

4.2 Comparison to Approval Ballots

Bounded approval ballots are our proposal to provide voters more expressive ballots. It is clear that simple approval ballots cannot express more than bounded approval ballots, since the latter generalizes the former. But are approval ballots really so much weaker than bounded approval ballots? In the following we illustrate that approval ballots—even strategic ones—can lead to much worse results for the voters than bounded approval ballots.

Bounded approval ballots also impose a restriction on the preferences that can be expressed. This is why classical measures like *distortion* [17] cannot be used here to compare bounded and standard approval ballots. We will instead show with the following examples that *(i)* standard approval ballots are not sufficiently expressive, compared to bounded ones, especially in cases where communication between the voters is impossible, and *(ii)* the loss of expressiveness can largely impact on the voters' satisfaction.

Example 14. *(Pure Substitution)* Assume that for every voter i, their preferences are defined such that there exists a set of alternatives $A_i \subseteq \mathcal{A}$ for which i is unsatisfied whenever $\pi \cap A_i = \emptyset$ and fully satisfied as soon as $\pi \cap A_i \neq \emptyset$. Note that i's preferences can easily be expressed by a single bounded set $\langle A_i,1,1,|A_i|\rangle$. Now, if voter i were asked to submit a standard approval ballot, the only reasonable ballot to submit would be A_i.

Let the number of voters n be such that n is divisible by k, and let $\mathcal{A} = \{a_1,\dots,a_{k^2}\}$. Consider the profile $\mathfrak{B}$ of bounded approval ballots in which n/k voters submit $\langle\{a_1,\dots,a_k\},1,1,k\rangle$, n/k voters submit $\langle\{a_{k+1},\dots,a_{2k}\},1,1,k\rangle$, and so on. If standard approval ballots were used, the first group of voters would approve $\{a_1,\dots,a_k\}$, the second group $\{a_{k+1},\dots,a_{2k}\}$, and so on. Overall, all alternatives would be approved by the same number of voters. Thus, if we were to select a committee of size k that maximizes the social welfare[3], for a suitable tie-breaking rule[4], $\{a_1,\dots,a_k\}$ would be selected using standard approval ballots. This fully satisfies the first voter block, but no other voters. In the case of bounded approval ballots, we have $score(\mathfrak{B},\{a_1,\dots,a_k\}) = n/k$, but $score(\mathfrak{B},\{a_k,a_{2k},\dots,a_{k^2}\}) = n$ for any φ-based scoring function *score*. Thus, $\{a_k,a_{2k},\dots,a_{k^2}\}$ may satisfy *all* voters. △

The example above shows that already for preferences incorporating approval and substitution, it is possible that only a fraction of the voters that could be fully satisfied are satisfied in simple approval voting. This becomes even worse with incompatibilities.

Example 15. *(Pure Incompatibility)* Assume that for every voter i, their preferences are defined such that there exists a set of alternatives $A_i \subseteq \mathcal{A}$ for which i is unsatisfied whenever $|\pi \cap A_i| \neq 1$ and fully satisfied otherwise. Note that voter i's preferences can be expressed by a bounded set $\langle A_i,1,1,1\rangle$.

Let $n \geq 3$, $k = 2$, and $\mathcal{A} = \{a,b,c,d\}$. Assume that the first $n-1$ voters submit the ballot $\langle\{a,b\},1,1,1\rangle$, and the last voter submits $\langle\{c,d\},1,1,1\rangle$. It is clear that according to each of our scoring functions the committees maximizing the social welfare are $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, or $\{b,d\}$. Each of them fully satisfies *all* voters. Under standard approval ballots it is reasonable to assume that the first $n-1$ voters would submit either $\{a\}$, $\{b\}$, or $\{a,b\}$, and the last one either $\{c\}$, $\{d\}$, or $\{c,d\}$. Then, unless the first $n-1$ voters all approve of only a or only b (which is unlikely if communication is impossible), the committee $\{a,b\}$ would maximize the social welfare. Note that it satisfies no voter at all. △

As voters cannot express incompatibilities in approval ballots, it is possible that all voters dislike the outcome, but there exists an outcome fully satisfying every voter. This massive difference comes solely from the bit of extra information in the bounded ballots.

5 CONCLUSIONS

We proposed bounded approval ballots as an extension to standard approval ballots. Bounded ballots are cognitively simple to use, and provide a reasonable surplus in expressiveness. Voters can easily express not only approval, but also substitution effects, incompatibilities, and dependencies between alternatives. We believe that these are the most common inter-alternative effects which voters want to express in multiwinner voting. Voters have indeed a high incentive to provide the extra information in bounded approval ballots, as it may greatly improve their satisfaction with the outcome.

We defined several scoring functions to evaluate bounded approval ballots. Our axiomatic study discovered that maintaining a behavior similar to that of standard approval ballots and simultaneously capturing substitution effects is generally impossible. This however seems to be a general problem that is not specific to bounded approval ballots. The most convenient way to circumvent the impossibility is to require bounded sets within a voter's ballots to be disjoint.

Of course, we want our ballot format to be used and tested in real elections. This will be the ultimate test to find out whether the ballot format is both simple and expressive enough to provide a real benefit for the voters. To allow interested research teams to make their own tests, we provide a prototype web-application in the following GitHub repository: github.com/claussmann/GoodVotes.

[3]For a profile of standard approval ballots $(A_i)_{i\in\mathcal{N}}$, the social welfare for a committee π is defined as $\sum_{i\in\mathcal{N}} |A_i \cap \pi|$. For a profile of bounded approval ballots $\mathfrak{B}$ with scoring function *score*, the social welfare of a committee π is defined as $\sum_{i\in\mathcal{N}} score(\mathfrak{B},\pi)$.
[4]Note that we could also add an extra voter to the first group to make tie-breaking unnecessary. However, the result would then only hold asymptotically for large n.

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